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EXISTENCE OF SOLUTIONS FOR TWO-POINT BOUNDARY-VALUE PROBLEMS WITH SINGULAR DIFFERENTIAL EQUATIONS OF VARIABLE ORDER

SHUQIN ZHANG

ABSTRACT. In this work, we show the existence of a solution for a two-point boundary-value problem having a singular differential equation of variable order. We use some analysis techniques and the Arzela-Ascoli theorem, and then illustrate our results with examples.

1. Introduction

Fractional calculus (fractional derivatives and integrals) refer to the differential and integral operators of arbitrary order, and fractional differential equations refer to those containing fractional derivatives. The former are the generalization of integer-order differential and integral operators and the latter, the generalization of differential equations of integer order. The derivatives and integrals of variable-order, which fall into a more complex category, are those whose orders are the functions of certain variables. Recently, derivatives and integrals and differential equations of variable-order have been considered, see the references in this article. In these works, authors consider the applications of variable-order derivatives in various topics, such as anomalous diffusion modeling, mechanical applications, multifractional Gaussian noises. Moreover, a physical experimental study of calculus of variable-order has been considered in [10], a comparative study of constant-order and variable-order models has been considered in [17].

The nonlinear functional analysis methods (such as some fixed point theorems) have played a very important role in considering existence of solutions to differential equations of integer order and fractional order (constant order, such as 1/3). For such applications, because differential equations can be transformed into integral equations, by means of some fundamental properties of differential and integral calculus of integer order and fractional calculus (constant order). But, in general, we find that calculus of variable-order lacks these fundamental properties, thereby making it difficult to apply nonlinear functional analysis methods to consider existence of solution to problems for differential equations of variable-order. The following are several definitions of derivatives and integrals of variable-order for a

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function f, which can be founded in for example in [10, 20],

$$I_{a+}^{p(t)}f(t) = \int_a^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} f(s)ds, \quad p(t) > 0,\ t > a, \tag{1.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function, $-\infty < a < +\infty$, provided that the right-hand side is pointwise defined.

$$I_{a+}^{p(t)}f(t) = \int_a^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} f(s)ds, \quad p(t) > 0,\ t > a, \tag{1.2}$$

provided that the right-hand side is pointwise defined.

$$I_{a^+}^{p(t)}f(t) = \int_a^t \frac{(t-s)^{p(t-s)-1}}{\Gamma(p(t-s))} f(s)ds, \quad p(t) > 0,\ t > a, \tag{1.3}$$

provided that the right-hand side is pointwise defined.

$$D_{a+}^{p(t)}f(t) = \frac{d^n}{dt^n} I_{a+}^{n-p(t)}f(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-s)^{n-1-p(t)}}{\Gamma(n-p(t))} f(s)ds, \quad t > a, \tag{1.4}$$

where $n-1 < p(t) < n, t > a, n \in \mathbb{N}$, provided that the right-hand side is pointwise defined.

$$D_{a+}^{p(t)}f(t) = \frac{d^n}{dt^n} I_{a+}^{n-p(t)}f(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-s)^{n-1-p(s)}}{\Gamma(n-p(s))} f(s)ds, \quad t > a, \tag{1.5}$$

where $n-1 < p(t) < n, t > a, n \in \mathbb{N}$, provided that the right-hand side is pointwise defined.

$$D_{a^+}^{p(t)}f(t) = \frac{d^n}{dt^n} I_{a+}^{n-p(t)}f(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-s)^{n-1-p(t-s)}}{\Gamma(n-p(t-s))} f(s)ds, \quad t > a, \tag{1.6}$$

where $n-1 < p(t) < n$, $t > a$, $n \in \mathbb{N}$, provided that the right-hand side is pointwise defined.

In particular, when $p(t)$ is a constant function, $p(t) \equiv q$, where q is a finite positive constant, then $I_{a+}^{p(t)}, D_{a+}^{p(t)}$ are usual Riemann-Liouville fractional integral I_{a+}^q and derivative D_{a+}^q, see [6]. It is well known that fractional calculus I_{a+}^q, D_{a+}^q have the following very important properties, which play a very important role in considering existence of solutions of fractional differential equation denoted by D_{a+}^q, by means of some fixed point theorems.

Proposition 1.1 ([6])**.** *The equality* $I_{a+}^{\gamma} I_{a+}^{\delta} f(t) = I_{a+}^{\gamma+\delta} f(t)$, $\gamma > 0$, $\delta > 0$ *holds for* $f \in L(a,b)$.

Proposition 1.2 ([6])**.** *The equality* $D_{a+}^{\gamma} I_{a+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ *holds for* $f \in L(a,b)$.

Proposition 1.3 ([6])**.** *Let* $\alpha > 0$. *Then the differential equation* $D_{a+}^{\alpha} u = 0$ *has unique solution*

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, *and* $n-1 < \alpha \le n$.

Proposition 1.4 ([6])**.** *Let* $\alpha > 0$, $u \in L(a,b)$, $D_{a+}^{\alpha} u \in L(a,b)$. *Then the following equality holds*

$$I_{a+}^{\alpha} D_{a+}^{\alpha} u(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, *and* $n-1 < \alpha \le n$.

In general, these properties do not hold for derivatives and integrals of variable-order $D_{a+}^{p(t)}, I_{a+}^{p(t)}$ defined by (1.1)–(1.6). For example, when $p(t), q(t)$ are not constant functions, we have that

$$I_{a+}^{p(t)} I_{a+}^{q(t)} f(t) \neq I_{a+}^{p(t)+q(t)} f(t), p(t) > 0, q(t) > 0, \quad f \in L(a,b). \tag{1.7}$$

Example 1.5. Let $p(t) = t$, $0 \le t \le 6$,

$$q(t) = \begin{cases} 2, & 0 \le t \le 2 \\ 1, & 2 < t \le 3, \\ t, & 3 < t \le 6, \end{cases}$$

$f(t) = 1$, $0 \le t \le 6$. We calculate $I_{0+}^{p(t)} f(t)$ and $I_{0+}^{p(t)+q(t)}$ defined by (1.3).

$$\begin{aligned}
&I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \\
&= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\
&= \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\
&= \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{2-1}}{\Gamma(2)} d\tau ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\
&= \int_0^2 \frac{(t-s)^{p(t)-1} s^2}{2\Gamma(p(t))} ds + \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds,
\end{aligned}$$

$$I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} f(s) ds,$$

we see that

$$\begin{aligned}
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} &= \int_0^2 \frac{(3-s)^{3-1} s^2}{2\Gamma(3)} ds + \int_2^3 \frac{(3-s)^{3-1}}{\Gamma(3)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds \\
&= \frac{8}{5} + \int_2^3 \frac{(3-s)^{3-1} s}{\Gamma(3)} ds = \frac{8}{5} + \frac{9}{24} = \frac{79}{40},
\end{aligned}$$

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=3} = \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(3)+q(3))} f(s) ds = \int_0^3 \frac{(3-s)^{3+1-1}}{\Gamma(3+1)} ds = \frac{27}{8}$$

we see easily that

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}.$$

According to (1.7), we can see that Propositions 1.2–1.4 do not hold for $D_{a+}^{p(t)}$ and $I_{a+}^{p(t)}$ defined by (1.1)-(1.6).

Remark 1.6. For integral of variable-order defined by (1.5)-(1.6), we can not easily calculate out fractional integral $I_{a+}^{p(t)}$ of some functions $f(t)$, for example, we do not know that what $I_{a+}^{p(t)} 1 = \int_a^t \frac{(t-s)^{p(s)-1}}{\Gamma(p(s))} ds$ and $I_{a+}^{p(t)} 1 = \int_a^t \frac{(t-s)^{p(t-s)-1}}{\Gamma(p(t-s))} ds$ equal.

There also has more complex integrals and derivatives of variable-order, whose order function $p(t)$ of (1.1)–(1.6) is replaced by $p(t, f(t))$; see [6, 9, 10]. For example, for given a function f, its integral and derivative of variable order $p(t,f(t))$ ($1 < p(t,f(t)) < 2$) can be defined as follows:

$$I_{a+}^{p(t,f(t))}f(t) = \int_a^t \frac{(t-s)^{p(s,f(s))-1}}{\Gamma(p(s,f(s)))}f(s)ds, \quad t > a, \tag{1.8}$$

$$D_{a+}^{p(t,f(t))}f(t) = \frac{d^2}{dt^2}I_{a+}^{2-p(t,f(t))}f(t) = \frac{d^2}{dt^2}\int_a^t \frac{(t-s)^{1-p(s,f(s))}}{\Gamma(2-p(s,f(s)))}f(s)ds, \quad t > a, \tag{1.9}$$

provided that the right-hand side is pointwise defined.

Of course, Propositions 1.1–1.4 do not usually hold for integral and derivative of variable-order defined by (1.8), (1.9). Therefore, without those properties, a variable-order differential equation cannot be transformed into an equivalent integral equation, so that one can consider existence of solutions of a differential equation of variable-order, by means of some fixed point theorems.

In this paper, we will consider the existence of solutions to the following singular two-point boundary-value problem for differential equation of variable order

$$D_{0+}^{q(t,x(t))}x(t) = f(t,x), \quad 0 < t < T, \quad 0 < T < +\infty, \tag{1.10}$$

$$x(0) = 0, \quad x(T) = 0, \tag{1.11}$$

where $D_{0+}^{q(t,x(t))}$ denotes derivative of variable-order defined by (1.9), $1 < q(t,x(t)) \le q^* < 2$, $0 \le t \le T$, $x \in \mathbb{R}$, and $t^r f : [0,T] \times R \to R$ is a continuous function, here $0 \le r < 1$.

Due to the properties of variable-order calculus, we do not transform problem $(1.10) - (1.11)$ to an integral equation, but, through the use of analysis techniques and the Arzela-Ascoli theorem to consider existence of solution to (1.10)–(1.11).

2. Preliminaries

Through this paper, we assume that:

(H1) $q : [0,T] \times R \to (1, q^*]$ is a continuous function, here $1 < q^* < 2$;

(H2) $t^r f : [0,T] \times R \to R$ is a continuous function, $0 \le r < 1$.

It follows from the continuity of compose functions that $\Gamma(q(t,x(t)))$ is continuous on $[0,T] \times R$, when q satisfies assumption condition (H1).

We assume $\delta > 0$ to be an arbitrary small number, which is important for the next step in the analysis.

Lemma 2.1. *Let* (H1) *hold. And let $x_n, x \in C[0,T]$, assume that $x_n(t) \to x(t), t \in [0,T]$ as $n \to \infty$, then*

$$\int_0^{t-\delta} \frac{(t-s)^{1-q(s,x_n(s))}}{\Gamma(2-q(s,x_n(s)))}x_n(s)ds \to \int_0^{t-\delta} \frac{(t-s)^{1-q(s,x(s))}}{\Gamma(2-q(s,x(s)))}x(s)ds, \tag{2.1}$$

for $t \in [\delta, T]$, as $n \to \infty$.

Proof. For $x_n, x \in C[0,T]$, we see that

$$\text{if } 0 < T \le 1, \text{ then } T^{1-q(s,x_n(s))} \le T^{1-q^*},\ T^{1-q(s,x(s))} \le T^{1-q^*}, \tag{2.2}$$

$$\text{if } 1 < T < +\infty, \text{ then } T^{1-q(s,x_n(s))} < 1,\ T^{1-q(s,x(s))} < 1. \tag{2.3}$$

Thus, for $0 < T < +\infty$, we let

$$T^* = \max\{T^{1-q^*}, 1\}. \tag{2.4}$$

Let

$$M=\max_{0\le t\le T}|x(t)|+1,\quad M_1=\max_{0\le t\le T}|x_n(t)|+1,$$
$$L=\max_{0\le t\le T,\|x_n\|\le M_1}|\frac{1}{\Gamma(2-q(t,x_n(t)))}|+1.$$

By the convergence of x_n, for $\frac{(2-q^*)\varepsilon}{3LT^*T}$ (ε is arbitrary small positive number), there exists $N_0\in\mathbb{N}$ such that

$$|x_n(t)-x(t)|<\frac{(2-q^*)\varepsilon}{3LT^*T},\quad t\in[0,T],\ n\ge N_0.$$

Since $(t-s)^{1-q(s,x(s))},\delta\le t-s\le T$, is continuous with respect to its exponent $1-q(s,x(s))$, for $\frac{\varepsilon}{3MLT}$, when $n\ge N_0$, it holds

$$|(t-s)^{1-q(s,x_n(s))}-(t-s)^{1-q(s,x(s))}|<\frac{\varepsilon}{3MLT},\delta\le t-s\le T, \tag{2.5}$$

also, by continuity of $\frac{1}{\Gamma(2-q(s,x(s)))}$, for $\frac{(2-q^*)\varepsilon}{3MT^{2-q^*}}$, when $n\ge N_0$, it holds

$$|\frac{1}{\Gamma(2-q(s,x_n(s)))}-\frac{1}{\Gamma(2-q(s,x(s)))}|<\frac{(2-q^*)\varepsilon}{3MT^*T},0\le s\le T. \tag{2.6}$$

Hence, from (2.2), (2.3), (2.4), (2.5), (2.6), for $\forall\varepsilon>0$, when $n>N_0$, we have that

$$\begin{aligned}
&|\int_0^{t-\delta}\frac{(t-s)^{1-q(s,x_n(s))}}{\Gamma(2-q(s,x_n(s)))}x_n(s)ds-\int_0^{t-\delta}\frac{(t-s)^{1-q(s,x(s))}}{\Gamma(2-q(s,x(s)))}x(s)ds|\\
&\le\int_0^{t-\delta}|\frac{(t-s)^{1-q(s,x_n(s))}}{\Gamma(2-q(s,x_n(s)))}||x_n(s)-x(s)|ds\\
&\quad+\int_0^{t-\delta}|\frac{(t-s)^{1-q(s,x_n(s))}-(t-s)^{1-q(s,x(s))}}{\Gamma(2-q(s,x_n(s)))}||x(s)|ds\\
&\quad+\int_0^{t-\delta}|(t-s)^{1-q(s,x(s))}||\frac{1}{\Gamma(2-q(s,x_n(s)))}-\frac{1}{\Gamma(2-q(s,x(s)))}||x(s)|ds\\
&\le\frac{L(2-q^*)\varepsilon}{3LT^*T}\int_0^{t-\delta}(t-s)^{1-q(s,x_n(s))}ds+\frac{ML\varepsilon}{3MLT}\int_0^{t-\delta}ds\\
&\quad+\frac{M(2-q^*)\varepsilon}{3MT^*T}\int_0^{t-\delta}(t-s)^{1-q(s,x(s))}ds\\
&=\frac{(2-q^*)\varepsilon}{3T^*T}\int_0^{t-\delta}T^{1-q(s,x_n(s))}(\frac{t-s}{T})^{1-q(s,x_n(s))}ds+\frac{\varepsilon}{3T}\int_0^{t-\delta}ds\\
&\quad+\frac{(2-q^*)\varepsilon}{3T^*T}\int_0^{t-\delta}T^{1-q(s,x(s))}(\frac{t-s}{T})^{1-q(s,x(s))}ds\\
&\le\frac{(2-q^*)\varepsilon}{3T^*T}\int_0^{t-\delta}T^*(\frac{t-s}{T})^{1-q^*}ds+\frac{\varepsilon}{3T}\int_0^{t-\delta}ds\\
&\quad+\frac{(2-q^*)\varepsilon}{3T^*T}\int_0^{t-\delta}T^*(\frac{t-s}{T})^{1-q^*}ds\\
&=\frac{(2-q^*)\varepsilon}{3T^{2-q^*}}\int_0^{t-\delta}(t-s)^{1-q^*}ds+\frac{\varepsilon}{3T}\int_0^{t-\delta}ds+\frac{(2-q^*)\varepsilon}{3T^{2-q^*}}\int_0^{t-\delta}(t-s)^{1-q^*}ds\\
&=\frac{\varepsilon}{3T^{2-q^*}}(t^{2-q^*}-\delta^{2-q^*})+\frac{\varepsilon}{3T}(t-\delta)+\frac{\varepsilon}{3T^{2-q^*}}(t^{2-q^*}-\delta^{2-q^*})
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\varepsilon T^{2-q^*}}{3T^{2-q^*}} + \frac{T\varepsilon}{3T} + \frac{\varepsilon T^{2-q^*}}{3T^{2-q^*}} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}
$$

which implies that (2.1) holds. □

By a similar argument, we can show the following result.

Lemma 2.2. *Let* (H1), (H2) *hold. And let* $x_n, x \in C[0,T]$*, assume that* $x_n(t) \to x(t), t \in [0,T]$ *as* $n \to \infty$*, then*

$$
\int_0^{t-\delta} (t-s) f(s, x_n(s)) ds \to \int_0^{t-\delta} (t-s) f(s, x(s)) ds, \quad t \in [\delta, T], \tag{2.7}
$$

as $n \to \infty$.

Proof. By the convergence of x_n, for $\zeta > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$
|x_n(t) - x(t)| < \zeta, \quad t \in [0,T],\ n \geq N_0,
$$

by the continuity of tf, for $\frac{2\varepsilon}{T^2}$ (where ε is arbitrary small number), when $n \geq N_0$, it holds

$$
s^r |f(s, x_n(s)) - f(s, x(s))| < \frac{\Gamma(3-r)\varepsilon}{T^{2-r}\Gamma(1-r)}, \quad s \in [0,T].
$$

Thus, we have

$$
\begin{aligned}
&| \int_0^{t-\delta} (t-s)(f(s, x_n(s)) - f(s, x(s))) ds | \\
&\leq \int_0^{t-\delta} (t-s) s^{-r} s^r |f(s, x_n(s)) - f(s, x(s))| ds \\
&< \frac{\Gamma(3-r)\varepsilon}{T^{2-r}\Gamma(1-r)} \int_0^{t-\delta} (t-s) s^{-r} ds \\
&\leq \frac{\Gamma(3-r)\varepsilon}{T^{2-r}\Gamma(1-r)} \int_0^{t} (t-s) s^{-r} ds \\
&= \frac{\Gamma(3-r)\Gamma(1-r)\varepsilon}{T^{2-r}\Gamma(1-r)\Gamma(3-r)} t^{2-r} \leq \varepsilon,
\end{aligned}
$$

which implies that (2.7) holds. □

Lemma 2.3 ([6])**.** *Let* $[a,b]$ *be a finite interval and let* $AC[a,b]$ *be the space of functions which are absolutely continuous on* $[a,b]$*. It is known that* $AC[a,b]$ *coincides with the space of primitives of Lebesgue summable functions:*

$$
f(t) \in AC[a,b] \Leftrightarrow f(t) = c + \int_0^t \varphi(s) ds, \quad \varphi \in L(a,b),\ c \in \mathbb{R},
$$

and therefore the absolutely continuous function $f(t)$ *has a summable derivative* $f'(t) = \varphi(t)$ *almost everywhere on* $[a,b]$.

3. Existence result

By the definition of derivative of variable order, defined by (1.9), we see that problem (1.10)-(1.11) is equivalent to the equation

$$\int_0^t \frac{(t-s)^{1-q(s,x(s))}}{\Gamma(2-q(s,x(s)))}x(s)ds = c_1 + c_2 t + \int_0^t (t-s)f(s,x(s))ds, \tag{3.1}$$

for $t \in [0,T]$, where $c_1, c_2 \in \mathbb{R}$ such that $x(0) = x(T) = 0$ holds.

Theorem 3.1. *Assume that* (H1), (H2) *hold. Then problem* (1.10)-(1.11) *exists one solution* $x^* \in C[0,T]$.

Proof. To obtain the existence result for (1.10)-(1.11), we firstly verify the following sequence has convergent subsequence,

$$x_k(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ x_{k-1}(t) + \int_0^{t-\delta} \frac{(t-s)^{1-q(s,x_{k-1}(s))}}{\Gamma(2-q(s,x_{k-1}(s)))}x_{k-1}(s)ds & \\ -c_{2,k-1}(t-\delta) - \int_0^{t-\delta}(t-s)f(s,x_{k-1}(s))ds, & \delta < t \le T, \end{cases} \tag{3.2}$$

for $k = 1, 2, \dots$, where $x_0(t) = 0$, $t \in [\delta, T]$, δ is an arbitrary small number, and

$$c_{2,k-1} = \frac{\int_0^{T-\delta} \frac{(T-s)^{1-q(s,x_{k-1}(s))}}{\Gamma(2-q(s,x_{k-1}(s)))}x_{k-1}(s)ds - \int_0^{T-\delta}(T-s)f(s,x_{k-1}(s))ds}{T-\delta}, \tag{3.3}$$

such that

$$x_k(\delta) = x_k(T) = 0, \quad k = 1, 2, \dots. \tag{3.4}$$

To apply the Arzela-Ascoli theorem to consider the existence of convergent subsequence of sequence x_k defined by (3.2), firstly, we prove the uniformly bounded of sequence x_k on $[0,T]$.

We find that x_k is uniformly bounded on $[0,\delta]$. Now, we will verify sequence x_k is uniformly bounded on $[\delta,T]$. Since $x_0 = 0$ is uniformly bounded on $[0,T]$, we have that

$$\begin{aligned}
&\Big|\int_0^{T-\delta} \frac{(T-s)^{1-q(s,x_0(s))}}{\Gamma(2-q(s,x_0(s)))}x_0(s)ds - \int_0^{T-\delta}(T-s)f(s,x_0(s))ds\Big| \\
&= \Big|\int_0^{T-\delta}(T-s)f(s,0)ds\Big| \\
&= \Big|\int_0^{T-\delta}(T-s)s^{-r}s^r f(s,0)ds\Big| \\
&\le M\int_0^{T-\delta}(T-s)s^{-r}ds \\
&\le M\int_0^{T}(T-s)s^{-r}ds \\
&= \frac{M\Gamma(1-r)}{\Gamma(3-r)}T^{2-r},
\end{aligned}$$

where $M = \max_{0\le t\le T} t^r|f(t,0)| + 1$, which implies that $|c_{2,0}| \le \frac{M\Gamma(1-r)}{(T-\delta)\Gamma(3-r)}T^{2-r}$. Then, for $t \in [\delta,T]$, we have

$$|x_1(t)| = |x_0(t) + \int_0^{t-\delta} \frac{(t-s)^{1-q(s,x_0(s))}}{\Gamma(2-q(s,x_0(s)))}x_0(s)ds - c_{2,0}(t-\delta)$$

$$
\begin{aligned}
&\quad - \int_0^{t-\delta}(t-s)f(s,0)ds| \\
&= |c_{2,0}(t-\delta) - \int_0^{t-\delta}(t-s)f(s,0)ds| \\
&\le |c_{2,0}|(T-\delta) + M\int_0^{t-\delta}(t-s)s^{-r}ds \\
&\le |c_{2,0}|(T-\delta) + M\int_0^{t}(t-s)s^{-r}ds \\
&\le \frac{M\Gamma(1-r)}{\Gamma(3-r)}T^{2-r} + \frac{M\Gamma(1-r)}{\Gamma(3-r)}T^{2-r} \doteq M_1,
\end{aligned}
$$

which implies that x_1 is uniformly bounded on $[\delta,T]$, together with $x_1(t)=0$ for $t\in[0,\delta]$, we obtain that x_1 is uniformly bounded on $[0,T]$.

From (2.2), (2.3), (2.4), it holds that

$$
\begin{aligned}
&|\int_0^{T-\delta}\frac{(T-s)^{1-q(s,x_1(s))}}{\Gamma(2-q(s,x_1(s)))}x_1(s)ds - \int_0^{T-\delta}(T-s)f(s,x_1(s))ds| \\
&\le M_1\int_0^{T-\delta}|\frac{T^{1-q(s,x_1(s))}}{\Gamma(2-q(s,x_1(s)))}||(\frac{T-s}{T})^{1-q(s,x_1(s))}|ds + M_f\int_0^{T-\delta}(T-s)s^{-r}ds \\
&\le M_1L\int_0^{T-\delta}T^*(\frac{T-s}{T})^{1-q^*}ds + M_f\int_0^{T}(T-s)s^{-r}ds \\
&= \frac{M_1LT^*T^{q^*-1}}{2-q^*}(T^{2-q^*}-\delta^{2-q^*}) + \frac{M_f\Gamma(1-r)}{\Gamma(3-r)}T^{2-r} \\
&\le \frac{M_1LT^*T}{2-q^*} + \frac{M_f\Gamma(1-r)}{\Gamma(3-r)}T^{2-r} := \widetilde{M},
\end{aligned}
$$

where

$$
L = \max_{0\le t\le T,\|x_1\|\le M_1}|\frac{1}{\Gamma(2-q(t,x_1(t)))}|+1, \quad M_f = \max_{0\le t\le T,\|x_1\|\le M_1}t^r|f(t,x_1(t))|+1,
$$

which implies that $|c_{2,1}|\le\frac{\widetilde{M}}{T-\delta}$. Also, for $t\in[\delta,T]$, by (2.2), (2.3), (2.4), we have that

$$
\begin{aligned}
|x_2(t)| &\le |x_1(t)| + |c_{2,1}|(T-\delta) + \int_0^{t-\delta}|\frac{(t-s)^{1-q(s,x_1(s))}}{\Gamma(2-q(s,x_1(s)))}||x_1(s)|ds \\
&\quad + \int_0^{t-\delta}(t-s)|f(s,x_1(s))|ds \\
&\le M_1 + |c_{2,1}|(T-\delta) + M_1L\int_0^{t-\delta}T^{1-q(s,x_1(s))}(\frac{t-s}{T})^{1-q(s,x_1(s))}ds \\
&\quad + \frac{M_f\Gamma(1-r)}{\Gamma(3-r)}T^{2-r} \\
&\le M_1 + |c_{2,1}|(T-\delta) + M_1L\int_0^{t-\delta}T^*(\frac{t-s}{T})^{1-q^*}ds + \frac{M_f\Gamma(1-r)}{\Gamma(3-r)}T^{2-r} \\
&= M_1 + |c_{2,1}|(T-\delta) + \frac{M_1LT^*T^{q^*-1}}{2-q^*}(t^{2-q^*}-\delta^{2-q^*}) + \frac{M_f\Gamma(1-r)}{\Gamma(3-r)}T^{2-r}
\end{aligned}
$$

$$\leq M_1 + \widetilde{M} + \frac{M_1 L T^* T}{2 - q^*} + \frac{M_f \Gamma(1-r)}{\Gamma(3-r)} T^{2-r} =: M_2,$$

which implies that x_2 is uniformly bounded on $[\delta, T]$, together with $x_2(t) = 0$ for $t \in [0, \delta]$, we obtain that x_2 is uniformly bounded on $[0, T]$. Continuous this process, we can obtain that sequence x_k is uniformly bounded on $[0, T]$.

Now, we consider the equicontinuous of sequence x_k on $[0, T]$. Firstly, we can know that

$$\text{the function } k(t) = a^t - b^t \text{ is decreasing for } t \in (-1, 0) \text{ and } 0 < a < b < 1. \tag{3.5}$$

Indeed, since $\ln a < \ln b < 0$, $a^t > b^t > 0$, we have that

$$k'(t) = a^t \ln a - b^t \ln b < b^t \ln a - b^t \ln b = b^t(\ln a - \ln b) < 0,$$

which implies that $k(t)$ is decreasing function. Thus, for

$$l(s) = \big(\frac{t_1 - s}{T}\big)^{1-q(s,x(s))} - \big(\frac{t_2 - s}{T}\big)^{1-q(s,x(s))}$$

where $0 < \frac{t_1 - s}{T} < \frac{t_2 - s}{T} < 1$, we may look $l(s)$ as the same type as $k(s)$, then $l(s)$ is decreasing with respect to its exponent $1 - q(s, x(s))$.

In the next analysis, we will use the Minkowsk's inequality: for a, b non-negative, and any $R \geq 0$, it holds

$$(a + b)^R \leq c_R(a^R + b^R), \quad \text{where } c_R = \max\{1, 2^{R-1}\}.$$

As a result, for a, b non negative, and any $0 < \mu < 1$, it holds

$$(a + b)^\mu \leq c_\mu(a^\mu + b^\mu) = \max\{1, 2^{\mu-1}\}(a^\mu + b^\mu) = a^\mu + b^\mu. \tag{3.6}$$

Obviously, x_0 is equicontinuous on $[0, T]$. We let $M = \max_{0 \leq t \leq T} s^r |f(s, 0)| + 1$. For all $\varepsilon > 0$, and all $t_1, t_2 \in [0, T]$, $t_1 < t_2$. we consider result in two cases.

Case I: $0 \leq t_1 \leq \delta < t_2 \leq T$. We take $\eta_{1,I} = \min\{\frac{\varepsilon}{2(|c_{2,0}|+1)}, (\frac{\varepsilon(1-r)}{2MT})^{\frac{1}{1-r}}\}$, when $t_2 - t_1 < \eta_{1,I}$, we have

$$\begin{aligned}
|x_1(t_2) - x_1(t_1)| &= |c_{2,0}(t_2 - \delta) + \int_0^{t_2-\delta} (t_2 - s) f(s, 0) ds| \\
&\leq |c_{2,0}|(t_2 - \delta) + M \int_0^{t_2-\delta} (t_2 - s) s^{-r} ds \\
&\leq |c_{2,0}|(t_2 - \delta) + MT \int_0^{t_2-\delta} s^{-r} ds \\
&= |c_{2,0}|(t_2 - \delta) + \frac{MT}{1-r}(t_2 - \delta)^{1-r} \\
&\leq (|c_{2,0}| + 1)|(t_2 - t_1) + \frac{MT}{1-r}(t_2 - t_1)^{1-r} \\
&< (|c_{2,0}| + 1)|\eta_{1,I} + \frac{MT}{1-r}\eta_{1,I}^{1-r} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Case II: $\delta \leq t_1 < t_2 \leq T$. We take

$$\eta_{1,II} = \min\big\{\frac{\varepsilon(1-r)}{2((|c_{2,0}| + 1)(1-r) + MT^{1-r}}, \big(\frac{\varepsilon(1-r)}{2MT}\big)^{\frac{1}{1-r}}\big\},$$

when $t_2 - t_1 < \eta_{1,II}$, by (3.6), we have

$$
\begin{aligned}
&|x_1(t_2) - x_1(t_1)| \\
&= |c_{2,0}(t_1 - t_2) + \int_0^{t_1-\delta}(t_1 - s)f(s,0)ds - \int_0^{t_2-\delta}(t_2 - s)f(s,0)ds| \\
&\le |c_{2,0}|(t_2 - t_1) + \int_0^{t_1-\delta}|t_1 - t_2||f(s,0)|ds + \int_{t_1-\delta}^{t_2-\delta}(t_2 - s)|f(s,0)|ds \\
&\le |c_{2,0}|(t_2 - t_1) + M\int_0^{t_1-\delta}(t_2 - t_1)s^{-r}ds + M\int_{t_1-\delta}^{t_2-\delta}(t_2 - s)s^{-r}ds \\
&\le \frac{(|c_{2,0}| + 1)(1 - r) + MT^{1-r}}{1 - r}(t_2 - t_1) + \frac{MT}{1 - r}((t_2 - \delta)^{1-r} - (t_1 - \delta)^{1-r}) \\
&= \frac{(|c_{2,0}| + 1)(1 - r) + MT^{1-r}}{1 - r}(t_2 - t_1) + \frac{MT}{1 - r}((t_2 - t_1 + t_1 - \delta)^{1-r} \\
&\quad - (t_1 - \delta)^{1-r}) \\
&\le \frac{(|c_{2,0}| + 1)(1 - r) + MT^{1-r}}{1 - r}(t_2 - t_1) + \frac{MT}{1 - r}((t_2 - t_1)^{1-r} + (t_1 - \delta)^{1-r} \\
&\quad - (t_1 - \delta)^{1-r}) \\
&= \frac{(|c_{2,0}| + 1)(1 - r) + MT^{1-r}}{1 - r}(t_2 - t_1) + \frac{MT}{1 - r}(t_2 - t_1)^{1-r} \\
&< \frac{(|c_{2,0}| + 1)(1 - r) + MT^{1-r}}{1 - r}\eta_{1,II} + \frac{MT}{1 - r}\eta_{1,II}^{1-r} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

These imply that $x_1(t)$ is equicontinuous on $[0,T]$, the same result can be obtained when $t_2 < t_1$.

We let

$$
M_f = \max_{0\le s\le T, \|x_1\|\le M_1} s^r|f(s,x_1)| + 1, \quad L = \max_{0\le s\le T, \|x_1\|\le M_1} |\frac{1}{\Gamma(2 - q(s,x_1(s)))}| + 1.
$$

For all $\varepsilon > 0$, and all $t_1, t_2 \in [0,T]$, $t_1 < t_2$. We consider result in two cases.
Case I: $0 \le t_1 \le \delta < t_2 \le T$. We take

$$
\eta_{2,I} = \min\{\eta_{1,I}, \frac{\varepsilon}{4(|c_{2,0}| + 1)}, (\frac{2 - q^*}{4M_1LT^*T^{q^*-1}})^{\frac{1}{2-q^*}}, (\frac{\varepsilon(1 - r)}{4M_fT})^{\frac{1}{1-r}}\},
$$

when $t_2 - t_1 < \eta_{2,I}$, by (2.2), (2.3), (3.6) and the previous arguments, we have

$$
\begin{aligned}
&|x_2(t_2) - x_2(t_1)| \\
&= |x_1(t_2) - c_{2,0}(t_2 - \delta) + \int_0^{t_2-\delta}\frac{(t_2 - s)^{1-q(s,x_1(s))}}{\Gamma(2 - q(s,x_1(s)))}x_1(s)ds \\
&\quad - \int_0^{t_2-\delta}(t_2 - s)f(s,x_1)ds| \\
&\le |x_1(t_2)| + |c_{2,0}|(t_2 - \delta) + M_1L\int_0^{t_2-\delta}(t_2 - s)^{1-q(s,x_1(s))}ds \\
&\quad + M_f\int_0^{t_2-\delta}(t_2 - s)s^{-r}ds
\end{aligned}
$$

$$
\begin{aligned}
&\le |x_1(t_2)| + |c_{2,0}|(t_2-\delta) + M_1L\int_0^{t_2-\delta} T^{1-q(s,x_1(s))}(\frac{t_2-s}{T})^{1-q(s,x_1(s))}ds \\
&\quad + M_fT\int_0^{t_2-\delta} s^{-r}ds \\
&\le |x_1(t_2)| + |c_{2,0}|(t_2-\delta) + M_1L\int_0^{t_2-\delta} T^*(\frac{t_2-s}{T})^{1-q^*}ds \\
&\quad + \frac{M_fT}{1-r}(t_2-\delta)^{1-r} \\
&= |x_1(t_2)| + |c_{2,0}|(t_2-\delta) + \frac{M_1LT^*T^{q^*-1}}{2-q^*}(t_2^{2-q^*}-\delta^{2-q^*}) + \frac{M_fT}{1-r}(t_2-\delta)^{1-r} \\
&= |x_1(t_2)-x_1(t_1)| + |c_{2,0}|(t_2-\delta) + \frac{M_1LT^*T^{q^*-1}}{2-q^*}((t_2-\delta+\delta)^{2-q^*}-\delta^{2-q^*}) \\
&\quad + \frac{M_fT}{1-r}(t_2-\delta)^{1-r} \\
&\le |x_1(t_2)-x_1(t_1)| + |c_{2,0}|(t_2-\delta) + \frac{M_1LT^*T^{q^*-1}}{2-q^*}((t_2-\delta)^{2-q^*}+\delta^{2-q^*}-\delta^{2-q^*}) \\
&\quad + \frac{M_fT}{1-r}(t_2-\delta)^{1-r} \\
&\le |x_1(t_2)-x_1(t_1)| + (|c_{2,0}|+1)(t_2-t_1) + \frac{M_1LT^*T^{q^*-1}}{2-q^*}(t_2-t_1)^{2-q^*} \\
&\quad + \frac{M_fT}{1-r}(t_2-t_1)^{1-r} \\
&< |x_1(t_2)-x_1(t_1)| + (|c_{2,0}|+1)\eta_{2,I} + \frac{M_1LT^*T^{q^*-1}}{2-q^*}\eta_{2,I}^{2-q^*} + \frac{M_fT}{1-r}\eta_{2,I}^{1-r} \\
&< \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{7\varepsilon}{4}.
\end{aligned}
$$

Case II: $\delta \le t_1 < t_2 \le T$. We take

$$
\eta_{2,II} = \min\{\eta_{1,II}, \frac{\varepsilon}{4(|c_{2,0}|+1)}, (\frac{(2-q^*)\varepsilon}{8M_1LT^*T^{q^*-1}})^{\frac{1}{2-q^*}}, \frac{\varepsilon(1-r)}{4M_fT^{1-r}}, (\frac{\varepsilon(1-r)}{4M_fT})^{\frac{1}{1-r}}\},
$$

when $t_2 - t_1 < \eta_{2,I}$, by (2.2), (2.3), (2.4), (3.5), (3.6) and the previous arguments, we have

$$
\begin{aligned}
&|x_2(t_2) - x_2(t_1)| \\
&= |x_1(t_2) - x_1(t_1) - c_{2,1}(t_2-t_1) + \int_0^{t_2-\delta} \frac{(t_2-s)^{1-q(s,x_1(s))}}{\Gamma(2-q(s,x_1(s)))}x_1(s)ds \\
&\quad - \int_0^{t_1-\delta} \frac{(t_1-s)^{1-q(s,x_1(s))}}{\Gamma(2-q(s,x_1(s)))}x_1(s)ds - \int_0^{t_2-\delta}(t_2-s)f(s,x_1)ds \\
&\quad + \int_0^{t_1-\delta}(t_1-s)f(s,x_1)ds| \\
&\le |x_1(t_2) - x_1(t_1)| + |c_{2,1}|(t_2-t_1) + \int_{t_1-\delta}^{t_2-\delta} |\frac{(t_2-s)^{1-q(s,x_1(s))}}{\Gamma(2-q(s,x_1(s)))}||x_1(s)|ds \\
&\quad + \int_0^{t_1-\delta} |\frac{1}{\Gamma(2-q(s,x_1(s)))}||(t_2-s)^{1-q(s,x_1(s))} - (t_1-s)^{1-q(s,x_1(s))}||x_1(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_1-\delta}|t_2-t_1||f(s,x_1(s))|ds+\int_{t_1-\delta}^{t_2-\delta}(t_2-s)|f(s,x_1(s))|ds\\
&\le |x_1(t_2)-x_1(t_1)|+|c_{2,1}|(t_2-t_1)\\
&\quad+M_1L\int_0^{t_1-\delta}((t_1-s)^{1-q(s,x_1(s))}-(t_2-s)^{1-q(s,x_1(s))})ds\\
&\quad+M_1L\int_{t_1-\delta}^{t_2-\delta}(t_2-s)^{1-q(s,x_1(s))}ds+M_f\int_0^{t_1-\delta}(t_2-t_1)s^{-r}ds\\
&\quad+M_fT\int_{t_1-\delta}^{t_2-\delta}s^{-r}ds\\
&=|x_1(t_2)-x_1(t_1)|+|c_{2,1}|(t_2-t_1)\\
&\quad+M_1L\int_{t_1-\delta}^{t_2-\delta}T^{1-q(s,x_1(s))}(\frac{t_2-s}{T})^{1-q(s,x_1(s))}ds\\
&\quad+M_1L\int_0^{t_1-\delta}T^{1-q(s,x_1(s))}((\frac{t_1-s}{T})^{1-q(s,x_1(s))}-(\frac{t_2-s}{T})^{1-q(s,x_1(s))})ds\\
&\quad+\frac{M_f(t_1-\delta)^{1-r}}{1-r}(t_2-t_1)+\frac{M_fT}{1-r}((t_2-\delta)^{1-r}-(t_1-\delta)^{1-r})\\
&\le |x_1(t_2)-x_1(t_1)|+|c_{2,1}|(t_2-t_1)\\
&\quad+M_1L\int_0^{t_1-\delta}T^*((\frac{t_1-s}{T})^{1-q^*}-(\frac{t_2-s}{T})^{1-q^*})ds\\
&\quad+M_1L\int_{t_1-\delta}^{t_2-\delta}T^*(\frac{t_2-s}{T})^{1-q^*}ds+\frac{M_fT^{1-r}}{1-r}(t_2-t_1)\\
&\quad+\frac{M_fT}{1-r}((t_2-\delta)^{1-r}-(t_1-\delta)^{1-r})\\
&=|x_1(t_2)-x_1(t_1)|+|c_{2,1}|(t_2-t_1)+\frac{M_1LT^*T^{q^*-1}}{2-q^*}(t_1^{2-q^*}-\delta^{2-q^*}\\
&\quad+2(t_2-t_1+\delta)^{2-q^*}-t_2^{2-q^*}-\delta^{2-q^*})+\frac{M_fT^{1-r}}{1-r}(t_2-t_1)\\
&\quad+\frac{M_fT}{1-r}((t_2-\delta)^{1-r}-(t_1-\delta)^{1-r})\\
&\le |x_1(t_2)-x_1(t_1)|+|c_{2,1}|(t_2-t_1)+\frac{M_1LT^*T^{q^*-1}}{2-q^*}\Big(t_2^{2-q^*}-2\delta^{2-q^*}\\
&\quad+2(t_2-t_1)^{2-q^*}+2\delta^{2-q^*}-t_2^{2-q^*}\Big)+\frac{M_fT^{1-r}}{1-r}(t_2-t_1)+\frac{M_fT}{1-r}((t_2-t_1)^{1-r}\\
&\quad+(t_1-\delta)^{1-r}-(t_1-\delta)^{1-r})\\
&=|x_1(t_2)-x_1(t_1)|+|c_{2,1}|(t_2-t_1)+\frac{2M_1LT^*T^{q^*-1}}{2-q^*}(t_2-t_1)^{2-q^*}\\
&\quad+\frac{M_fT^{1-r}}{1-r}(t_2-t_1)+\frac{M_fT}{1-r}(t_2-t_1)^{1-r}\\
&<|x_1(t_2)-x_1(t_1)|+(|c_{2,1}|+1)\eta_{2,II}+\frac{2M_1LT^*T^{q^*-1}}{2-q^*}\eta_{2,II}^{2-q^*}+\frac{M_fT^{1-r}}{1-r}\eta_{2,II}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{M_f T}{1-r}\eta_{2,II}^{1-r} \\
&< \varepsilon + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = 2\varepsilon.
\end{aligned}
$$

These imply that $x_2(t)$ is equicontinuous on $[0,T]$, the same result can be obtained when $t_2 < t_1$. Continue these process, we can obtain that x_k, $k = 1, 2\dots$, is equaicontinuous on $[0,T]$.

By the arguments of equicontinuity of x_k, we can show that $x_k \in C[0,T]$, for $k = 1, 2, \dots$. Then, from the Arzela-Ascoli theorem, sequence x_k exists a convergent subsequence x_{m_k}. From (3.2), x_{m_k} should satisfy

$$
x_{m_k}(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ x_{m_{k-1}}(t) + \int_0^{t-\delta} \frac{(t-s)^{1-q(s,x_{m_{k-1}}(s))}}{\Gamma(2-q(s,x_{m_{k-1}}(s)))} x_{m_{k-1}}(s)ds & \\ -c_{2,m_{k-1}}(t-\delta) - \int_0^{t-\delta}(t-s)f(s,x_{m_{k-1}}(s))ds, & \delta < t \le T, \end{cases} \tag{3.7}
$$

where

$$
c_{2,m_{k-1}} = \frac{\int_0^{T-\delta} \frac{(T-s)^{1-q(s,x_{m_{k-1}}(s))}}{\Gamma(2-q(s,x_{m_{k-1}}(s)))} x_{m_{k-1}}(s)ds - \int_0^{T-\delta}(T-s)f(s,x_{m_{k-1}}(s))ds}{T-\delta}, \tag{3.8}
$$

such that

$$
x_{m_k}(\delta) = x_{m_k}(T) = 0, \quad k = 1, 2, \dots. \tag{3.9}
$$

Now, we prove that the continuous limit of x_{m_k}, denoted by x^* is one solution of problem (1.10)-(1.11).

Let $k \to +\infty$ in (3.7), (3.8), (3.9), by Lemmas 2.1, 2.2, we have

$$
x^*(t) = \begin{cases} 0, & 0 \le t \le \delta, \\ x^*(t) + \int_0^{t-\delta} \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds - c_2(t-\delta) & \\ -\int_0^{t-\delta}(t-s)f(s,x^*(s))ds, & \delta < t \le T, \end{cases} \tag{3.10}
$$

$$
x^*(\delta) = x^*(T) = 0. \tag{3.11}
$$

where

$$
c_2 = \frac{\int_0^{T-\delta} \frac{(T-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds - \int_0^{T-\delta}(T-s)f(s,x^*(s))ds}{T-\delta}, \tag{3.12}
$$

Thus, we find that, for $t \in [0,\delta]$, $x^* = 0$; for $t \in [\delta, T]$, x^* satisfies relation

$$
\int_0^{t-\delta} \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds - c_2(t-\delta) - \int_0^{t-\delta}(t-s)f(s,x^*(s))ds = 0, \tag{3.13}
$$

for $\delta \le t \le T$.

To verify x^* is one solution of problem (1.10)-(1.11), we let $\delta \to 0$ in (3.11), (3.12), (3.13). Now, for all $\varepsilon > 0$, take

$$
\delta_0 = \min\{(\frac{\varepsilon(2-q^*)}{MLT^*T^{q^*-1}})^{\frac{1}{2-q^*}}, (\frac{\varepsilon(1-r)}{M_f T})^{\frac{1}{1-r}}\}
$$

where

$$
M = \max_{0 \le t \le T} |x^*(t)| + 1, \quad L = \max_{0 \le t \le T, \|x^*\| \le M} |\frac{1}{\Gamma(2-q(t,x^*(t)))}| + 1,
$$

when $\delta < \delta_0$, by (2.2), (2.3), (2.4), (3.6), we have

$$\begin{aligned}
&|\int_0^{t-\delta} \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds - \int_0^t \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds| \\
&= |\int_{t-\delta}^t \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds| \\
&= |\int_{t-\delta}^t \frac{T^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} (\frac{t-s}{T})^{1-q(s,x^*(s))} x^*(s)ds| \\
&\le ML \int_{t-\delta}^t T^* (\frac{t-s}{T})^{1-q^*} ds \\
&= \frac{MLT^*T^{q^*-1}}{2-q^*} \delta^{2-q^*} \\
&< \frac{MLT^*T^{q^*-1}}{2-q^*} \delta_0^{2-q^*} = \varepsilon,
\end{aligned}$$

which implies that

$$\lim_{\delta\to 0} \int_0^{t-\delta} \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds = \int_0^t \frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds. \tag{3.14}$$

By the same arguments, we have that

$$\lim_{\delta\to 0} \int_0^{T-\delta} \frac{(T-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds = \int_0^T \frac{(T-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))} x^*(s)ds. \tag{3.15}$$

Similarly, we have

$$\begin{aligned}
&|\int_0^{t-\delta} (t-s)f(s,x^*(s)) - \int_0^t (t-s)f(s,x^*(s))ds| \\
&= |\int_{t-\delta}^t (t-s)f(s,x^*(s))ds| \\
&\le M_f \int_{t-\delta}^t (t-s)s^{-r} ds \\
&\le M_f T \int_{t-\delta}^t s^{-r} ds \\
&= \frac{M_f T}{1-r} (t^{1-r} - (t-\delta)^{1-r}) \\
&= \frac{M_f T}{1-r} ((t-\delta+\delta)^{1-r} - (t-\delta)^{1-r}) \\
&\le \frac{M_f T}{1-r} ((t-\delta)^{1-r} + \delta^{1-r} - (t-\delta)^{1-r}) \\
&= \frac{M_f T}{1-r} \delta^{1-r} \\
&< \frac{M_f T}{1-r} \delta_0^{1-r} < \varepsilon,
\end{aligned}$$

which implies

$$\lim_{\delta\to 0} \int_0^{t-\delta} (t-s)f(s,x^*(s))ds = \int_0^t (t-s)f(s,x^*(s))ds. \tag{3.16}$$

By the same arguments, we also have

$$\lim_{\delta\to 0}\int_0^{T-\delta}(T-s)f(s,x^*(s))ds=\int_0^T(T-s)f(s,x^*(s))ds. \tag{3.17}$$

Now, we let $\delta\to 0$ in (3.11), (3.12), (3.13), by (3.14), (3.15), (3.16) and (3.17), we obtain

$$x^*(0)=x^*(T)=0, \tag{3.18}$$

$$\int_0^t\frac{(t-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))}x^*(s)ds=\widetilde{c}t+\int_0^t(t-s)f(s,x^*(s))ds,\quad 0\le t\le T. \tag{3.19}$$

where

$$\widetilde{c}=\frac{\int_0^T\frac{(T-s)^{1-q(s,x^*(s))}}{\Gamma(2-q(s,x^*(s)))}x^*(s)ds-\int_0^T(T-s)f(s,x^*(s))ds}{T}.$$

Differentiating on both sides of (3.19), we obtain

$$\frac{d}{dt}I_{0+}^{2-q(t,x^*(t))}x^*(t)=\widetilde{c}+\int_0^t f(t,x^*),\quad 0<t<T, \tag{3.20}$$

From the continuity of t^rf and Lemma 2.3 it follows that $\int_0^t f(s,x^*(s))ds$ is in $AC[0,T]$; consequently, from (3.20), we obtain

$$\int_0^t f(s,x^*(s))ds=\frac{d}{dt}I_{0+}^{2-q(t,x^*(t))}x^*(t)-\widetilde{c}\in AC[0,T]. \tag{3.21}$$

As a result, differentiating on both sides of (3.21), by definition of derivative of variable-order (1.9), we obtain

$$D_{0+}^{q(t,x^*(t))}x^*(t)=f(t,x^*),0<t\le T, \tag{3.22}$$

which together with (3.18) yields that x^* is a solution of (1.10)-(1.11). Thus the proof is complete. □

Example 3.2. Consider the problem

$$\begin{gathered}D_{0+}^{q(t,x(t))}x(t)=f(t,x),\quad 0<t<1,\\ x(0)=x(1)=0,\end{gathered} \tag{3.23}$$

where $q(t,x)=1+\frac{t^3}{3}+\frac{1}{3(1+x^2)}$ is a continuous function on $[0,1]\times R$, $f(t,x)=t^{-\frac{1}{2}}+x^3$ is a continuous function on $(0,1]\times R$. Clearly, for $(t,x)\in[0,1]\times R$, we have $1<q(t,x)<1+\frac{1}{3}+\frac{1}{3}=\frac{5}{3}$. Therefore Theorem 3.1 implies that (3.22) has one solution $x^*\in C[0,1]$.

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Shuqin Zhang

Department of Mathematics, China University of Mining and Technology, Beijing 100083, China

E-mail address: `zsqjk@163.com, Tel +86 10 62331118, Fax +86 10 62331465`