POINTWISE ESTIMATES FOR SOLUTIONS TO A SYSTEM OF NONLINEAR DAMPED WAVE EQUATIONS

WENJUN WANG

Abstract. In this article, we consider the existence of global solutions and pointwise estimates for the Cauchy problem of a nonlinear damped wave equation. We obtain the existence by using the approach introduced by Li and Chen in [7] and some estimates of the solution. The proofs of the estimates are based on a detailed analysis of the Green function of the linear damped wave equations. Also, we show the $L^p$ convergence rate of the solution.

1. Introduction

In this paper, we consider the nonlinear damped wave equation

$$\begin{align*}
\partial_t^2 u - \Delta u + \partial_t u &= F(u), \quad t > 0, x \in \mathbb{R}^n, \\
u(0, x) &= a(x), \quad \partial_t u(0, x) = b(x), \quad x \in \mathbb{R}^n,
\end{align*}$$

(1.1)

where $u(t, x) = (u_1(t, x), u_2(t, x), \ldots, u_m(t, x)) : (0, T) \times \mathbb{R}^n \to \mathbb{R}^m$ is the unknown vector valued function and $a(x) = (a_1(x), \ldots, a_m(x))$ and $b(x) = (b_1(x), \ldots, b_m(x))$ are given initial data. The nonlinear smooth vector function $F : \mathbb{R}^m \to \mathbb{R}^m$, $F(u) = (F_1(u), \ldots, F_m(u))$ such that

$$F_j(u) = O \left( \prod_{k=1}^l a_k^{p_{j,k}} \right),$$

(1.2)

with $p_{j,k} \geq 1$ or $p_{j,k} = 0$ for $j, k = 1, \ldots, m$.

The first aim of this paper is to obtain the existence of classical global solutions to system (1.1). We show the existence directly by using the Banach fixed point theorem with a detailed analysis of the Green function. At the same time, we have the following decay rates of the solutions

$$\|u_j(t)\|_{L^\infty} \leq C(1 + t)^{-n/2}, \quad \|u_j(t)\|_{L^2} \leq C(1 + t)^{-n/4}, \quad j = 1, \ldots, m.$$  

(1.3)

The second aim is to get the pointwise estimate of the solutions to system (1.1). With the help of the pointwise estimates of the Green function and using the method of the Green function, we show the pointwise estimates of the solutions to system (1.1). This estimates represent a clear decaying structure of the solutions. Furthermore, we get the optimal $L^p$ decay estimates of the solutions.
There are many authors working in this field. For the single nonlinear damped wave equation
\[
\partial_t^2 u - \Delta u + \partial_t u = f(u), \quad t > 0, x \in \mathbb{R}^n, \\
u(0, x) = a_1(x), \quad \partial_t u(0, x) = b_1(x), \quad x \in \mathbb{R}^n, 
\]
many results have been published. For the case \( f(u) = -|u|^{\theta} u \), Kawashima, Nakao and Ono [6] studied the decay properties of solutions to (1.4) by using the energy method combined with \( L^p-L^q \) estimates. Ono [19] derived sharp decay rates in the subcritical case of solutions in unbounded domains in \( \mathbb{R}^n \). Nakao and Ono in [12] proved the existence and decay of global solutions weak solutions for (1.4) by using the potential well method. By employing the weighted \( L^2 \) energy method, Nishihara and Zhao [16] obtained that the behavior of solutions to (1.4) as \( t \to \infty \) is expected to be same as that for the corresponding heat equation. The global asymptotic behaviors were studied by Nishihara [14, 15] for \( n = 3, 4 \) and Ikehata, Nishihara and Zhao [4] for \( n \geq 1 \). In [9, 11], the pointwise estimates of classical solutions to (1.4) were obtained.

For the case of \( f(u) = |u|^{\theta} u \), Ikehata, Miyaoka and Nakatake [3] obtained the global existence of weak solutions to (1.4). Furthermore, Hosono and Ogawa [11] obtained that the behavior of solutions to (1.4) as \( t \to \infty \) is expected to be same as that for the corresponding heat equation. The global asymptotic behaviors were studied by Nishihara [14, 15] for \( n = 3, 4 \) and Ikehata, Nishihara and Zhao [4] for \( n \geq 1 \). In [9, 11], the pointwise estimates of classical solutions to (1.4) were obtained.

For the general case \( f(u) = O(u^{\theta+1}) \), Wang and Wang [25] proved the pointwise estimates of classical solutions to (2.1). There also have been a lot of investigations for those cases. For detail results, please refer to [2, 5, 8, 20, 21, 24, 28].

For the system of the nonlinear damped wave equations
\[
\partial_t^2 u_1 - \Delta u_1 + \partial_t u_1 = |u_m|^{p_1}, \quad t > 0, x \in \mathbb{R}^n, \\
\partial_t^2 u_2 - \Delta u_2 + \partial_t u_2 = |u_1|^{p_2}, \quad t > 0, x \in \mathbb{R}^n, \\
\ldots \\
\partial_t^2 u_m - \Delta u_m + \partial_t u_m = |u_{m-1}|^{p_m}, \quad t > 0, x \in \mathbb{R}^n, \\
u_j(0, x) = a_j(x), \quad \partial_t u_j(0, x) = b_j(x), \quad x \in \mathbb{R}^n, \quad (1 \leq j \leq m), 
\]
Sun and Wang [22] for \( m = 2 \) and Takeda [23] for \( m \geq 2 \) obtained global weak solutions to system (1.5).

For the general case (1.1), Ogawa and Takeda in [17] obtained the existence of the global solution under some conditions, which include the results of [22] and [23]. Recently, Ogawa and Takeda in [18] proved the asymptotic behavior of solutions to the problem (1.1) by using the \( L^p-L^q \) type decomposition of the fundamental solution of the linear damped wave equations into the dissipative part and hyperbolic part.

However, there are few studies concerning the global existence and decay property of classical solutions to the Cauchy problem of the nonlinear damped wave system. In this paper, we investigate the global existence and pointwise estimates of classical solution to system (1.1). First of all, we employ the Green function of the linear damped wave equation to express the solution of system (1.1). Then, we obtain the global solution directly by using the method introduced by Li and Chen in [7]. Unlike the usual energy method, this method needn’t to prove the local
existence and extend the local solution to the global one in time. In this process, the decay properties of the Green function play an important role. We employ $G_1$ and $G_2$ to define the Green function of linear equation. By a detailed analysis of the Green function, we obtain the pointwise estimates of the Green function. Compared with the methods in [11, 13, 14], the method of dealing with the existence theory in this paper is more useful to show a clear decaying structure of the solution. Secondly, with the obtained pointwise estimates of the Green function, we give the pointwise estimates of the solution to (1.4) by the method of the Green function. Finally, as a corollary of the pointwise estimates, the optimal $L^p$ ($1 \leq p \leq \infty$) convergence rate can be obtained easily.

Throughout this paper, we assume that the nonlinear term $\{F_j(u)\}_{j=1}^m$ satisfies the following conditions, for $p_{j,k} \in [0, +\infty) \cup \{0\}$, $(j = 1, \ldots, m; k = 1, \ldots, m)$,

$$|\partial^{\tilde{a}_j}F_j(u)| \leq C_{\tilde{a}_j,\delta} \sum_{\alpha_j,1+\cdots+\alpha_j,m=\tilde{a}_j} \prod_{k=1}^m |u_k|^{(p_{j,k}-\alpha_j,k)_+}, \quad |u_j| \leq \delta, \quad 0 \leq \tilde{a}_j \leq \tilde{p}_j,$$

$$|\partial^{\tilde{a}_j}F_j(u)| \leq C_{\tilde{a}_j,\delta}, \quad |u_j| \leq \delta, \quad \tilde{p}_j \leq \tilde{a}_j \leq l,$$

and for $|u_j| \leq \delta$, $|v_j| \leq \delta$, $\tilde{a}_j \leq l$,

$$|\partial^{\tilde{a}_j}F_j(u) - \partial^{\tilde{a}_j}F_j(v)| \leq C_{\tilde{a}_j,\delta} \sum_{\alpha_j,1+\cdots+\alpha_j,m=\tilde{a}_j} \prod_{k=1}^m |u_k|^{(p_{j,k}-\alpha_j,k)_+} \prod_{k=s+1}^m |v_k|^{(p_{j,k}-\alpha_j,k)_+} \times \left(|u_l|^{(p_{j,l}-\alpha_j,l)_+} + |v_l|^{(p_{j,l}-\alpha_j,l)_+}\right)|u_s - v_s| \right),$$

where

$$\tilde{p}_j = \sum_{k=1}^m p_{j,k}, \quad \tilde{a}_j = \sum_{k=1}^m \alpha_j,k$$

with $\alpha_j,k \geq 0$ and $(a)_+ = \max\{a,0\}$.

Our main results are the following two theorems.

**Theorem 1.1.** Assume that $\tilde{p}_j \geq 2$, $\tilde{p}_j > 1 + \frac{2}{n}$, $l \geq n+1$ and the initial data $\{(a_j,b_j)\}_{j=1}^m \subset (H^{l+1}(R^n) \cap W^{l,1}(R^n)) \times (H^l(R^n) \cap W^{l,1}(R^n))$ and

$$N_0 := \sum_{j=1}^m (\|a_j\|_{H^{l+1}(R^n) \cap W^{l,1}(R^n)} + \|b_j\|_{H^l(R^n) \cap W^{l,1}(R^n)}),$$

is sufficiently small and the nonlinear coupling $F(u)$ satisfies the assumptions [1.6], [1.7] and [1.8]. Then there exists a unique global classical solution $\{u_j(t)\}_{j=1}^m$ of system (1.1).

Moreover, for $j = 1, 2, \ldots, m$, we have the decay estimates

$$\|u_j\|_{W^{l-1,\infty}(R^n)} \leq C(1+t)^{-n/2}, \quad \text{and} \quad \|u_j\|_{H^l} \leq C(1+t)^{-n/4}.$$  \hspace{1cm} (1.10)

For the solution in the above theorem, we have the following pointwise estimates.

**Theorem 1.2.** Under the assumptions of Theorem 1.1, if for any multi-index $\alpha$, $|\alpha| < l$, there exist some constant $r > n$ and a small positive constant $\varepsilon_0$, such that

$$|\partial^{\alpha}a_j| + |\partial^{\alpha}b_j| \leq \varepsilon_0(1 + |x|^2)^{-r},$$

\hspace{1cm} (1.11)
then for $|\alpha| < l - n$ the solution to (1.1) satisfies:
\[
|D_x^\alpha u_j(t)| \leq C(1 + t)^{-\frac{n + |\alpha|}{2}} B_{\frac{4}{n}}(|x|, t),
\]  
where $B_N(|x|, t) = (1 + \frac{|x|^2}{1 + t})^{-N}$. And for $p \in [1, \infty]$, we have
\[
\|D_x^\alpha u_j(t)\|_{L^p(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n}{p}(1 - \frac{1}{p}) - \frac{|\alpha|}{2}}.
\]  

**Notation.** Various positive constants will be denoted by $C$. $W^{m,p}(\mathbb{R}^n)$, with $m \in \mathbb{Z}_+$ and $p \in [1, \infty]$, denotes the usual Sobolev space with the norm
\[
\|f\|_{W^{m,p}(\mathbb{R}^n)} := \sum_{k=0}^m \|\partial_x^k f\|_{L^p(\mathbb{R}^n)}.
\]  
In particular, we use $W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^n)}$, $\|\cdot\|_{m,p} = \|\cdot\|_{W^{m,p}(\mathbb{R}^n)}$ and $\|\cdot\|_m = \|\cdot\|_{H^m(\mathbb{R}^n)}$.

The rest of this article is organized as follows. In the next section, we show the pointwise estimates of the solutions for nonlinear equations are obtained in Section 4. Furthermore, the pointwise estimates of solutions for nonlinear equations are obtained in Section 4.

## 2. Green Function

To obtain the global existence and pointwise estimates of the solutions, we should first derive representation formulas of the solutions through the Green function. The single linearized equation of (1.1) is
\[
\partial_t^2 u_j - \Delta u_j + \partial_t u_j = 0,
\]
\[
|\partial = 0, \quad u_j| = a_j, \quad u_j| = b_j.
\]  
Then, the Green function of (2.1) can be defined as follows:
\[
\partial_t^2 G_1 - \Delta G_1 + \partial_t G_1 = 0,
\]
\[
G_1| = \delta(x), \quad G_1| = 0,
\]
\[
G_2| = 0, \quad G_2| = \delta(x).
\]

Now, we show the formulas of the solutions in the following theorem. The proof of the theorem is similar to that of [27 Theorem 2.5]. We show the proof here for the convenience of the readers.

**Theorem 2.1.** The solution of (2.1) is
\[
u_j(x, t) = G_1 * a_j + G_2 * b_j + \int_0^t G_2(t - s) * F_j(u)(s)ds.
\]  

**Proof.** It is obvious that
\[
u_j(x, 0) = G_1(x, 0) * a_j + G_2(x, 0) * b_j = \delta(x) * a_j = a_j.
\]
\[
u_j(x, t) = G_1(x, t) * a_j + G_2(x, 0) * b_j + G_2(t - t) * F_j(u)(t)
\]
\[
+ \int_0^t G_2(t - s) * F_j(u)(s)ds.
\]
Then, we have
\begin{align}
  u_{j1}(x, 0) = \delta(x) * b_j = b_j, \\
  \Delta u_j = \Delta G_1 * a_j + \Delta G_2 * b_j + \int_0^t \Delta G_2(t - s) * F_j(u)(s)ds, \\
  \partial_t u_j = \partial_t G_1 * a_j + \partial_t G_2 * b_j + G_2(t - t) * F_j(u)(t) + \int_0^t \partial_t G_2(t - s) * F_j(u)(s)ds, \\
  \partial^2_t u_j = \partial^2_t G_1 * a_j + \partial^2_t G_2 * b_j + G_2(t - t) * F_j(u)(t) \\
  + \int_0^t \partial^2_t G_2(t - s) * F_j(u)(s)ds \\
  = \partial^2_t G_1 * a_j + \partial^2_t G_2 * b_j + G_2(u)(t) + \int_0^t \partial^2_t G_2(t - s) * F_j(u)(s)ds.
\end{align}

Then, by the definition of the Green function, we obtain
\begin{align}
  \partial^2_t u_j - \Delta u_j + \partial_t u_j &= (\partial^2_t G_1 - \Delta G_1 + \partial_t G_1) * a_j + (\partial^2_t G_2 - \Delta G_2 + \partial_t G_2) * b_j \\
  &+ F_j(u)(t) + \int_0^t (\partial^2_t G_2 - \Delta G_2 + \partial_t G_2)(t - s) * F_j(u)(s)ds \\
  &= F_j(u)(t).
\end{align}

The proof is complete.

By the Fourier transform and a direct calculation, we have
\begin{align}
  \hat{G}_1(\xi, t) &= -\frac{\tau-}{\tau+ - \tau-} e^{\tau+ t} + \frac{\tau+}{\tau+ - \tau-} e^{\tau- t}, \\
  \hat{G}_2(\xi, t) &= \frac{1}{\tau+ - \tau-} e^{\tau+ t} - \frac{1}{\tau+ - \tau-} e^{\tau- t},
\end{align}

where $\hat{G}_i(\xi, t) (i = 1, 2)$ are the Fourier transform corresponding to $G_i(x, t)$ and
\begin{align}
  \tau_{\pm} = -1 \pm \sqrt{1 - 4|\xi|^2}. 
\end{align}

In what follows, we show the asymptotic behavior of $u$ by using the pointwise estimates of $G_1, G_2$. First, we divide $|\xi|$ into three cases: $|\xi|$ is small, bounded and big enough. Here, we set
\begin{align}
  \chi_1(\xi) = \begin{cases} 
  1, & \text{if } \xi < \epsilon, \\
  0, & \text{if } \xi > 2\epsilon,
\end{cases} \quad \chi_3(\xi) = \begin{cases} 
  1, & \text{if } \xi > R, \\
  0, & \text{if } \xi < R - 1,
\end{cases}
\end{align}

where $\epsilon$ is small enough, $R$ is large enough, and $\chi_1, \chi_3$ are smooth functions. We denote $\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi)$. Then, we set $\hat{G}_{i,j}(\xi, t) = \chi_j(\xi) G_i(\xi, t)$ where $i = 1, 2; j = 1, 2, 3$.

To deal with the coupling of Green function, we need the following two lemmas. The following lemma corresponds to [26] Lemma 3.2. We omit its proof here.

**Lemma 2.2.** Assume that $\text{supp} \hat{f} \subset \{ \xi : |\xi| > R \}$ with $R$ large enough, $|\partial^\beta \hat{f}| \leq C|\xi|^{-|\beta|-1}$, then there exist distributions $f_1(x), f_2(x)$, such that $f(x) = f_1(x) + f_2(x)$.
f_2(x). And |\partial_x^\alpha f_1(x)| \leq C(1+|x|^2)^{-N} for positive number 2N > n + |\alpha|, \|f_2\|_{L^1} \leq C, \text{ supp } f_2(x) \subset \{x, |x| < 2\varepsilon\} \text{ with } \varepsilon \text{ being sufficiently small.}

The proof of the following lemma can be founded in [10]. We omit it here.

**Lemma 2.3.** If the functions \( H(x, t) \) and \( S(x, t) \) satisfy

\[
|\partial_x^\alpha H(x, t)| \leq C(1 + t)^{-(n + |\alpha|)/2} B_N(|x|, t),
\]

\[
|\partial_x^\alpha S(x, t)| \leq C(1 + t)^{-(2n + |\alpha|)/2} B_n(|x|, t),
\]

then

\[
|\partial_x^\alpha \int_0^t (H(t - \tau) \ast S(\tau))d\tau| \leq C(1 + t)^{-(n + |\alpha|)/2} B_2^N(|x|, t).
\]

Next we estimate \( G_{i,j}(x, t), (i = 1, 2; j = 1, 2, 3) \) which are the inverse Fourier transform corresponding to \( \hat{G}_{i,j}(\xi, t) \). First of all, for \( G_{i,j}, (i = 1, 2; j = 1, 2) \), we can use the following results from [11].

**Proposition 2.4.** For any positive number \( N \), we have

\[
|\partial_x^\alpha G_{i,1}| \leq C(1 + t)^{-n + |\alpha|/2} B_N(|x|, t), \quad i = 1, 2.
\]

**Proposition 2.5.** There exists a positive number \( c_0 \), such that

\[
|\partial_x^\alpha G_{i,2}| \leq Cc_0 t B_N(|x|, t), \quad i = 1, 2.
\]

For \( G_{i,3}, (i = 1, 2) \), we show a subtle analysis as follows. When \( |\xi| \) is large enough, using the Taylor expansion, we have

\[
\sqrt{1 - 4|\xi|^2} = |\xi|\sqrt{|\xi|^{-2} - 4} = 2\sqrt{-1}|\xi| - \sqrt{-1} \frac{1}{4}|\xi|^{-1} + O(|\xi|^{-3}), \quad (2.14)
\]

\[
\frac{1}{\sqrt{1 - 4|\xi|^2}} = |\xi|^{-1} \frac{1}{\sqrt{|\xi|^{-2} - 4}} = -\frac{\sqrt{-1}}{2}|\xi|^{-1} + \frac{\sqrt{-1}}{16} |\xi|^{-3} + O(|\xi|^{-5}), \quad (2.15)
\]

By using the Taylor expansion, we have

\[
e^{\tau \pm t} = e^{-\frac{1}{2} \pm \sqrt{-1}|\xi|0} \left( 1 + \sum_{j=1}^{k-1} (\pm a_j) |\xi|^{1-2j} t + \ldots + \frac{1}{k!} \sum_{j=1}^{k-1} (\pm a_j) |\xi|^{1-2j} t^k + R^\pm(\xi, t) \right), \quad (2.16)
\]

where \( R^\pm(\xi, t) \leq (1 + t)^{k+1}(1 + |\xi|)^{1-2k} \).

Then, by using (2.15) and (2.16), we have

\[
\frac{1}{\tau_+ - \tau_-} e^{\tau_+ t} = e^{-\frac{1}{2} + \sqrt{-1}|\xi|0} \left( \sum_{j=1}^{2k-2} p_{j}^+ (t)|\xi|^{-j} + p_{2k-1}^+ (t) O(|\xi|^{1-2k}) \right), \quad (2.17)
\]

\[
\frac{1}{\tau_+ - \tau_-} e^{\tau_- t} = e^{-\frac{1}{2} - \sqrt{-1}|\xi|0} \left( \sum_{j=1}^{2k-2} p_{j}^- (t)|\xi|^{-j} + p_{2k-1}^- (t) O(|\xi|^{1-2k}) \right), \quad (2.18)
\]

\[
\frac{\tau_+}{\tau_+ - \tau_-} e^{\tau_+ t} = e^{-\frac{1}{2} - \sqrt{-1}|\xi|0} \left( \sum_{j=1}^{2k-2} q_{j}^- (t)|\xi|^{-j} + q_{2k-1}^- (t) O(|\xi|^{1-2k}) \right), \quad (2.19)
\]
\[ \frac{\tau_-}{\tau_+ - \tau_-} e^{\tau_- t} = e^{(-\frac{1}{2} + \sqrt{\tau} \xi) t} \left( \sum_{j=0}^{2k-2} q_j^+ (t) |\xi|^{-j} + q_{2k-1}^+ (t) O(|\xi|^{1-2k}) \right). \]  

(2.20)

Here, \( p_j^\pm, q_j^\pm \) are polynomials in \( t \) with degree no more than \( j \).

Since \( |\xi| > R \), it is observed that

\[ |\hat{G}_{1,3}(\xi, t)| + |\hat{G}_{2,3}(\xi, t)| \leq Ce^{-t/4}. \]  

(2.21)

Now, we take

\[ \hat{F}_{1\alpha} = -\chi_3(\xi)e^{(-\frac{1}{2} + \sqrt{\tau} \xi) t} \sum_{j=0}^{|\alpha|+n+1} q_j^+(t)|\xi|^{-j} \]  

(2.22)

and

\[ \hat{F}_{2\alpha} = \chi_3(\xi)e^{(-\frac{1}{2} + \sqrt{\tau} \xi) t} \sum_{j=1}^{|\alpha|+n+1} p_j^+(t)|\xi|^{-j} \]  

(2.23)

Then, for the high frequency part, we have the following result.

**Proposition 2.6.** There exists a positive number \( c_1 \), such that

\[ |\partial_x^\alpha (G_{i,3} - F_{i\alpha})| \leq Ce^{-c_1 t} B_N(|x|, t), \quad i = 1, 2. \]  

(2.24)

**Proof.** It is obvious that

\[ |x^\beta (\partial_x^\alpha (G_{i,3} - F_{i\alpha}))| \leq \int |\partial_x^\beta (\xi^\alpha (\hat{G}_{i,3} - \hat{F}_{i\alpha})))|d\xi \]  

(2.25)

\[ \leq Ce^{-c_1 t} \int |\xi|^{a-|\beta|-(a+n+1)-1}d\xi \]  

\[ \leq Ce^{-c_1 t}. \]

Take \( |\beta| = 0 \) or \( |\beta| = 2N \), we obtain the the statement of Proposition 2.6. \( \square \)

3. Global classical solutions

The solution can be constructed in the complete metric space

\[ X = \{ u(t) = (u_1(t), u_2(t), \ldots, u_m(t)) \| u \|_X \leq E \}, \]  

(3.1)

where \( E \) is a positive constant and

\[ \| u \|_X = \sup_{t \geq 0} \sum_{j=1}^m (1 + t)^{\frac{2}{q}} \| u_j(\cdot, t) \|_{W^{1-\alpha, \infty}(\mathbb{R}^n)} + \sup_{t \geq 0} \sum_{j=1}^m (1 + t)^{\frac{2}{q}} \| u_j(\cdot, t) \|_{L^1(\mathbb{R}^n)}. \]

Then \( (X, \| \cdot \|_X) \) is a Banach space. Let

\[ T[u](t) := (T_1[u](t), T_2[u](t), \ldots, T_m[u](t)), \]  

(3.2)

where

\[ T_j[u](t) = G_1 * a_j + G_2 * b_j + \int_0^t G_2(t-s) * F_j(u(s))(x)ds, \quad (1 \leq j \leq m). \]  

(3.3)
In the following lemma, we show that $T$ is a map from $X$ to itself.

**Lemma 3.1.** If $E$ and $N_0$ are sufficiently small with $N_0 \ll E$, then $T$ is a map from $X$ to $X$.

**Proof.** Firstly, we note that

$\|T_j[u](t)\|_{t-n-1, \infty} \leq \|(G_1 - F_{1\alpha})(t) * a_j\|_{t-n-1, \infty} + \|(G_2 - F_{2\alpha})(t) * b_j\|_{t-n-1, \infty} + \|F_{1\alpha}(t) * a_j\|_{t-n-1, \infty} + \|F_{2\alpha}(t) * b_j\|_{t-n-1, \infty} + \int_0^t \|(G_2 - F_{2\alpha})(t - \tau) * F_j(\tau)\|_{t-n-1, \infty} d\tau$ 

$\quad + \int_0^t \|F_{2\alpha}(t - \tau) * F_j(\tau)\|_{t-n-1, \infty} d\tau$

$:= \sum_{i=1}^6 I_i.$

For $I_1$, it follows from the Young inequality and Propositions 2.4, 2.0 that

$I_1 \leq \|(G_1 - F_{1\alpha})(t)\|_{L^\infty} |a_j|_{t-n-1, 1} \leq C(1 + t)^{-n/2} \|a_j\|_{t-n-1, 1}.$  \hspace{1cm} (3.4)

Similarly to the estimates of $I_1$, we obtain

$I_2 \leq C(1 + t)^{-n/2} \|b_j\|_{t-n-1, 1}.$  \hspace{1cm} (3.5)

By noticing $|\alpha| \leq l - n - 1$ and the definition of $F_{1\alpha}$, for some positive number $c_2$, we have

$|\partial_x^\alpha F_{1\alpha} * a_j| \leq \int |\xi^\alpha \hat{F}_{1\alpha} \hat{a}_j| d\xi$

$\leq C \|a_j\|_{L^1} e^{-c_2 t} \int |\chi(\xi)| |\xi|^{|\alpha|} |\xi|^{-l} d\xi$

$\leq C \|a_j\|_{L^1} e^{-c_2 t}.$  \hspace{1cm} (3.6)

Similarly, we have

$|\partial_x^\alpha F_{2\alpha} * b_j| \leq C \|b_j\|_{t-1, 1} e^{-c_2 t}.$  \hspace{1cm} (3.7)

Then, we obtain

$I_3 \leq C(1 + t)^{-n/2} \|a_j\|_{t, 1}$ \hspace{0.5cm} and \hspace{0.5cm} $I_4 \leq C(1 + t)^{-n/2} \|b_j\|_{t-1, 1}.$  \hspace{1cm} (3.8)

To estimate $I_5$ and $I_6$, we give the estimates of $F_j(\tau)(t)$ as follows:

$|\partial_x^\alpha F_j(\tau)(t)| \leq |\partial_x^\alpha F_j(\tau)(t)| \sum_{i=1}^m |\partial_x^\alpha u_i|

+ |\partial_x^\alpha F_j(\tau)(t)| \sum_{1 \leq k_1, k_2 \leq m, \alpha_1 + \alpha_2 = \alpha} |\partial_x^{\alpha_1} u_{k_1} \partial_x^{\alpha_2} u_{k_2}| + \ldots

+ |\partial_x^\alpha F_j(\tau)(t)| \sum_{1 \leq k_1, \ldots, k_m \leq m, \alpha_1 + \ldots + \alpha_m = \alpha} |\partial_x^{\alpha_1} u_{k_1} \ldots \partial_x^{\alpha_m} u_{k_m}|,$

where $0 \leq \alpha_k \leq \alpha, (k = 1, \ldots, m)$.

Then, by using 1.6, 1.7, 3.9, the Hölder inequality ($\|fg\|_{L^1} \leq \|f\|_{L^2} \|g\|_{L^2}$) and the assumption $\|u\|_{X} \leq E$, we have

$\|F_j(\tau)(t)\|_{t-n-1, 1} \leq C(1 + t)^{-\frac{n}{2}} \gamma_j \hat{E} \hat{\gamma}_j,$  \hspace{1cm} (3.9)
Thus, the combination of (3.4)-(3.12) gives
\[ \|F_j(u)(t)\| \leq C(1 + t)^{-\frac{2}{3}(\bar{\rho}_j^{1/2}) - \frac{4}{3} E^{\bar{\rho}_j}}. \] (3.11)

Using the Young inequality, (3.10) and Proposition 2.4 and noticing \( \bar{\rho}_j > 1 + \frac{2}{n} \), for \( I_5 \), we have
\[
I_5 \leq \int_0^t \|(G_1 - F_{1,\alpha})(t - \tau)\| \|F_j(u)(\tau)\|_{l-n-1,1} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-n/2} \|F_j(u)(\tau)\|_{l-n-1,1} d\tau \\
\leq CE^{\bar{\rho}_j} \int_0^t (1 + t - \tau)^{-n/2} (1 + \tau)^{-\frac{2}{3}(\bar{\rho}_j^{1/2})} d\tau \\
\leq C(1 + t)^{-n/2} E^{\bar{\rho}_j}.
\]

For \( I_6 \), it follows from Lemma 2.2 (3.11) and the Sobolev inequality that
\[
I_6 \leq C \int_0^t e^{-(t-\tau)/4} \|(f_1 + f_2) \ast F_j(u)(\tau)\|_{l-n-1,1} d\tau \\
\leq C \int_0^t e^{-(t-\tau)/4} (\|f_1\|_{L^1} + \|f_2\|_{L^1}) \|F_j(u)(\tau)\|_{l-n-1,\infty} d\tau \\
\leq C \int_0^t e^{-(t-\tau)/4} \|F_j(u)(\tau)\|_{l-n-1,\infty} d\tau \\
\leq C \int_0^t e^{-(t-\tau)/4} \|F_j(u)(\tau)\|_{l-n(\frac{1}{2})} d\tau \\
\leq CE^{\bar{\rho}_j} \int_0^t e^{-(t-\tau)/4} (1 + \tau)^{-\frac{2}{3}(\bar{\rho}_j^{1/2})} d\tau \\
\leq C(1 + t)^{-n/2} E^{\bar{\rho}_j}.
\]

Thus, the combination of (3.4)-(3.12) gives
\[
\|T_j[u](t)\|_{l-n-1,\infty} \\
\leq C(1 + t)^{-n/2} (\|a_j\|_{l,1} + \|b_j\|_{l-1,1} + \|a_j\|_{l-n-1,1} + \|b_j\|_{l-n-1,1} + E^{\bar{\rho}_j}). \] (3.13)

To estimate \( H^l \) norm of \( T_j[u](t) \), we consider
\[
\|T_j[u](t)\|_l \leq \|(G_1 - G_{1,3})(t) \ast a_j\|_l + \|\partial_t (G_2 - G_{2,3})(t) \ast b_j\|_l \\
+ \|G_{1,3}(t) \ast a_j\|_l + \|G_{2,3}(t) \ast b_j\|_l \\
+ \int_0^t \|(G_2 - G_{2,3})(t - \tau) \ast F_j(u)(\tau)\|_l d\tau \\
+ \int_0^t \|G_{2,3}(t - \tau) \ast F_j(u)(\tau)\|_l d\tau \\
:= \sum_{i=1}^6 J_i.
\]

By using Propositions 2.4, 2.5 and the Young inequality, for \( J_1 \), we obtain
\[
J_1 \leq \|(G_1 - G_{1,3})(t)\|_{l,1} \|a_j\|_{l,1} \leq C(1 + t)^{-n/4} \|a_j\|_{l,1}.
\] (3.14)
For $J_3$, it follows from the Plancherel theorem and (2.21) that

$$J_3 \leq \sum_{|\alpha|=0}^{l} \|G_{1,3}(t) \ast \partial_x^\alpha a_j\| = \sum_{|\alpha|=0}^{l} \|\hat{G}_{1,3}(t)\hat{\partial_x^\alpha a_j}\| \leq C e^{-t/4} \|a_j\|_t. \quad (3.15)$$

Similar to the estimates of $J_1$ and $J_3$, we obtain

$$J_2 \leq C(1 + t)^{-n/4} \|b_j\|_{l,1}, \quad \text{and} \quad J_4 \leq C e^{-t/4} \|b_j\|_t. \quad (3.16)$$

Using Propositions 2.4, 2.5, the Young inequality and noticing $\tilde{p}_j > 1 + \frac{2}{n}$, we have

$$J_5 \leq C \int_0^t \|(G_2 - G_{2,3})(t - \tau)\| F_j(\tau) \|d\tau
\leq C E^{\tilde{p}_j} \int_0^t (1 + \tau - \tau)^{-n/4} (1 + t)^{-\frac{n}{2}(\tilde{p}_j - 1) - \frac{n}{2}} d\tau
\leq C (1 + t)^{-n/4} E^{\tilde{p}_j}. \quad (3.17)$$

For $J_6$, by using the Plancherel theorem and (2.21), we get

$$J_6 \leq C \int_0^t e^{-(t-\tau)/4} \|F_j(\tau)\|_l d\tau \leq C (1 + t)^{-n/4} E^{\tilde{p}_j}. \quad (3.18)$$

Thus, we obtain

$$\|T_j[u](t)\|_l \leq C(1 + t)^{-n/4} (\|a_j\|_l + \|b_j\|_l + \|a_j\|_{l,1} + \|b_j\|_{l,1} + E^{\tilde{p}_j}). \quad (3.19)$$

By using (3.13), (3.19), $\tilde{p}_j > 1 + \frac{2}{n}$ and the small assumptions of $E$ and $N_0$ with $N_0 \ll E$, we get $\|T[u](t)\|_X \leq E$. Thus, the proof of Lemma 3.1 is complete. \(\square\)

Next, we prove that this map $T$ is a contraction mapping.

**Lemma 3.2.** Assume $u, v \in X$ and $E > 0$ is sufficiently small, then there exists a constant $\gamma$ with $0 < \gamma < 1$, such that

$$\|T[u] - T[v]\|_X \leq \gamma \|u - v\|_X.$$

**Proof.** By the Duhamel principle and the triangle inequality, we have

$$\|T[u] - T[v]\|_{l^{-n-1,\infty}}$$

$$\leq \int_0^t \|(G_2 - F_{2\alpha})(t - \tau) * (F_j(u) - F_j(v))(\tau)\|_{l^{-n-1,\infty}} d\tau$$

$$+ \int_0^t \|F_{2\alpha}(t - \tau) * (F_j(u) - F_j(v))(\tau)\|_{l^{-n-1,\infty}} d\tau$$

$$:= H_1 + H_2. \quad (3.20)$$
By a directly calculation, we have

\[ |\partial_x^a F(u) - \partial_x^a F(v)| \]

\[ \leq \sum_{k=1}^{m} \partial_{u_k}^1 F(u) \partial_x^a u_k - \sum_{s=1}^{m} \partial_{v_s}^1 F(v) \partial_x^a v_s | + \]

\[ \sum_{1 \leq k_1, k_2 \leq m; \alpha_k + \alpha = \alpha} \partial_{u_{k_1} u_{k_2}}^2 F(u) \partial_x^{\alpha_{k_1}} u_{k_1} \partial_x^{\alpha_{k_2}} u_{k_2} - \]

\[ \sum_{1 \leq s_1, s_2 \leq m; \alpha_{s_1} + \alpha_{s_2} = \alpha} \partial_{v_{s_1} v_{s_2}}^2 F(v) \partial_x^{\alpha_{s_1}} v_{s_1} \partial_x^{\alpha_{s_2}} v_{s_2} | + \ldots \]

\[ + \sum_{1 \leq k_1, m, i = 1, \ldots, \alpha; \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{m} = \alpha} \partial_{u_{k_1} u_{k_2} \ldots u_{k_m}}^a F(u) \partial_x^{\alpha_{k_1}} u_{k_1} \ldots \partial_x^{\alpha_{k_m}} u_{k_m} - \]

\[ \partial_x^a F(v) \partial_x^{\alpha_{k_1}} v_{s_1} \ldots \partial_x^{\alpha_{k_m}} v_{s_m} | \quad (3.21) \]

Using (3.21) and the assumption (1.8), we have

\[ \|F_j(u) - F_j(v)\|_{L_1} \]

\[ \leq C \sum_{s=1}^{m} \sum_{\alpha_j = 1}^{l} \left( \sum_{\alpha_1 + \ldots + \alpha_j = \alpha} \prod_{k=1}^{s-1} \|u_k\|_{L_\infty}^{p_{j,k} - \alpha_j,k} \right) \]

\[ \times \prod_{k=s+1}^{m} \|v_k\|_{L_\infty}^{p_{j,k} - \alpha_j,k} \left( \|u_s\|_{L_\infty}^{p_{j,s} - \alpha_j,s - 1} + \|v_s\|_{L_\infty}^{p_{j,s} - \alpha_j,s - 1} \right) \]

\[ \times \left( \|u_s - v_s\|_{L_1} \sum_{i=1}^{m} (\|u_i\|_{L_1} + \|v_i\|_{L_1}) \right) \]

\[ + C \sum_{s=1}^{m} \sum_{\alpha_j = 1}^{l} \left( \sum_{\alpha_1 + \ldots + \alpha_j = \alpha} \prod_{k=1}^{s-1} \|v_k\|_{L_\infty}^{p_{j,k} - \alpha_j,k} \right) \]

\[ \times \prod_{k=s+1}^{m} \|v_k\|_{L_\infty}^{p_{j,k} - \alpha_j,k} \left( \|u_s\|_{L_\infty}^{p_{j,s} - \alpha_j,s - 1} + \|v_s\|_{L_\infty}^{p_{j,s} - \alpha_j,s - 1} \right) \]

\[ \times \left( \sum_{s=1}^{m} \sum_{\alpha_j = 1}^{l} \left( \sum_{\alpha_1 + \ldots + \alpha_j = \alpha} \prod_{k=1}^{s-1} \|\partial_x^{\alpha_k} u_k\|_{L_\infty} \prod_{k=s+1}^{m} \|\partial_x^{\alpha_k} v_k\|_{L_\infty} \right) \|u_s - v_s\|_{L_1} \right), \]

(3.22)
For $H_1$, by using the Young inequality and Propositions 2.4–2.6 and noticing the definition of $\| \cdot \|_X$ and $\tilde{p}_j > 1 + \frac{2}{n}$, we have

$$
H_1 \leq \int_0^t \|(G_2 - F_{2\alpha})(t - \tau)\|_{L^\infty} \|(F_j(u) - F_j(v))(\tau)\|_{l^{-n-1,1}} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-n/2} \|(F_j(u) - F_j(v))(\tau)\|_{l^{-n-1,1}} d\tau \\
\leq CE^{\tilde{p}_j-1} \int_0^t (1 + t - \tau)^{-n/2} (1 + t)^{-n/2(\tilde{p}_j-2)-n/4} \|u - v\|_X d\tau \\
\leq CE^{\tilde{p}_j-1}(1 + t)^{-n/2} \|u - v\|_X.
$$

(3.23)

Similarly, by using (3.21), we have

$$
\|F_j(u) - F_j(v)\|_l \\
\leq C \sum_{s=1}^m \sum_{l} \left( \sum_{\alpha_j = 1}^{s-1} \prod_{k=1}^{j=m} \|u_k\|_{L^\infty}^{P_{j,k} - \alpha_j} \right) \\
\|v_k\|_{L^\infty}^{(p_{j,k} - \alpha_j,k)} \left( \sum_{s=1}^m \sum_{l} \prod_{k=1}^{j=m} \|v_k\|_{L^\infty}^{(p_{j,k} - \alpha_j,k)} + \sum_{s=1}^m \sum_{l} \|\partial_x^m u_k\|_{L^\infty} \prod_{k=s+1}^{m} \|\partial_x^m v_k\|_{L^\infty} \|u_s - v_s\|_l \right). \\
$$

(3.24)

For $H_2$, similar to the estimate of $I_6$, it follows from Lemma 2.2 and the Sobolev inequality that

$$
H_2 \leq C \int_0^t e^{-(t-\tau)/4} \|(F_j(u) - F_j(v))(\tau)\|_{l^{-n-1,\infty}} d\tau \\
\leq C \int_0^t e^{-(t-\tau)/4} \|(F_j(u) - F_j(v))(\tau)\|_{l^{-n/2|2\tau|}} d\tau \\
\leq CE^{\tilde{p}_j} \int_0^t e^{-(t-\tau)/4} (1 + \tau)^{-n/2(\tilde{p}_j-2)-n/4} \|u - v\|_X d\tau \\
\leq CE^{\tilde{p}_j}(1 + t)^{-n/2} \|u - v\|_X,
$$

where we used $\tilde{p}_j > 1 + \frac{2}{n}$. Then, we obtain

$$
\|T[u] - T[v]\|_{l^{-n-1,\infty}} \leq CE^{\tilde{p}_j}(1 + t)^{-n/2} \|u - v\|_X.
$$

(3.25)

On the other hand, by using the Young inequality, the Plancherel theorem, (2.21) and Propositions 2.4–2.5, we have

$$
\|T[u] - T[v]\|_l \leq \int_0^t \|(G_2 - G_{2,3})(t - \tau) * (F_j(u) - F_j(v))(\tau)\|_l d\tau
$$

\text{(3.24)}
Take equation (1.11) with 

For some positive number \( b \) \( \) data

Proof of Theorem 1.1.

3.2. \( \square \)

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Similarly, we have

point theorem, there exists a fixed point \( u \)

\[ \{ u_j \} \]

Then

\[ \| T[u] - T[v] \|_X \leq C E \bar{p}_j (1 + t)^{-n/4} \| u - v \|_X. \]

Combining (3.25) with (3.26), we obtain

\[ \| T^j \|_X \leq C E \bar{p}_j \| u - v \|_X. \]

Since the smallness assumption of \( E \) and \( \bar{p}_j > 1 + \frac{2}{n} \), we complete the proof of Lemma 3.2. \( \square \)

Proof of Theorem 1.1. Lemmas 3.1 and 3.2 show that for sufficiently small initial data

\[ (a_j, b_j) \in (W^{l,1}(\mathbb{R}^n) \cap H^{l+1}(\mathbb{R}^n)) \times (W^{l,1}(\mathbb{R}^n) \cap H^l(\mathbb{R}^n)) \]

for \( j = 1, 2, \ldots, m \), \( T : X \to X \) is a contraction mapping. By the Banach fixed point theorem, there exists a fixed point \( u \in X \). Here, we obtain the solution \( \{ u_j(t) \}_{j=1}^m \) to system (1.1) satisfies \( \| u \|_X \leq E \). Then, the proof is complete. \( \square \)

4. Pointwise estimates

In this section, we show the pointwise estimates of the solutions to system (1.1). First of all, we recall

\[ \partial^\alpha_x u_j(x, t) \]

\[ \partial^\alpha_x G_1 * a_j + \partial^\alpha_x G_2 * b_j + \partial^\alpha_x \int_0^t G_2(t - s) * F_j(u)(s)ds \]

\[ = (\partial^\alpha_x (G_1 - F_{1\alpha})(t)) * a_j + (\partial^\alpha_x (G_2 - F_{2\alpha})(t)) * b_j + \partial^\alpha_x F_{1\alpha}(t) * a_j + \partial^\alpha_x F_{2\alpha}(t) * b_j \]

\[ \quad + \int_0^t (\partial^\alpha_x (G_2 - F_{2\alpha})(t - \tau)) * F_j(u)(\tau)d\tau + \int_0^t \partial^\alpha_x F_{2\alpha}(t - \tau) * F_j(u)(\tau)d\tau. \]

(4.1)

From Propositions 2.4, 2.5 and the assumption (1.11), by using Lemma 2.3, we have

\[ |(\partial^\alpha_x (G_1 - F_{1\alpha})(t)) * a_j + (\partial^\alpha_x (G_2 - F_{2\alpha})(t)) * b_j| \]

\[ \leq C \varepsilon_0 (1 + t)^{-\frac{n+\alpha}{2}} B_r(|x|, t). \]

For some positive number \( b \), by noticing the definition of \( F_{1\alpha} \), \( |\alpha| < l \) and assumption (1.11) with \( r > n \), we have

\[ |x^\beta \partial^\alpha_x F_{1\alpha} * a_j| \leq \int |x^\beta \xi^\alpha \hat{F}_{1\alpha} \hat{a}_j|d\xi \leq C \varepsilon_0 e^{-bt}. \]

(4.3)

Take \( |\beta| = 0 \) or \( |\beta| = n \), we obtain

\[ |\partial^\alpha_x F_{1\alpha} * a_j| \leq C \varepsilon_0 e^{-\frac{2t}{n}} B_{\frac{2}{n}}(|x|, t). \]

(4.4)

Similarly, we have

\[ |\partial^\alpha_x F_{2\alpha} * b_j| \leq C \varepsilon_0 e^{-bt/2} B_r(|x|, t). \]

(4.5)

To estimate the other parts in (4.1), we set

\[ \varphi_\alpha(x, t) = (1 + t)\frac{n+|\alpha|}{2} (B_{\frac{2}{n}}(|x|, t))^{-1}, \]

(4.6)
and

\[ M(t) = \sup_{0 \leq s, \tau \leq t, |\alpha| \leq t} \sum_{j=1}^{m} |\partial_x^2 u_j(x, \tau)| \varphi_\alpha(x, s). \quad (4.7) \]

When \(|\alpha| \leq l - 1\), from the assumptions \(|1.6|, (1.7)\) and the definition of \( M \), we have

\[ |\partial_x^2 F_j(u)(x, t)| \leq M^2(t)(1 + t)^{-n - |\alpha|} B_n(|x|, t). \quad (4.8) \]

When \(|\alpha| = l\), from the definition of \( M \), by using Theorem \(|1.1|\) and Lemma \(|3.1|\) we have

\[ |\partial_x^2 F_j(u)(x, s)| \leq M^2(t)(1 + t)^{-n - |\alpha|} B_n(|x|, t) + EM(t)(1 + t)^{-n - |\alpha|} B_2^\alpha(|x|, t). \quad (4.9) \]

Set

\[ R^\alpha = \left| \int_0^t F_{2\alpha}(t - s) \star \partial_x^2 F_j(u)(s) ds \right|. \]

From Lemma \(|2.2|\) and \(|4.8|\) and \(|4.9|\), we obtain

\[ R^\alpha \leq \left| \int_0^t e^{-b(t-s)} f_1 + f_2 \star \partial_x^2 F_j(u)(s) ds \right| \]

\[ \leq \left| \int_0^t e^{-b(t-s)} f_1 \star \partial_x^2 F_j(u)(s) ds \right| + \left| \int_0^t e^{-b(t-s)} f_2 \star \partial_x^2 F_j(u)(s) ds \right| \quad (4.10) \]

\[ := R_1^\alpha + R_2^\alpha. \]

The right-hand side of the above inequality can be estimated as follows.

\[ R_2^\alpha \]

\[ \leq \left| \int_0^t e^{-b(t-s)} \int_{\mathbb{R}^n} f_2(x - y) \partial_x^2 F_j(u)(y, s) \chi_{y \in \mathbb{R}^n} dy ds \right| \]

\[ \leq \int_0^t e^{-b(t-s)} \int_{\mathbb{R}^n} |f_2(x - y)|(M^2(t) + EM(t))(1 + t)^{-n - |\alpha|} B_2^\alpha(|y|, t) \chi_{y \in \mathbb{R}^n} dy ds \]

\[ \leq \int_0^t e^{-b(t-s)} \|f_2\|_{L^1}(M^2(t) + EM(t))(1 + t)^{-n - |\alpha|} B_2^\alpha(|x|, t) ds \]

\[ \leq C(1 + t)^{-n - |\alpha|} \frac{M^2(t) + EM(t)}{B_2^\alpha(|x|, t)}. \]

Since

\[ |\partial_x^2 (e^{-bt} f_1(x))| \leq C(1 + t)^{-n - \frac{|\alpha| + 1}{2}} B_N(|x|, t), \]

we have

\[ R_1^\alpha = \left| \int_0^t e^{-b(t-s)} f_1 \star F_j(u)(s) ds \right| \leq C(1 + t)^{-n - \frac{|\alpha|}{2}} (M^2(t) + M(t)E) B_2^\alpha(|x|, t). \]

From the definition of \( M \), we have

\[ |\partial_x^2 F_j(u)(x, s)| \leq CM^2(t)(1 + s)^{-n - |\alpha|} B_n(|x|, s). \quad (4.11) \]

From Proposition \(|2.6|\) and Lemma \(|2.3|\) we obtain

\[ \left| \int_0^t \partial_x^2 (G_2(t - s) - F_{2\alpha}) \star F_j(u)(s) ds \right| \]

\[ \leq C(M^2(t) + M(t)E)(1 + t)^{-n - |\alpha|} B_2^\alpha(|x|, t). \quad (4.12) \]
From the above inequalities and the definition of $M$, we obtain

$$
M(t) \leq C(M^2(t) + EM(t) + \varepsilon_0).
$$

(4.13)

Since $E$ and $\varepsilon_0$ are small enough, we have $M(t) \leq C$. It yields that

$$
|\partial_x^\alpha u_j(t)| \leq C(1 + t)^{-\frac{n+|\alpha|}{2}} B_2^\alpha(|x|, t).
$$

(4.14)

Thus, we can easily obtain the optimal $L^p$, $1 \leq p \leq \infty$, convergence rate as follows.

**Corollary 4.1.** Under the assumptions of Theorem 1.1, for $p \in [1, \infty]$, $|\alpha| \leq l$, we have

$$
\|\partial_x^\alpha u_j(\cdot, t)\|_{L^p} \leq C(1 + t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \quad j = 1, \ldots, m.
$$

(4.15)

Thus, we have complete the proof of Theorem 1.2.

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**References**


Wenjun Wang
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: wwj001373@hotmail.com