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EIGENVALUE PROBLEMS FOR $p(x)$-KIRCHHOFF TYPE EQUATIONS

GHASEM A. AFROUZI, MARYAM MIRZAPOUR

Abstract. In this article, we study the nonlocal $p(x)$-Laplacian problem

$$-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

By means of variational methods and the theory of the variable exponent Sobolev spaces, we establish conditions for the existence of weak solutions.

1. Introduction

The purpose of this article is to show the existence of solutions of the $p(x)$-Kirchhoff type eigenvalue problem

$$\begin{gathered}-M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$, $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function, p, q are continuous functions on $\overline{\Omega}$ such that $1 < p(x) < N$ and $q(x) > 1$ for any $x \in \overline{\Omega}$ and λ is a positive number. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [28], electrorheological fluids [1] or image restoration [6].

Equation (1.1) is called a nonlocal problem because of the the term M, which implies that the equation in (1.1) is no longer a pointwise equation. This causes some mathematical difficulties which make the study of such a problem particularly interesting. Nonlocal differential equations are also called Kirchhoff-type equations because Kirchhoff [23] investigated an equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinct feature

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is that the (1.2) contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2\,dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$, and hence the equation is no longer a pointwise equation. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [25] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions [25], various equations of Kirchhoff-type have been studied extensively, see e.g. [3, 5] and [9]-[14]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [13, 14, 9, 10]) and $p(x)$-Laplacian (see [4, 8, 11, 12, 22]). Motivated by the above papers and the results in [7, 26], we consider (1.1) to study the existence of weak solutions.

2. Preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [15, 16, 18, 19] for details.

Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \text{for all } x \in \overline{\Omega}\};$$
$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x); x \in \overline{\Omega}\};$$
$$L^{p(x)}(\Omega) = \big\{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}dx < \infty\big\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\big\}$$

becomes a Banach space [24]. We also define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u(x)|_{L^{p(x)}(\Omega)} + |\nabla u(x)|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. In this paper, we denote by $X = W_0^{1,p(x)}(\Omega)$.

Proposition 2.1 ([15, 19]). *(i) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv|dx \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}$$

(ii) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ and $p_1(x) \le p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2 ([20]). *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}dx$, then for $u \in X$ and $(u_k) \subset X$, we have*

1. *$\|u\| < 1$ (respectively$= 1; > 1$) if and only if $\rho(u) < 1$ (respectively$= 1; > 1$);*
2. *for $u \ne 0$, $\|u\| = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*

(3) *if* $\|u\| > 1$, *then* $\|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$;
(4) *if* $\|u\| < 1$, *then* $\|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$;
(5) $\|u_k\| \to 0$ *(respectively* $\to \infty$*) if and only if* $\rho(u_k) \to 0$ *(respectively* $\to \infty$*).*

For $x \in \Omega$, let us define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3 ([19]). *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$, *then there is a continuous (compact) embedding* $X \hookrightarrow L^{q(x)}(\Omega)$.

Lemma 2.4 ([21]). *Denote*

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad \text{for all} \quad u \in X,$$

then $I(u) \in C^1(X, R)$ *and the derivative operator* I' *of* I *is*

$$\langle I'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \text{for all } u,\ v \in X,$$

and we have

(1) I *is a convex functional;*
(2) $I' : X \to X^*$ *is a bounded homeomorphism and strictly monotone operator;*
(3) I' *is a mapping of type* (S_+), *namely:* $u_n \rightharpoonup u$ *and* $\limsup_{n\to+\infty} I'(u_n)(u_n - u) \le 0$, *imply* $u_n \to u$.

Definition 2.5. A function $u \in X$ is said to be a weak solution of (1.1) if

$$M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \Big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{q(x)-2} uv \, dx = 0,$$

for all $v \in X$.

The Euler-Lagrange functional associated to (1.1) is

$$J_\lambda(u) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \Big) - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx,$$

where $\widehat{M}(t) = \int_0^t M(\tau) d\tau$. Then

$$\langle J'_\lambda(u), v\rangle = M\Big(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \Big) \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{q(x)-2} uv \, dx,$$

for all $u, v \in X$, then we know that the weak solution of (1.1) corresponds to the critical point of the functional J_λ. Hereafter $M(t)$ is supposed to verify the following assumptions:

(M1) There exists $m_2 \ge m_1 > 0$ and $\beta \ge \alpha > 1$ such that $m_1 t^{\alpha-1} \le M(t) \le m_2 t^{\beta-1}$.
(M2) For all $t \in \mathbb{R}^+$, $\widehat{M}(t) \ge M(t)t$.

For simplicity, we use c_i, to denote the general nonnegative or positive constant (the exact value may change from line to line).

3. Main results and proofs

Theorem 3.1. *Assume that M satisfies* (M1) *and* (M2) *and the function $q \in C(\overline{\Omega})$ satisfies*

$$\beta p^{+} < q^{-} \leq q^{+} < p^{*}(x). \tag{3.1}$$

Then for any $\lambda > 0$ problem (1.1) *possesses a nontrivial weak solution.*

Lemma 3.2. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in X$ with $\|u\| = \eta$.*

Proof. First, we point out that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^{-}} + |u(x)|^{q^{+}}, \quad \text{for all } x \in \overline{\Omega}.$$

Using the above inequality and (M1), we find that

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big) - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^{\alpha} - \frac{\lambda}{q^{-}}\Big(|u|_{q^{-}}^{q^{-}} + |u|_{q^{+}}^{q^{+}}\Big). \end{aligned}$$

From the assumptions of Theorem 3.1, X is continuously embedded in $L^{q^{-}}(\Omega)$ and $L^{q^{+}}(\Omega)$. Then, there exist two positive constants c_1 and c_2 such that

$$|u(x)|_{q^{-}} \leq c_1\|u\|, \quad |u(x)|_{q^{+}} \leq c_2\|u\|, \quad \text{for all } u \in X.$$

Hence, for any $u \in X$ with $\|u\| < 1$, we obtain

$$J_\lambda(u) \geq \frac{m_1}{\alpha (p^{+})^{\alpha}}\|u\|^{\alpha p^{+}} - \frac{\lambda}{q^{-}}\Big(c_1^{q^{-}}\|u\|^{q^{-}} + c_2^{q^{+}}\|u\|^{q^{+}}\Big).$$

Since the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = \frac{m_1}{\alpha (p^{+})^{\alpha}} - \frac{\lambda c_1^{q^{-}}}{q^{-}} t^{q^{-} - \alpha p^{+}} - \frac{\lambda c_2^{q^{+}}}{q^{-}} t^{q^{+} - \alpha p^{+}},$$

is positive in a neighborhood of the origin, the proof is complete. $\square$

Lemma 3.3. *There exists $e \in X$ with $\|e\| > \eta$ (where η is given in Lemma 3.2) such that $J_\lambda(e) < 0$.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and $\psi \neq 0$ and $t > 1$. By (M1) we have

$$\begin{aligned} J_\lambda(t\psi) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\psi|^{p(x)}dx\Big) - \lambda \int_\Omega \frac{1}{q(x)}|t\psi|^{q(x)}dx \\ &\leq \frac{m_2}{\beta}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\psi|^{p(x)}dx\Big)^{\beta} - \lambda \frac{t^{q^{-}}}{q^{+}}\int_\Omega |\psi|^{q(x)}dx \\ &\leq \frac{m_2}{\beta (p^{-})^{\beta}} t^{\beta p^{+}}\Big(\int_\Omega |\nabla \psi|^{p(x)}dx\Big)^{\beta} - \lambda \frac{t^{q^{-}}}{q^{+}}\int_\Omega |\psi|^{q(x)}dx. \end{aligned}$$

Since $\beta p^{+} < q^{-}$, we obtain $\lim_{t\to\infty} J_\lambda(t\psi) = -\infty$. Then for $t > 1$ large enough, we can take $e = t\psi$ such that $\|e\| > \eta$ and $J_\lambda(e) < 0$. $\square$

Proof of Theorem 3.1. By Lemmas 3.2–3.3 and the mountain pass theorem of Ambrosetti and Rabinowitz [2], we deduce the existence of a sequence $(u_n) \subset X$ such that

$$J_\lambda(u_n) \to c_3 > 0, \quad J'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

We prove that (u_n) is bounded in X. Arguing by contradiction. We assume that, passing eventually to a subsequence, still denote by (u_n), $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n. By (3.2) and (M1)-(M2), for n large enough, we have

$$\begin{aligned}
&1 + c_3 + \|u_n\| \\
&\geq J_\lambda(u_n) - \frac{1}{q^-}\langle J'_\lambda(u_n), u_n\rangle \\
&\geq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \lambda\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx \\
&\quad - \frac{1}{q^-}M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)}dx + \frac{\lambda}{q^-}\int_\Omega |u_n|^{q(x)}dx \\
&\geq \frac{m_1}{\alpha(p^+)^{\alpha-1}}\Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\|u_n\|^{\alpha p^-} + \lambda\Big(\frac{1}{q^-} - \frac{1}{q(x)}\Big)\int_\Omega |u_n|^{q(x)}dx \\
&\geq \frac{m_1}{\alpha(p^+)^{\alpha-1}}\Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\|u_n\|^{\alpha p^-} + \lambda\Big(\frac{1}{q^-} - \frac{1}{q(x)}\Big)\Big(c_1\|u_n\|^{q^-} + c_2\|u_n\|^{q^+}\Big).
\end{aligned}$$

Dividing the above inequality by $\|u_n\|^{\alpha p^-}$, taking into account (3.1) holds true and passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that (u_n) is bounded in X. This information, combined with the fact that X is reflexive, implies that there exists a subsequence, still denote by (u_n) and $u_1 \in X$ such that (u_n) converges weakly to u_1 in X. Note that Proposition 2.3 yields that X is compactly embedded in $L^{q(x)}(\Omega)$, it follows that (u_n) converges strongly to u_1 in $L^{q(x)}(\Omega)$. Then by Hölder inequality we deduce

$$\lim_{n\to\infty}\int_\Omega |u_n|^{q(x)-2}u_n(u_n - u_1)dx = 0. \tag{3.3}$$

Using (3.2), we infer that

$$\lim_{n\to\infty}\langle J'_\lambda(u_n), u_n - u_1\rangle = 0. \tag{3.4}$$

Since (u_n) is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to t_0 \geq 0 \quad \text{as } n \to \infty.$$

If $t_0 = 0$ then (u_n) converges strongly to $u_1 = 0$ in X and the proof is complete. If $t_0 > 0$ then since the function M is continuous, we obtain

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big) \to M(t_0) \geq 0 \quad \text{as } n \to \infty.$$

Thus, by (M1), for sufficiently large n, we have

$$0 < c_4 \leq M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big) \leq c_5. \tag{3.5}$$

From (3.3)-(3.5), we deduce that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u_1)dx = 0. \tag{3.6}$$

Using Lemma 2.4, we deduce that actually (u_n) converges strongly to u_1 in X. Then by relation (3.2) we have

$$J_\lambda(u_1) = c_3 > 0, \quad J'_\lambda(u_1) = 0;$$

that is, u_1 ia a nontrivial weak solution of (1.1). □

Theorem 3.4. *If we assume that* (M1)–(M2) *hold and* $q \in C_+(\overline{\Omega})$ *satisfies*

$$1 < q^- \le q^+ < \alpha p^-, \tag{3.7}$$

then there exists $\lambda^* > 0$ *such that for any* $\lambda > \lambda^*$*, problem* (1.1) *possesses a nontrivial weak solution.*

Under the theorem's conditions, we want to construct a global minimizer of the functional. We start with the following auxiliary result.

Lemma 3.5. *The functional* J_λ *is coercive on* X*.*

Proof. By Theorem 3.1 and Proposition 2.2, we deduce that for all $u \in X$,

$$J_\lambda(u) \ge \frac{m_1}{\alpha (p^+)^\alpha}\Big(\int_\Omega |\nabla u|^{p(x)}dx\Big)^\alpha - \frac{\lambda}{q^-}\Big(c_1\|u\|^{q^-} + c_2\|u\|^{q^+}\Big).$$

Now we set $\|u\| > 1$, then

$$J_\lambda(u) \ge \frac{m_1}{\alpha (p^+)^\alpha}\|u\|^{\alpha p^-} - \frac{\lambda}{q^-}\Big(c_1\|u\|^{q^-} + c_2\|u\|^{q^+}\Big).$$

Since by relation (3.7) we have $\alpha p^- > q^+ \ge q^-$, we infer that $J_\lambda(u) \to \infty$ as $\|u\| \to \infty$. In other words, J_λ is coercive in X. □

Proof of Theorem 3.4. $J_\lambda(u)$ is a coercive functional and weakly lower semi-continuous on X. These two facts enable us to apply [27, Theorem 1.2] in order to find that there exists $u_\lambda \in X$ a global minimizer of J_λ and thus a weak solution of problem (1.1).

We show u_λ is not trivial for λ large enough. Letting $t_0 > 1$ be a constant and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$, we assume that $v_0 \in C_0^\infty(\overline{\Omega})$ is such that $v_0(x) = t_0$ for any $x \in \overline{\Omega_1}$ and $0 \le v_0(x) \le t_0$ in $\Omega \backslash \Omega_1$. We have

$$\begin{aligned} J_\lambda(v_0) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla v_0|^{p(x)}dx\Big) - \lambda\int_\Omega \frac{1}{q(x)}|v_0|^{q(x)}dx \\ &\le c_6 - \frac{\lambda}{q^+}\int_\Omega |v_0|^{q(x)}dx \le c_6 - \frac{\lambda}{q^+}t_0^{q^-}|\Omega_1|. \end{aligned}$$

So there exists $\lambda^* > 0$ such that $J_\lambda(v_0) < 0$ for any $\lambda \in [\lambda^*, +\infty)$. It follows that for any $\lambda \ge \lambda^*$, u_λ is a nontrivial weak solution of problem (1.1) for λ large enough. □

Theorem 3.6. *If* $q \in C_+(\overline{\Omega})$ *with*

$$1 < q(x) < p(x) < p^*(x), \tag{3.8}$$

then there exists $\lambda^{**} > 0$ *such that for any* $\lambda \in (0, \lambda^{**})$*, problem* (1.1) *possesses a nontrivial weak solution.*

We plan to apply Ekeland variational principle [17] to get a nontrivial solution to problem (1.1). We start with two auxiliary results.

Lemma 3.7. *There exists* $\lambda^{**} > 0$ *such that for any* $\lambda \in (0, \lambda^{**})$ *there are* $\rho, a > 0$ *such that* $J_\lambda(u) \ge a > 0$ *for any* $u \in X$ *with* $\|u\| = \rho$*.*

Proof. Under the assumption of Theorem 3.6, X is continuously embedded in $L^{q(x)}(\Omega)$. Thus, there exists a positive constant c_7 such that

$$|u|_{q(x)} \le c_7\|u\| \quad \text{for all } u \in X. \tag{3.9}$$

Now, Let us assume that $\|u\| < \min\{1, \frac{1}{c_7}\}$, where c_7 is the positive constant from above. Then we have $|u|_{q(x)} < 1$. Using Proposition 2.2 we obtain

$$\int_\Omega |u|^{q(x)}dx \le |u|_{q(x)}^{q^-} \quad \text{for all } u \in X \text{ with } \|u\| = \rho \in (0,1). \tag{3.10}$$

Relations (3.9) and (3.10) imply

$$\int_\Omega |u|^{q(x)}dx \le c_7^{q^-}\|u\|^{q^-} \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{3.11}$$

Using the hypotheses (M1) and (3.11), we deduce that for any $u \in X$ with $\|u\| = \rho$, the following hold

$$\begin{aligned}
J_\lambda(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big) - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\
&\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \frac{\lambda}{q^-}c_7^{q^-}\|u\|^{q^-} \\
&= \rho^{q^-}\Big(\frac{m_1}{\alpha(p^+)^\alpha}\rho^{\alpha p^+ - q^-} - \frac{\lambda}{q^-}c_7^{q^-}\Big).
\end{aligned}$$

By (3.8) we have $q^- \le q^+ < p^- \le p^+ < \alpha p^+$. So, if we take

$$\lambda^{**} = \frac{m_1 q^-}{2\alpha(p^+)^\alpha}\rho^{\alpha p^+ - q^-}, \tag{3.12}$$

then for any $\lambda \in (0,\lambda^{**})$ and $u \in X$ with $\|u\| = \rho$, there exists $a = \frac{\rho^{\alpha p^+}}{2\alpha(p^+)^\alpha}$ such that $J_\lambda(u) \ge a > 0$. $\square$

Lemma 3.8. *For any $\lambda \in (0,\lambda^{**})$ given by (3.12), there exists $\varphi \in X$ such that $\varphi \ge 0$, $\varphi \ne 0$ and $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough.*

Proof. Assumption (3.8) implies that $q(x) < \beta p(x)$. Let $\epsilon_0 > 0$ such that $q^- + \epsilon_0 < \beta p^-$. Since $q \in C(\overline{\Omega})$, there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. It follows that $q(x) < q^- + \epsilon_0 < \beta p^-$ for all $x \in \Omega_0$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}(\varphi) \supset \overline{\Omega_0}$, $\varphi(x) = 1$ for all $x \in \overline{\Omega_0}$ and $0 \le \varphi \le 1$ in Ω. Then for any $t \in (0,1)$, we have

$$\begin{aligned}
J_\lambda(t\varphi) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}dx\Big) - \lambda\int_\Omega \frac{1}{q(x)}|t\varphi|^{q(x)}dx \\
&\le \frac{m_2}{\beta}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}dx\Big)^\beta - \lambda\int_\Omega \frac{1}{q(x)}t^{q(x)}|\varphi|^{q(x)}dx \\
&\le \frac{m_2}{\beta(p^-)^\beta}t^{\beta p^-}\Big(\int_\Omega |\nabla\varphi|^{p(x)}dx\Big)^\beta - \frac{\lambda}{q^+}t^{q^- + \epsilon_0}\int_{\Omega_0}|\varphi|^{q(x)}dx < 0,
\end{aligned}$$

for all $t < \delta^{\frac{1}{\beta p^- - q^- - \epsilon_0}}$ with

$$0 < \delta < \min\Big\{1, \frac{\lambda\beta(p^-)^\beta}{m_2 q^+}\frac{\int_{\Omega_0}|\varphi|^{q(x)}dx}{\big(\int_\Omega |\nabla\varphi|^{p(x)}dx\big)^\beta}\Big\}.$$

$\square$

Proof of Theorem 3.6. Let λ^{**} be defined as in (3.12) and $\lambda \in (0, \lambda^{**})$. By Lemma 3.7, it follows that on the boundary of the ball centered at the origin and of radius ρ in X, we have

$$\inf_{\partial B_\rho(0)} J_\lambda(u) > 0.$$

On the other hand, by Lemma 3.8, there exists $\varphi \in X$ such that

$$J_\lambda(t\varphi) < 0 \quad \text{for } t > 0 \text{ small enough.}$$

Moreover, for $u \in B_\rho(0)$,

$$J_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} - \frac{\lambda}{q^-} c_7^{q^-} \|u\|^{q^-}.$$

It follows that

$$-\infty < c_8 = \inf_{\overline{B_\rho(0)}} J_\lambda(u) < 0.$$

We let now $0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland variational principle [17] to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon$$
$$J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

Since

$$J_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} J_\lambda + \varepsilon \leq \inf_{B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $K_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $K_\lambda(u) = J_\lambda(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of K_λ and thus

$$\frac{K_\lambda(u_\varepsilon + tv) - K_\lambda(u_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and $v \in B_\rho(0)$. The above relation yields

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle J'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$ and we infer that $\|J'_\lambda(u_\varepsilon)\| \leq \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_1(0)$ such that

$$J_\lambda(v_n) \to c_8, \quad J'_\lambda(v_n) \to 0. \tag{3.13}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_2 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_2 in X. Actually, with similar arguments as those used in the proof Theorem 3.1, we can show that $v_n \to u_2$ in X. Thus, by relation (3.13),

$$J_\lambda(u_2) = c_8 < 0, \quad J'_\lambda(u_2) = 0;$$

i.e., u_2 is a nontrivial weak solution for problem (1.1). $\square$

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4. Corrigendum posted in August 27, 2014

A reader pointed out that no function $M(t)$ can satisfy both hypotheses (M1) and (M2). In response, we present a proof of our results by adding the following assumption

$$m_1 q^-(p^-)^\alpha > m_2 \alpha p^-(p^+)^\alpha. \tag{4.1}$$

and without assumption (M2).

Modified assumptions. We delete the assumption (M2) and re-state the following:

(M1) There exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}, \quad \forall t \in \mathbb{R}^+$$

(The original (M1) implies $\alpha = \beta$, so we rename constant α.);

In the proof of Theorem 3.1, By (3.2) and (M1), for n large enough, we can write

$$\begin{aligned}
&1 + c_3 + \|u_n\| \\
&\geq J_\lambda(u_n) - \frac{1}{q^-}\langle J'_\lambda(u_n), u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big) - \lambda\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}\,dx \\
&\quad - \frac{1}{q^-}M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big) + \frac{\lambda}{q^-}\int_\Omega |u_n|^{q(x)}\,dx \\
&\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)^\alpha - \frac{m_2}{q^-}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)^{\alpha-1}\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&\quad + \lambda\int_\Omega\Big(\frac{1}{q^-} - \frac{1}{q(x)}\Big)|u_n|^{q(x)}\,dx \\
&\geq \Big(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{q^-(p^-)^{\alpha-1}}\Big)\|u_n\|^{\alpha p^-} + \lambda\Big(c_1\|u_n\|^{q^-} + c_2\|u_n\|^{q^+}\Big).
\end{aligned}$$

Dividing the above inequality by $\|u_n\|^{\alpha p^-}$, taking into account (3.1) and (4.1) hold true and passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that (u_n) is bounded in X.

Theorem 3.6 remains unchanged. However, Theorems 3.1 and 3.4 need to be stated without assumption (M2). Relation (3.1) need to be changed by $\alpha p^+ < q^- \leq q^+ < p^*(x)$. The proofs of Theorems and Lemmas are similar to the original proofs, but replacing β by α.

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Ghasem Alizadeh Afrouzi
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: afrouzi@umz.ac.ir

Maryam Mirzapour
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: mirzapour@stu.umz.ac.ir