GLOBAL SOLVABILITY FOR INVOLUTIVE SYSTEMS
ON THE TORUS

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Abstract. In this article, we consider a class of involutive systems of \( n \) smooth vector fields on the torus of dimension \( n + 1 \). We prove that the global solvability of this class is related to an algebraic condition involving Liouville forms and the connectedness of all sublevel and superlevel sets of the primitive of a certain 1-form associated with the system.

1. Introduction

In this article we study the global solvability of a system of vector fields on \( \mathbb{T}^{n+1} \cong (\mathbb{R}/2\pi\mathbb{Z})^{n+1} \) given by

\[
L_j = \frac{\partial}{\partial t_j} + (a_j + ib_j)(t) \frac{\partial}{\partial x}, \quad j = 1, \ldots, n,
\]

where \( (t_1, \ldots, t_n, x) = (t, x) \) denotes the coordinates on \( \mathbb{T}^{n+1} \), \( a_j, b_j \in C^\infty(\mathbb{T}^n; \mathbb{R}) \) and for each \( j \) we consider \( a_j \) or \( b_j \) identically zero.

We assume that the system (1.1) is involutive (see [1, 12]) or equivalently that the 1-form \( c(t) = \sum_{j=1}^{n} (a_j + ib_j)(t) dt_j \in \bigwedge^1 C^\infty(\mathbb{T}^n_t) \) is closed.

When the 1-form \( c(t) \) is exact the problem was treated by Cardoso and Hounie in [9]. Here, we will consider that only the imaginary part of \( c(t) \) is exact, that is, the real 1-form \( b(t) = \sum_{j=1}^{n} b_j(t) dt_j \) is exact.

The system (1.1) gives rise to a complex of differential operators \( L \) which at the first level acts in the following way

\[
\mathbb{L}u = d_t u + c(t) \wedge \frac{\partial}{\partial x} u, \quad u \in C^\infty(\mathbb{T}^{n+1}) \text{ or } \mathcal{D}'(\mathbb{T}^{n+1}),
\]

where \( d_t \) denotes the exterior differential on the torus \( \mathbb{T}^n_t \). Our aim is to carry out a study of the global solvability at the first level of this complex. In other words we study the global solvability of the equation \( \mathbb{L}u = f \) where \( u \in \mathcal{D}'(\mathbb{T}^{n+1}) \) and \( f \in C^\infty(\mathbb{T}^n_t \times \mathbb{T}^1_x, \bigwedge^{1,0}) \).

Note that if the equation \( \mathbb{L}u = f \) has a solution \( u \) then \( f \) must be of the form

\[
f = \sum_{j=1}^{n} f_j(t, x) dt_j.
\]

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The local solvability of this complex of operators was studied by Treves in his seminal work [11].

When each function $b_j \equiv 0$, the global solvability was treated by Bergamasco and Petronilho in [8]. In this case the system is globally solvable if and only if the real 1-form $a(t) = \sum_{j=1}^{n} a_j(t)dt_j$ is either non-Liouville or rational (see definition in [2]).

When $c(t)$ is exact the problem was solved by Cardoso and Hounie in [9]. In this case the 1-form $c$ has a global primitive $C$ defined on $T^n$ and global solvability is equivalent to the connectedness of all sublevels and superlevels of the real function $\text{Im}(C)$.

We are interested in global solvability when at least one of the functions $b_j \neq 0$ and $c(t)$ is not exact. Moreover, we suppose that $\text{Im}(c)$ is exact and for each $j$, $a_j \equiv 0$ or $b_j \equiv 0$.

We prove that system (1.1) is globally solvable if and only if the real 1-form $a$ is either non-Liouville or rational and any primitive of the 1-form $b(t)$ has only connected sublevels and superlevels on $T^n$ (see Theorem 2.2).

The articles [3, 4, 5, 6, 7, 10] deal with similar questions.

2. Preliminaries and statement of the main result

There are natural compatibility conditions on the 1-form $f$ for the existence of a solution $u$ to the equation $\mathbb{L}u = f$. We now move on to describing them.

If $f \in C^\infty(T^n_x \times T^1_x; \wedge^1,0)$ we consider the $x$-Fourier series

$$f(t, x) = \sum_{\xi \in \mathbb{Z}} \hat{f}(t, \xi)e^{i\xi x},$$

where $\hat{f}(t, \xi) = \sum_{j=1}^{n} \hat{f}_j(t, \xi)dt_j$ and $\hat{f}_j(t, \xi)$ denotes the Fourier transform with respect to $x$.

Since $b$ is exact there exists a function $B \in C^\infty(T^n_x; \mathbb{R})$ such that $dt_B = b$. Moreover, we may write $a = a_0 + \alpha A$ where $A \in C^\infty(T^1_x; \mathbb{R})$ and $a_0 \in \wedge^1 \mathbb{R}^n \simeq \mathbb{R}^n$.

Thus, we may write $c(t) = a_0 + \alpha C$ where $C(t) = A(t) + iB(t)$.

We will identify the 1-form $a_0 \in \wedge^1 \mathbb{R}^n$ with the vector $a_0 := (a_{10}, \ldots, a_{n0})$ in $\mathbb{R}^n$ consisting of the periods of the 1-form $a$ given by

$$a_{j0} = \frac{1}{2\pi} \int_0^{2\pi} a_j(0, \ldots, \tau_j, \ldots, 0)d\tau_j.$$  

Thus, if $f \in C^\infty(T^n_x \times T^1_x; \wedge^1,0)$ and if there exists $u \in \mathcal{D}'(T^{n+1})$ such that $\mathbb{L}u = f$ then, since $\mathbb{L}$ defines a differential complex, $Lf = 0$ or equivalently $L_j f_k = L_k f_j$, $j, k = 1, \ldots, n$; also

$$\hat{f}(t, \xi)e^{i\xi(a_0 + t + C(t))}$$

is exact when $\xi a_0 \in \mathbb{Z}$.  \hspace{1cm} (2.1)

We define now the set

$$\mathcal{E} = \{f \in C^\infty(T^n_x \times T^1_x; \wedge^1,0); \mathbb{L}f = 0 \text{ and (2.1) holds}\}.$$

**Definition 2.1.** The operator $\mathbb{L}$ is said to be globally solvable on $T^{n+1}$ if for each $f \in \mathcal{E}$ there exists $u \in \mathcal{D}'(T^{n+1})$ satisfying $\mathbb{L}u = f$. 


Given $\alpha \notin \mathbb{Q}^n$ we say that $\alpha$ is \textit{Liouville} when there exists a constant $C > 0$ such that for each $N \in \mathbb{N}$ the inequality

$$\max_{j=1,...,n} |\alpha_j - \frac{p_j}{q}| \leq \frac{C}{q^N},$$

has infinitely many solutions $(p_1,\ldots,p_n,q) \in \mathbb{Z}^n \times \mathbb{N}$.

Let us consider the following two sets

$J = \{j \in \{1,\ldots,n\}; b_j \equiv 0\}$, $K = \{k \in \{1,\ldots,n\}; a_k \equiv 0\};$

and we will write $J = \{j_1,\ldots,j_m\}$ and $K = \{k_1,\ldots,k_p\}$. Under the above notation, the main result of this work is the following theorem.

\textbf{Theorem 2.2.} Let $B$ be a global primitive of the 1-form $b$. If $J \cup K = \{1,\ldots,n\}$ then the operator $L$ given in (1.2) is globally solvable if and only if one of the following two conditions holds:

(I) $J \neq \emptyset$ and $(a_{j_1},\ldots,a_{j_m}) \notin \mathbb{Q}^m$ is non-Liouville.

(II) The sublevels $\Omega_s = \{t \in \mathbb{T}^n; B(t) < s\}$ and superlevels $\Omega_s = \{t \in \mathbb{T}^n; B(t) > s\}$ are connected for every $s \in \mathbb{R}$ and $(a_{j_1},\ldots,a_{j_m}) \in \mathbb{Q}^m$ if $J \neq \emptyset$.

Note that if $J = \emptyset$ then $K = \{1,\ldots,n\}$ (since $J \cup K = \{1,\ldots,n\}$ by hypothesis). In this case each $a_k \equiv 0$ and Theorem 2.2 says that $L$ is globally solvable if and only if all the sublevels and superlevels of $B$ are connected in $\mathbb{T}^n$, which is according to [9].

When $J = \{1,\ldots,n\}$ we have that $b = 0$, hence any primitive of $b$ has only connected sublevels and superlevels on $\mathbb{T}^n$. In this case Theorem 2.2 says that $L$ is globally solvable if and only if either $a_0 \notin \mathbb{Q}^n$ is non-Liouville or $a_0 \in \mathbb{Q}^n$, which was proved in [8]. Thus, in order to prove Theorem 2.2 it suffices to consider the following situation $\emptyset \neq J \neq \{1,\ldots,n\}$.

\textbf{Remark 2.3.} As in [8], the differential operator $L$ is globally solvable if and only if the differential operator

$$d_t + (a_0 + ib(t)) \wedge \frac{\partial}{\partial x}$$

is globally solvable.

Indeed, consider the automorphism

$$S : \mathcal{D}'(\mathbb{T}^{n+1}) \rightarrow \mathcal{D}'(\mathbb{T}^{n+1})$$

$$\sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi)e^{i\xi x} \rightarrow \sum_{\xi \in \mathbb{Z}} \hat{u}(t,\xi)e^{i\xi A(t)}e^{i\xi x},$$

where $A$ is the previous smooth real valued function satisfying $d_t A = a(t) - a_0$.

Observe that following relation holds:

$$SLS^{-1} = d_t + (a_0 + ib(t)) \wedge \frac{\partial}{\partial x},$$

which ensures the above statement.

Therefore, it is sufficient to prove Theorem 2.2 for the operator (2.2). For the rest of this article, we will denote by $L$ the operator (2.2); that is,

$$L = d_t + (a_0 + ib(t)) \wedge \frac{\partial}{\partial x}$$

(2.3)
and by $E$ the corresponding space of compatibility conditions. The new operator $L$ is associated with the vector fields

$$L_j = \frac{\partial}{\partial t_j} + (a_{j0} + i b_j(t)) \frac{\partial}{\partial x}, \quad j = 1, \ldots, n. \quad (2.4)$$

3. Sufficiency part of Theorem 2.2

First assume that $(a_{j1}, \ldots, a_{jn}, 0) \notin Q^m$ is non-Liouville where

$$J = \{j_1, \ldots, j_m\} := \{j \in \{1, \ldots, n\}, \ b_j \equiv 0\}.$$  

Then, there exist a constant $C > 0$ and an integer $N > 1$ such that

$$\max_{j \in J} |\xi a_{j0} - \kappa_j| \geq C |\xi|^{-N}, \quad \forall (\kappa, \xi) \in Z^m \times N. \quad (3.1)$$

Consider the set $I$ where $I \cap J = \{1, \ldots, n\}$ and $I \cap J = \emptyset$. Remember that $\emptyset \neq J \neq \{1, \ldots, n\}$ then $I \neq \emptyset$ and $b_{\ell} \neq 0$ if $\ell \in I$.

We denote by $t_j$ the variables $t_{j_1}, \ldots, t_{j_m}$ and by $t_I$ the other variables on $T^n_I$. Let $f(t, x) = \sum_{j=1}^n f_j(t, x) dt_j \in E$. Consider the $(t_I, x)$-Fourier series as follows

$$u(t, x) = \sum_{(\kappa, \xi) \in Z^m \times Z} \hat{u}(t_I, \kappa, \xi) e^{i(\kappa \cdot t_I + \xi x)} \quad (3.2)$$

and for each $j = 1, \ldots, n$,

$$f_j(t, x) = \sum_{(\kappa, \xi) \in Z^m \times Z} \hat{f}_j(t_I, \kappa, \xi) e^{i(\kappa \cdot t_I + \xi x)}, \quad (3.3)$$

where $\kappa = (\kappa_{j_1}, \ldots, \kappa_{j_m}) \in Z^m$ and $\hat{u}(t_I, \kappa, \xi)$ and $\hat{f}_j(t_I, \kappa, \xi)$ denote the Fourier transform with respect to variables $(t_{j_1}, \ldots, t_{j_m}, x)$.

Substituting the formal series (3.2) and (3.3) in the equations $L_j u = f_j, \ j \in J$, we have for each $(\kappa, \xi) \neq (0, 0)$

$$i(\kappa_{j} + \xi a_{j0}) \hat{u}(t_I, \kappa, \xi) = \hat{f}_j(t_I, \kappa, \xi), \quad j \in J.$$  

Also, from the compatibility conditions $L_j f_{\ell} = L_{\ell} f_j$, for all $j, \ell \in J$, we obtain the equations

$$(\kappa_{j} + \xi a_{j0}) \hat{f}_{\ell}(t_I, \kappa, \xi) = (\kappa_{\ell} + \xi a_{00}) \hat{f}_j(t_I, \kappa, \xi), \quad j, \ell \in J.$$  

By the preceding equations we have

$$\hat{u}(t_I, \kappa, \xi) = \frac{1}{i(\kappa M + \xi a_{M0})} \hat{f}_M(t_I, \kappa, \xi), \quad (\kappa, \xi) \neq (0, 0), \quad (3.4)$$

where $M \in J, \ M = M(\xi)$ is such that

$$|\kappa M + \xi a_{M0}| = \max_{j \in J} |\kappa_j + \xi a_{j0}| \neq 0.$$  

If $(\kappa, \xi) = (0, 0)$, since $\hat{f}(t_I, 0, 0)$ is exact, there exists $v \in C^\infty(T^n_{t_I})$ such that $dv = f_I(t_I, 0, 0)$. Thus, we choose $\hat{u}(t_I, 0, 0) = v(t_I)$.

Given $\alpha \in Z^m_{+}$ we obtain from (3.1) and (3.4) the inequality

$$|\partial^\alpha \hat{u}(t_I, \kappa, \xi)| \leq \frac{1}{C} |\xi|^{N-1} |\partial^\alpha \hat{f}_M(t_I, \kappa, \xi)|.$$
Since each $f_j$ is a smooth function we conclude that
\[ u(t, x) = \sum_{(\kappa, \xi) \in \mathbb{Z} \times \mathbb{Z}} \hat{u}(t_I, \kappa, \xi) e^{i(\kappa \cdot t + \xi \cdot x)} \in C^\infty(\mathbb{T}^{n+1}). \]

By construction $u$ is a solution of
\[ L_j u = f_j, \quad j \in J. \]

Now, we will prove that $u$ is also a solution to the equations
\[ L \hat{u} = f, \quad \ell \in I. \]

Let $\ell \in I$. Given $(\kappa, \xi) \neq (0, 0)$ by the compatibility condition $L_M f_{\ell} = L_{\ell} f_M$ we have
\[ i(\kappa_M + \xi a_M) \hat{f}_I(t_I, \kappa, \xi) = \frac{\partial}{\partial t_I} \hat{f}_M(t_I, \kappa, \xi) - \xi b_I(t) \hat{f}_M(t_I, \kappa, \xi). \quad (3.5) \]

Therefore, (3.4) and (3.5) imply
\[
\begin{align*}
\frac{\partial}{\partial t_I} \hat{u}(t_I, \kappa, \xi) - \xi b_I(t) \hat{u}(t_I, \kappa, \xi) & = \frac{1}{i(\kappa_M + \xi a_M)} \frac{\partial}{\partial t_I} \hat{f}_M(t_I, \kappa, \xi) - \xi b_I(t) \hat{f}_M(t_I, \kappa, \xi) \\
& = \frac{1}{i(\kappa_M + \xi a_M)} \frac{\partial}{\partial t_I} \hat{f}_M(t_I, \kappa, \xi) - \xi b_I(t) \hat{f}_M(t_I, \kappa, \xi) \\
& = \hat{f}_I(t_I, \kappa, \xi).
\end{align*}
\]

If $(\kappa, \xi) = (0, 0)$ then $\frac{\partial}{\partial t_I} \hat{u}(t_I, 0, 0) = \hat{f}_I(t_I, 0, 0)$.

We have thus proved that condition (I) implies global solvability.

Suppose now that the condition (II) holds. Let $q_{J}$ be the smallest positive integer such that $q_J(a_{j_1}, \ldots, a_{j_m}) \in \mathbb{Z}^m$.

We denote by $\mathcal{A} := q_J \mathbb{Z}$ and $\mathcal{B} := \mathbb{Z} \setminus \mathcal{A}$ and define
\[ \mathcal{D}_A^J(\mathbb{T}^{n+1}) := \{ u \in \mathcal{D}(\mathbb{T}^{n+1}); \ u(t, x) = \sum_{\xi \in \mathcal{A}} \hat{u}(t, \xi) e^{i\xi x} \}. \]

Let $L_A$ be the operator $L$ acting on $\mathcal{D}_A^J(\mathbb{T}^{n+1})$. Similarly, we define $\mathcal{D}_B^J(\mathbb{T}^{n+1})$ and $L_B$.

Then $L$ is globally solvable if and only if $L_A$ and $L_B$ are globally solvable (see [3]).

**Lemma 3.1.** The operator $L_A$ is globally solvable.

**Proof.** Since $q_J a_0 \in \mathbb{Z}^n$, we define
\[ T : \mathcal{D}_A^J(\mathbb{T}^{n+1}) \longrightarrow \mathcal{D}_A^J(\mathbb{T}^{n+1}) \]
\[ \sum_{\xi \in \mathcal{A}} \hat{u}(t, \xi) e^{i\xi x} \longrightarrow \sum_{\xi \in \mathcal{A}} \hat{u}(t, \xi) e^{-i\xi a_0 \cdot t} e^{i\xi x}. \]

Note that $T$ is an automorphism of $\mathcal{D}_A^J(\mathbb{T}^{n+1})$ (and of $C_A^\infty(\mathbb{T}^{n+1})$). Furthermore the following relation holds:
\[ T^{-1} L_A T = L_{0,A}, \tag{3.6} \]
where $L_0 := d_t + i b(t) \cdot \frac{\partial}{\partial t}$.

Let $B$ be a global primitive of $b$ on $\mathbb{T}^n$. Since all the sublevels and superlevels of $B$ are connected in $\mathbb{T}^n$, by work [8] we have $L_0$ globally solvable, hence $L_{0,A}$ is
globally solvable. Since $T$ is an automorphism, from equality (3.6) we obtain that $L_{A}$ is globally solvable.

If $q_{J} = 1$ then $A = Z$ and the proof is complete. Otherwise we have:

**Lemma 3.2.** The operator $L_{B}$ is globally solvable.

**Proof.** Let $(\kappa, \xi) \in Z^{m} \times B$. Since $q_{J}$ is defined as the smallest natural such that $q_{J}(a_{j,0}, \ldots, a_{j,m}) \in Z^{m}$, there exists $\ell \in J$ such that

$$|a_{0} - \frac{\kappa_{\ell}}{\xi}| \geq \frac{C}{|\xi|},$$

where $C = 1/q_{J}$. Therefore

$$\max_{j \in J} |a_{0} - \frac{\kappa_{j}}{\xi}| \geq \frac{C}{|\xi|},$$

Note that if the denominators $\xi \in B$ then $(a_{j,0}, \ldots, a_{j,m})$ behaves as non-Liouville. Thus, the rest of the proof is analogous to the case where $(a_{j,0}, \ldots, a_{j,m})$ is non-Liouville. □

4. Necessity part of Theorem 2.2

Assume first that $(a_{j,0}, \ldots, a_{j,m}) \in Q^{m}$ and the global primitive $B : T^{n} \to \mathbb{R}$ of $b$ has a disconnected sublevel or superlevel on $T^{n}$.

By Lemma 3.1 we have that $L_{A}$ is globally solvable if and only if $L_{0,A}$ is globally solvable, where $A = q_{J}Z$ and $L_{0} = dt_{0} + ib(t) \wedge \frac{\partial}{\partial x}$. Since $B$ has a disconnected sublevel or superlevel, we have $L_{0,A}$ not globally solvable by [9]. Therefore $L$ is not globally solvable.

Suppose now that $(a_{j,0}, \ldots, a_{j,m}) \notin Q^{m}$ is Liouville. Therefore, by work [8] the involutive system $L_{J}$ generated by the vector fields

$$L_{j} = \frac{\partial}{\partial t_{j}} + a_{j,0} \frac{\partial}{\partial x}, \quad j \in J = \{j_{1}, \ldots, j_{m}\},$$

is not globally solvable on $T^{m+1}$.

As in the sufficiency part, we will consider the set $I$ such that $J \cup I = \{1, \ldots, n\}$ and $J \cap I = \emptyset$.

Consider the space of compatibility conditions $E_{J}$ associated to $L_{J}$. Since (4.1) is not globally solvable on $T^{m+1}$ there exists $g(t, x) = \sum_{j \in J} g_{j}(t, j, x)dt_{j} \in E$ such that

$$L_{J}v = g$$

has no solution $v \in D'(T^{m+1})$.

Now, we define smooth functions $f_{1}, \ldots, f_{n}$ on $T^{n+1}$ such that $f = \sum_{j=1}^{n} f_{j}dt_{j} \in E$ and $Lu = f$ has no solution $u \in D'(T^{n+1})$.

Let $B$ be a primitive of the 1-form $b$. Thus, we have $\frac{\partial}{\partial t_{j}}B = b_{j}$. Since for each $j \in J$ the function $b_{j} \equiv 0$ then $B$ depends only on the variables $t_{I}$; that is, $B = B(t_{I})$.

For $\ell \in I$ we choose $f_{\ell} \equiv 0$ and for $j \in J$ we define

$$f_{j}(t, x) := \sum_{\xi \in \mathbb{Z}} \tilde{f}_{j}(t, \xi)t^{\xi}_{x},$$
where
\[
\hat{f}_j(t, \xi) := \begin{cases} 
\hat{g}_j(t, \xi)e^{\xi(B(t,t)-M)} & \text{if } \xi \geq 0 \\
\hat{g}_j(t, \xi)e^{\xi(B(t,t)-\mu)} & \text{if } \xi < 0,
\end{cases}
\]
where \(M\) and \(\mu\) are, respectively, the maximum and minimum of \(B\) over \(\mathbb{T}^n\).

Given \(\alpha \in \mathbb{Z}_+^n\), for each \(j \in J\) we obtain
\[
\partial^\alpha \hat{f}_j(t, \xi) = \left[\partial^\alpha_j g_j(t, \xi)\right]\xi^{\left|\alpha\right|}\left[\partial^\alpha_j B(t_j)\right]e^{\xi(B(t,t)-M)}, \quad \xi \geq 0,
\]
and
\[
\partial^\alpha \hat{f}_j(t, \xi) = \left[\partial^\alpha_j g_j(t, \xi)\right]\xi^{\left|\alpha\right|}\left[\partial^\alpha_j B(t_j)\right]e^{\xi(B(t,t)-\mu)}, \quad \xi < 0,
\]
where \(\left|\alpha\right| := \sum_{i \in I} \alpha_i\). Since the derivatives of \(B\) are bounded on \(\mathbb{T}^n\) then there exists a constant \(C_\alpha > 0\) such that \(\left|\partial^\alpha_j B(t_j)\right| \leq C_\alpha\) for all \(t_j \in \mathbb{T}^n\). Therefore,
\[
\left|\partial^\alpha \hat{f}_j(t, \xi)\right| \leq C_\alpha|\xi|^{|\alpha|}\left|\partial^\alpha_j g_j(t_j, \xi)\right|, \quad \xi \in \mathbb{Z}.
\]

Since \(g_j\) are smooth functions it is possible to conclude by the above inequality that \(f_j, \ j \in J\), are smooth functions. Moreover, it is easy to check that \(f = \sum_{j=1}^n f_j dt_j \in E\).

Suppose that there exists \(u \in D'(\mathbb{T}^{n+1})\) such that \(\mathbb{L} u = f\). Then, if \(u(t, x) = \sum_{\xi \in \mathbb{Z}} \hat{u}(t, \xi)e^{i\xi x}\), for each \(\xi \in \mathbb{Z}\) we have
\[
\frac{\partial}{\partial t_j} \hat{u}(t, \xi) + i\xi a_{j0} \hat{u}(t, \xi) = \hat{f}_j(t, \xi), \quad j \in J \tag{4.2}
\]
and
\[
\frac{\partial}{\partial t_\ell} \hat{u}(t, \xi) - \xi b_\ell(t) \hat{u}(t, \xi) = 0, \quad \ell \in I \tag{4.3}
\]
Thus, for each \(\ell \in I\) we may write (4.3) as follows
\[
\frac{\partial}{\partial t_\ell} \left(\hat{u}(t, \xi)e^{-\xi(B(t,t)-M)}\right) = 0, \quad \text{if } \xi \geq 0,
\]
\[
\frac{\partial}{\partial t_\ell} \left(\hat{u}(t, \xi)e^{-\xi(B(t,t)-\mu)}\right) = 0, \quad \text{if } \xi < 0.
\]
Therefore,
\[
\hat{u}(t, \xi)e^{-\xi(B(t,t)-M)} := \varphi_\ell(t_j), \quad \xi \geq 0,
\]
\[
\hat{u}(t, \xi)e^{-\xi(B(t,t)-\mu)} := \varphi_\ell(t_j), \quad \xi < 0.
\]
Let \(t_j^*\) and \(t_{j*}\) such that \(B(t_j^*) = M\) and \(B(t_{j*}) = \mu\). Thus, \(\varphi_\ell(t_j) = \hat{u}(t_j, t_j^*, \xi)\) if \(\xi \geq 0\) and \(\varphi_\ell(t_j) = \hat{u}(t_j, t_{j*}, \xi)\) if \(\xi < 0\) for all \(t_j\). Since \(u \in D'(\mathbb{T}^{n+1})\) we have
\[
v(t_j, x) := \sum_{\xi \in \mathbb{Z}} \varphi_\ell(t_j)e^{i\xi x} \in D'(\mathbb{T}^{m+1}). \tag{4.5}
\]
On the other hand, by (4.2) and (4.4) we have for each \(j \in J\)
\[
\frac{\partial}{\partial t_j} (\varphi_\ell(t_j)e^{\xi(B(t,t)-M)}) + i\xi a_{j0}(\varphi_\ell(t_j)e^{\xi(B(t,t)-M)}) = \hat{f}_j(t, \xi), \quad \xi \geq 0,
\]
\[
\frac{\partial}{\partial t_j} (\varphi_\ell(t_j)e^{\xi(B(t,t)-\mu)}) + i\xi a_{j0}(\varphi_\ell(t_j)e^{\xi(B(t,t)-\mu)}) = \hat{f}_j(t, \xi), \quad \xi < 0,
\]
thus
\[
\frac{\partial}{\partial t_j} \varphi_\ell(t_j) + i\xi a_{j0}\varphi_\ell(t_j) = \hat{g}_j(t_j, \xi), \quad \xi \in \mathbb{Z}, \ j \in J.
\]
We conclude that the $v$ given by (4.5) is a solution of $L_J v = g$, which is a contradiction.

References


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