EXISTENCE AND GLOBAL ASYMPTOTIC STABILITY OF
POSITIVE PERIODIC SOLUTIONS OF A LOTKA-VOLterra
TYPE COMPETITION SYSTEMS WITH DELAYS AND
FEEDBACK CONTROLS

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Abstract. The existence of positive periodic solutions of a periodic Lotka-Volterra type competition system with delays and feedback controls is studied by applying the continuation theorem of coincidence degree theory. By contracting a suitable Liapunov functional, a set of sufficient conditions for the global asymptotic stability of the positive periodic solution of the system is given. A counterexample is given to show that the result on the existence of positive periodic solution in [3] is incorrect.

1. INTRODUCTION

In this paper, we consider the following non-autonomous Lotka-Volterra n-species competition system with delays and feedback controls

\[
\begin{align*}
\dot{x}_i(t) &= g_i(x_i(t)) \left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=k=1}^{n} b_{ijk}(t)x_j(t - \tau_{ijk}(t))
- \sum_{j=1}^{n} \sum_{k=1}^{m} \int_{-\infty}^{t} c_{ijk}(t,s)x_j(s)ds - d_i(t)u_i(t)
- \sum_{k=1}^{m} e_{ik}(t)u_i(t - \sigma_{ik}(t)) - \int_{-\infty}^{t} f_i(t,s)u_i(s)ds\right],
\end{align*}
\]

(1.1)

\[
\dot{u}_i(t) = -\alpha_i(t)u_i(t) + \beta_i(t)x_i(t) + \sum_{k=1}^{m} p_{ik}(t)x_i(t - \gamma_{ik}(t))
+ \int_{-\infty}^{t} \psi_i(t,s)x_i(s)ds,
\]

where \(i \in \{1, 2, \ldots, n\}\), \(u_i\) denote indirect feedback control variables. For system (1.1), we introduce the following hypotheses

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(H1) $r_i, \alpha_i \in C(\mathbb{R}, \mathbb{R})$, $a_{ij}, b_{ijk}, d_i, e_{ij}, p_{ik}, \beta_i \in C(\mathbb{R}, \mathbb{R}^+)$. are $\omega$-periodic ($\omega$ is a fixed positive number) with $\int_0^{\omega} r_i(t)dt > 0$, $\int_0^{\omega} \alpha_i(t)dt > 0$, $i, j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$.

(H2) $c_{ijk}, f_i, v_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ are $\omega$-periodic functions; i.e.,

$$c_{ijk}(t + \omega, s + \omega) = c_{ijk}(t, s), f_i(t + \omega, s + \omega) = f_i(t, s), v_i(t + \omega, s + \omega) = v_i(t, s)$$

and $\int_{-\infty}^{t} c_{ijk}(t, s)ds$, $\int_{-\infty}^{t} f_i(t, s)ds$, $\int_{-\infty}^{t} v_i(t, s)ds$ are continuous with respect to $t$. Moreover

$$\int_{-t}^{t} \int_{0}^{0} c_{ijk}(s + t, s)ds dt < +\infty, \quad \int_{-t}^{t} \int_{0}^{0} f_i(s + t, s)ds dt < +\infty,$$

$$\int_{-t}^{t} \int_{0}^{0} v_i(s + t, s)ds dt < +\infty, \quad i, j = 1, 2, \ldots, n; k = 1, 2, \ldots, m.$$

(H3) $g_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ is strictly increasing, $g_i(0) = 0$ and $\lim_{v \to -0^+} \frac{g_i(v)}{v}$ is a positive constant. Moreover, there are positive constants $l$ and $L$ such that $l \leq \frac{g_i(v)}{v} \leq L$ for all $v > 0$, $i = 1, 2, \ldots, n$.

(H4) $\tau_{ijk}, \sigma_{ik}, \gamma_{ik} \in C(\mathbb{R}, \mathbb{R}^+)$ are $\omega$-periodic for $i, j, k = 1, 2, \ldots, n, k = 1, 2, \ldots, m$.

(H5) $\tau_{ijk}, \sigma_{ik}, \gamma_{ik} \in C^1(\mathbb{R}, \mathbb{R}^+)$ and $\dot{\tau}_{ijk}(t) < 1$, $\dot{\sigma}_{ik}(t) < 1$, $\dot{\gamma}_{ik}(t) < 1$ for all $t \in \mathbb{R}, i, j = 1, 2, \ldots, n, k = 1, 2, \ldots, m$.

We consider (1.1) with the initial conditions

$$x_i(s) = \phi_i(s), s \in (-\infty, 0], \quad \phi_i \in C((-\infty, 0], \mathbb{R}^+), \phi_i(0) > 0$$

$$u_i(s) = \psi_i(s), s \in (-\infty, 0], \quad \psi_i \in C((-\infty, 0], \mathbb{R}^+), \quad \psi_i(0) > 0,$$

for $i = 1, 2, \ldots, n$. Throughout this paper, we use the following symbols: for an $\omega$-periodic function $f \in C(\mathbb{R}, \mathbb{R})$, we define

$$\bar{f} = \frac{1}{\omega} \int_{0}^{\omega} f(t)dt, \quad c_{ijk}(t) = \int_{-\infty}^{t} c_{ijk}(t, s)ds, \quad f_i(t) = \int_{-\infty}^{t} f_i(t, s)ds,$$

$$v_i(s) = \int_{-\infty}^{t} v_i(t, s)ds, \quad G_i(t, s) = \frac{\exp\{\int_{0}^{s} \alpha_i(v)dv\}}{\exp\{\int_{0}^{t} \alpha_i(v)dv\} - 1}, s \geq t,$$

$$P_i(t) = d_i(t) \int_{t}^{t+\omega} G_i(t, s) \left[\beta_i(s) + \sum_{k=1}^{m} p_{ik}(s) + v_i^*(s)\right]ds,$$

$$Q_i(t) = \sum_{j=1}^{m} c_{ij}(t) \int_{t}^{t+\omega} G_i(t, s) \left[\beta_i(s) + \sum_{k=1}^{m} p_{ik}(s) + v_i^*(s)\right]ds,$$

$$R_i(t) = \int_{-\infty}^{t} f_i(t, s) \int_{s}^{s+\omega} G_i(s, \tau) \left[\beta_i(\tau) + \sum_{k=1}^{m} p_{ik}(\tau) + v_i^*(\tau)\right]d\tau ds.$$

Fan et al. [1] studied system (1.1) in the case $k = 1$, $q_i(v_i) = v_i$, $f_i(t, s) = 0$ for $i = 1, 2, \ldots, n$ and obtained several results for the existence and global asymptotically stability of positive periodic solutions of the system. Recently, Yan and Liu [4] considered system (1.1) in the case $c_{ik}(t) = p_{ik}(t) = 0$ and $f_i(t, s) =$...
\[v_i(t, s) = 0 \text{ for } k = 2, 3, \ldots, m, i = 1, 2, \ldots, n,\]

\[\dot{x}_i(t) = g_i(x_i(t)) \left[ r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} \sum_{k=1}^{m} b_{ijk}(t)x_j(t) - \tau_{ij}(t)\right] - \sum_{j=1}^{m} \sum_{k=1}^{m} \int_{-\infty}^{t'} c_{ijk}(t, s)x_j(s)ds - d_i(t)u_i(t) - e_i(t)u_i(t) - \sigma_i(t))\]

\[\dot{u}_i(t) = -\alpha_i(t)u_i(t) + \beta_i(t)x_i(t) + p_i(t)x_i(t) - \gamma_i(t), \quad i = 1, 2, \ldots, n.\]

By employing fixed point index theory on cones, Yan and Liu [4] established the following result.

**Theorem 1.1** [4]. Assume that (H1)–(H4) hold. For system (1.4), to have at least one positive \(\omega\)-periodic solution, a necessary and sufficient condition is

\[\min_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \bar{a}_{ij} + \sum_{k=1}^{m} (\bar{b}_{ijk} + \bar{c}_{ijk}) + \bar{P}_i + \bar{Q}_i \right\} > 0 \quad (1.5)\]

and

\[\min_{1 \leq i \leq n} \{ \bar{\beta}_i + \bar{p}_{i1} \} > 0. \quad (1.6)\]

Unfortunately, the sufficient condition in the above theorem is incorrect, as shown by the following example.

**Example.** Consider the system

\[\dot{x}_1(t) = x_1(t) \left[ 1 - x_1(t) - 3x_2(t) - 2x_2(t) - \tau_1(t) - u_1(t) - u_1(t) - \sigma_1(t) \right],\]

\[\dot{x}_2(t) = x_1(t) \left[ 2 - 3x_1(t) - x_2(t) - 2x_2(t) - \tau_2(t) - u_2(t) - u_2(t) - \sigma_2(t) \right], \quad (1.7)\]

\[\dot{u}_1(t) = -u_1(t) + x_1(t) + x_1(t) - \gamma_1, \quad \dot{u}_2(t) = -u_2(t) + x_2(t) + x_2(t) - \gamma_2,\]

where \(\tau_i, \sigma_i, \gamma_i, i = 1, 2,\) are positive constants. It is easy to see that system (1.7) satisfies all hypotheses of Theorem 1.1 with \(\omega = 1\). On the other hand, if \((x_1^*(t), x_2^*(t), u_1^*(t), u_2^*(t))\) is a positive 1-periodic solution of system (1.7), then

\[\frac{d}{dt} \ln x_1^*(t) = \left[ 1 - x_1^*(t) - 3x_2^*(t) - 2x_2^*(t) - \tau_1(t) - u_1^*(t) - u_1^*(t) - \sigma_1(t) \right],\]

\[\frac{d}{dt} \ln x_2^*(t) = \left[ 2 - 3x_1^*(t) - x_2^*(t) - 2x_2^*(t) - \tau_2(t) - u_2^*(t) - u_2^*(t) - \sigma_2(t) \right], \quad (1.8)\]

\[\frac{d}{dt} u_1^*(t) = -u_1^*(t) + x_1^*(t) + x_1^*(t) - \gamma_1,\]

\[\frac{d}{dt} u_2^*(t) = -u_2^*(t) + x_2^*(t) + x_2^*(t) - \gamma_2,\]

Integrating (1.8) from 0 to 1 and simplifying, we obtain

\[\ddot{x}_1^* + 5\ddot{x}_2^* + 2\ddot{u}_1^* = 1, \quad 5\ddot{x}_1^* + \ddot{x}_2^* + 2\ddot{u}_2^* = 2, \quad \dddot{u}_1^* = 2\ddot{x}_1^*, \quad \dddot{u}_2^* = 2\ddot{x}_2^*. \quad (1.9)\]

This implies

\[5\ddot{x}_1^* + 5\ddot{x}_2^* = 1, \quad 5\ddot{x}_1^* + 5\ddot{x}_2^* = 2,\]

which is impossible. Thus, system (1.7) has no positive 1-periodic solution and then the sufficient condition in Theorem 1.1 (given in [4]) is incorrect.

In the proof of Theorem 1.1 (see [4]), authors considered a map \(\Phi : K \rightarrow K\), where

\[K = \{(x_1, \ldots, x_n) \in E : x_i(t) \geq \delta_i \|x_i\|_0, i = 1, 2, \ldots, n, \text{ } t \in [0, \omega]\}\]
(with $\delta_i = \exp\{-\bar{r}_i \omega_i\}$) is a cone of the Banach space $E = \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t) \text{ for all } t \in \mathbb{R}\}$ with the norm $\|x\|_0 = \sum_{i=1}^n \|x_i\|_0$ (where $\|x_i\|_0 = \max_{t \in [0, \omega]} |x_i(t)|$). By employing fixed point index theory on cone, it was proved in [4] that there exist positive constants $r$ and $R$ ($r < R$) such that $\Phi$ has at least one fixed point $x^*$ in $K_{r,R} = \{x \in K : r < \|x\|_0 \leq R\}$, and then it was concluded that $(x^*(t), u^*(t))$ is positive $\omega$-periodic solution of system (1.1). The mistake in the proof of Theorem 1.1 (see [4]) is that $x^* \in K_{r,R}$ does not imply that $x^*(t)$ is positive for $n \geq 2$.

Our purpose of this paper is by using the technique of coincidence degree theory developed by Gains and Mawhin in [2] to study the existence of positive periodic solutions of system (1.1). Moreover, by contracting a suitable Liapunov functional, we study the global asymptotic stability of the positive periodic solution of system (1.1). The remainder of this paper is organized as follows. Section 2 is preliminaries, in which we introduce the continuation theorem and some lemmas. Section 3 contains our main results on the existence and the global asymptotic stability of positive periodic solutions of system (1.1).

2. Preliminaries

Let $Y$ and $Z$ be two normed vector spaces, $L : \text{Dom} \ L \subset Y \to Z$ be a linear mapping, and $N : Y \to Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim} \text{ Im } L < +\infty$ and $\text{Im } L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P : Y \to Y$ and $Q : Z \to Z$ such that $\text{Im } P = \ker L$, $\text{Im } L = \ker Q = \text{Im}(I - Q)$, it follows that $L|_{\text{Dom} \ L \cap \ker P} : (I - P)X \to \text{Im } L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $Y$, the mapping $N$ will be called $L$-compact on $\Omega$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q) : \Omega \to Y$ is compact. Since $\text{Im } Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \text{Im } Q \to \ker L$.

Lemma 2.1 (Continuation theorem [2]). Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\bar{\Omega}$. Suppose that

(i) for each $\lambda \in (0, 1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \not\in \partial \Omega$;
(ii) $QN x \neq 0$ for each $x \in \partial \Omega \cap \ker L$ and $\deg(JQN, \Omega \cap \ker L, 0) \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution lying in $\text{Dom } L \cap \bar{\Omega}$.

Let us consider the system

$$
\dot{x}_i(t) = g_i(x_i(t)) - \sum_{j=1}^n a_{ij}(t)x_j(t) - \sum_{j=1}^n \sum_{k=1}^m b_{ijk}(t)x_j(t - \tau_{ijk}(t))
- \sum_{j=1}^n \sum_{k=1}^m \int_{-\infty}^t c_{ijk}(t,s)x_j(s)ds - d_i(t)(V_i x_i)(t)
- \sum_{k=1}^m e_{ik}(t)(V_i x_i)(t - \sigma_{ik}(t)) - \int_{-\infty}^t f_i(t,s)(V_i x_i)(s)ds,
$$

$$
\quad u_i(t) = (V_i x_i)(t), \quad i = 1, 2, \ldots, n,
$$

(2.1)
where
\[(V_ix_i)(t) = \int_t^{t+\omega} G_i(t, s) \left[ \beta_i(s)x_i(s) + \sum_{k=1}^{m} p_{ik}(s)x_i(s-\gamma_{ik}(s)) \right] ds + \int_{-\infty}^{s} v_i(s, \tau)x_i(\tau) d\tau, \quad i = 1, 2, \ldots, n. \]

(2.2)

Lemma 2.2. Assume \((H1)-(H4)\) hold. Then \((x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_n(t))\) is an \(\omega\)-periodic solution of system (1.1) if and only if it is an \(\omega\)-periodic solution of system (2.1).

Proof. \((\Rightarrow)\) Let \((x(t), u(t)) = (x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_n(t))\) be an \(\omega\)-periodic solution of system (1.1). It is easy to see from (1.1) that
\[u_i(t) = \exp\{-\int_t^{t+\omega} \alpha_i(\tau)d\tau\} \left[ u_i(t) + \int_t^{t+\omega} \exp\{\int_t^{\tau} \alpha_i(\tau)d\tau\} \left[ \beta_i(\tau)x_i(\tau) + \sum_{k=1}^{m} p_{ik}(\tau)x_i(\tau-\gamma_{ik}(\tau)) \right] dv \right] ds + \int_{-\infty}^{s} v_i(\tau, v)x_i(v) d\tau, \quad i = 1, 2, \ldots, n, \]
for \(\bar{t} \geq t, \ i = 1, 2, \ldots, n. \) Thus, since \(u_i(t) = u_i(t+\omega)\) for \(i = 1, 2, \ldots, n, \) it follows that
\[u_i(t) = u_i(t+\omega) = \exp\{-\int_t^{t+\omega} \alpha_i(\tau)d\tau\} \left[ u_i(t) + \int_t^{t+\omega} \exp\{\int_t^{\tau} \alpha_i(\tau)d\tau\} \left[ \beta_i(\tau)x_i(\tau) + \sum_{k=1}^{m} p_{ik}(\tau)x_i(\tau-\gamma_{ik}(\tau)) \right] dv \right] ds + \int_{-\infty}^{s} v_i(\tau, v)x_i(v) d\tau, \quad i = 1, 2, \ldots, n, \]
for \(t \in \mathbb{R}, \ i = 1, 2, \ldots, n. \) Hence,
\[u_i(t) = \int_t^{t+\omega} G_i(t, \tau) \left[ \beta_i(\tau)x_i(\tau) + \sum_{k=1}^{m} p_{ik}(\tau)x_i(\tau-\gamma_{ik}(\tau)) \right] ds + \int_{-\infty}^{s} v_i(\tau, v)x_i(v) d\tau = (V_ix_i)(t), \quad i = 1, 2, \ldots, n, \]
which implies that \((x(t), u(t))\) is an \(\omega\)-periodic solution of system (2.1).

\((\Leftarrow)\) Let \((x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_n(t))\) be an \(\omega\)-periodic solution of system (2.1). It is easy to see from (2.2) that \((u_1(t), \ldots, u_n(t))\) satisfies the system
\[\dot{u}_i(t) = -\alpha_i(t)u_i(t) + \beta_i(t)x_i(t) + \sum_{k=1}^{m} p_{ik}(t)x_i(t-\gamma_{ik}(t)) + \int_{-\infty}^{t} v_i(t, s)x_i(s) ds, \quad i = 1, 2, \ldots, n, \]
for \(i = 1, 2, \ldots, n. \) Thus, \((x(t), u(t))\) is an \(\omega\)-periodic solution of system (1.1). The proof is complete. \[\Box\]

Lemma 2.3. Suppose that \(\nu \in C^1(\mathbb{R}, \mathbb{R}_+ \) is \(\omega\)-periodic and \(\nu'(t) < 1\) for all \(t \in [0, \omega]. \) Then the function \(t-\nu(t)\) has a unique inverse \(\eta(t)\) satisfying \(\eta \in C(\mathbb{R}, \mathbb{R})\) with \(\eta(t+\omega) = \eta(t)\) for all \(t \in \mathbb{R}. \)

Lemma 2.4. Suppose that \(q_{ij} > 0, \delta_i > 0\) for \(i, j = 1, 2, \ldots, n. \) If
\[\delta_i - \sum_{j \neq i} q_{ij} \frac{\delta_j}{q_{jj}} > 0, \quad i = 1, 2, \ldots, n, \]
(2.3)
then the system of algebraic equations
\[ \delta_i - \sum_{j=1}^{n} q_{ij} x_j = 0, \quad i = 1, 2, \ldots, n \quad (2.4) \]
has a unique solution \( x^* = (x^*_1, \ldots, x^*_n) \in \mathbb{R}^n \). Moreover, \( x^*_i > 0 \) for \( i = 1, 2, \ldots, n \).

Proof. Let \( y_i = q_{ii} x_i \) for \( i = 1, 2, \ldots, n \), so that system \((2.4)\) becomes
\[ \delta_i - \sum_{j=1}^{n} q_{ij}^* y_j = 0, \quad i = 1, 2, \ldots, n, \quad (2.5) \]
where \( q_{ij}^* = q_{ij} / q_{jj} \), \( i, j = 1, 2, \ldots, n \). Clearly, \( q_{ii}^* = 1 \) for \( i = 1, 2, \ldots, n \). By \((2.3)\),
\[ \delta_i - \sum_{j \neq i} q_{ij}^* \delta_j > 0, \quad i = 1, 2, \ldots, n. \quad (2.6) \]
Let \( \epsilon \) be a positive number such that \( \epsilon < \min_{1 \leq i \leq n} [\delta_i - \sum_{j \neq i} q_{ij}^* \delta_j] \). Denote
\[ D = \{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n : \epsilon \leq y_i \leq \delta_i, i = 1, 2, \ldots, n \}, \]
\[ F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n, \]
and \( H = F + I \), where \( I \) is the identity operator on \( \mathbb{R}^n \). It is easy to see that \( \epsilon < H_i(y) < \delta_i \) for \( i = 1, 2, \ldots, n \), \( y \in D \); i.e., \( H(y) \) is in \( \text{int}(D) \)-the interior of \( D \) for all \( y \in D \). Thus, \( \text{deg}(H-I, \text{int} D, 0) = \text{deg}(F, \text{int} D, 0) = 1 \). This implies that the equation \( Fy = 0 \) has at least one solution \( y^* \) in \( \text{int} D \). Since \( Fy \neq 0 \) for all \( y \in \partial D \) and \( Fy = 0 \) is linear equation, it follows that \( y^* \) is the unique solution in \( \mathbb{R}^n \) of the equation \( Fy = 0 \). Thus, equation \((2.4)\) has a unique solution \( x^* = (x^*_1, \ldots, x^*_n) \in \mathbb{R}^n \). Moreover, \( x^*_i > 0 \) for \( i = 1, 2, \ldots, n \). The proof is complete. \( \square \)

Definition 2.5. Let \((\tilde{x}(t), \tilde{u}(t)) = (\tilde{x}_1(t), \ldots, \tilde{x}_n(t), \tilde{u}_1(t), \ldots, \tilde{u}_n(t))\) be a positive \( \omega \)-periodic solution of system \((1.1)\). It is said to be globally asymptotically stable if any positive solution \((x(t), u(t)) = (x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_n(t))\) of \((1.1)-(1.2)\) satisfies
\[ \lim_{t \to +\infty} \sum_{i=1}^{n} \left\{ |x_i(t) - \tilde{x}_i(t)| + |u_i(t) - \tilde{u}_i(t)| \right\} = 0. \]

Remark 2.6. Let us put \( y_i = h_i(x_i) := \int_1^{x_i} \frac{ds}{g(s)} \), \( i = 1, 2, \ldots, n \). By \((H3)\), it is easy to see that \( h_i : (0, +\infty) \to \mathbb{R} \), \( x_i \mapsto y_i = h_i(x_i) \) has a unique inverse \( \varphi_i : \mathbb{R} \to (0, +\infty) \), \( y_i \mapsto x_i = \varphi_i(y_i) \). Moreover, \( \varphi_i \in C^1(\mathbb{R}, (0, +\infty)) \) and \( \varphi_i \) is strictly monotone increasing.

Remark 2.7. By \((H5)\), Lemma \(2.3\) implies that the functions \( t - \tau_{ijk}(t), t - \sigma_{ik}(t) \) and \( t - \gamma_{ik}(t) \) have the unique inverses, respectively. Let \( \mu_{ijk}(t), \xi_{ik}(t) \) and \( \xi_{ik}(t) \) represent the inverses of functions \( t - \tau_{ijk}(t), t - \sigma_{ik}(t) \) and \( t - \gamma_{ik}(t) \), respectively. Obviously, \( \mu_{ijk}, \xi_{ik}, \xi_{ik} \in C(\mathbb{R}, \mathbb{R}) \) and \( \mu_{ijk}(t + \omega) = \mu_{ijk}(t), \xi_{ik}(t + \omega) = \xi_{ik}(t) \) for all \( t \in \mathbb{R} \).

Remark 2.8. It is easy from \((H1)-(H4)\) to show that solutions of \((1.1)-(1.2)\) are well defined for all \( t \geq 0 \) and satisfy \( x_i(t) > 0 \) and \( u_i(t) > 0 \) for all \( t \geq 0 \) and \( i = 1, 2, \ldots, n \).
3. Main Results

**Theorem 3.1.** Assume that (H1)–(H4) hold. Let

\[
\int_0^\omega \left[ \beta_i(s) + \sum_{k=1}^m p_{ik}(s) + u_i^*(s) \right] ds > 0, \quad i = 1, 2, \ldots, n, \tag{3.1}
\]

\[
A_i := a_i + \sum_{k=1}^m b_{iik} + \sum_{k=1}^m c_{iik} + \bar{P}_i + \bar{Q}_i + \bar{R}_i > 0, \quad i = 1, 2, \ldots, n, \tag{3.2}
\]

\[
\bar{r}_i > \sum_{j \neq i}^n \left( \bar{a}_{ij} + \sum_{k=1}^m \bar{b}_{ijk} + \sum_{k=1}^m \bar{c}_{ijk} + \bar{P}_j + \bar{Q}_j + \bar{R}_j \right) \varphi_j(B_j), \quad i = 1, 2, \ldots, n, \tag{3.3}
\]

where \( B_i = h_i(\bar{r}_i/A_i) + (\bar{r}_i + |\bar{r}_i|) \omega, \ i = 1, 2, \ldots, n. \) Then system (1.1) has at least one positive \( \omega \)-periodic solution.

**Proof.** Consider the system

\[
\dot{y}_i(t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) \varphi_j(y_j(t)) - \sum_{j=1}^n \sum_{k=1}^m b_{ijk}(t) \varphi_j(y_j(t - \tau_{ijk}(t))) - \sum_{j=1}^n \sum_{k=1}^m c_{ijk}(t, s) \varphi_j(y_j(s))ds - d_i(t)(V_i \varphi_i(y_i))(t) - \sum_{k=1}^m e_{ik}(t)(V_i \varphi_i(y_i))(t - \sigma_{ik}(t)) - \int_{-\infty}^t f_i(t, s)(V_i \varphi_i(y_i))(s)ds. \tag{3.4}
\]

By (1.3), (2.2) and (3.1), if system (3.4) has an \( \omega \)-periodic solution \((y_1^*(t), \ldots, y_n^*(t))\), then

\[
u_i^*(t) = (V_i \varphi_i(y_i^*))(t) \\
\geq \int_t^{t+\omega} \left( \frac{1}{\exp(\alpha_i \omega) - 1} \min_{\tau \in [0, \omega]} \varphi_i(y_i^*(\tau)) \right) \beta_i(s) + \sum_{k=1}^m p_{ik}(s) + u_i^*(s) ds > 0, \quad t \in \mathbb{R}, \ i = 1, 2, \ldots, n.
\]

Thus, \((x_1^*(t), \ldots, x_n^*(t), u_1^*(t), \ldots, u_n^*(t))\) with values \(x_i^*(t) = \varphi_i(y_i^*(t))\) and \(u_i^*(t) = (V_i \varphi_i(y_i^*))(t)\) for \( i = 1, 2, \ldots, n \) is a positive \( \omega \)-periodic solution of system (2.1). So, by Lemma 2.2, we only need to show that system (3.4) has at least one \( \omega \)-periodic solution in order to complete the proof. To apply the continuation theorem of coincidence degree theory to the existence of an \( \omega \)-periodic solution of system (3.4), we take

\[Y = Z = \left\{ y(t) = (y_1(t), \ldots, y_n(t)) \in C(\mathbb{R}, \mathbb{R}^n) : y(t + \omega) = y(t) \text{ for all } t \in \mathbb{R} \right\}.
\]

Denote \( \|y\|_0 = \sum_{i=1}^n \|y_i\|_0 \), where \( \|y_i\|_0 = \max_{t \in [0, \omega]} |y_i(t)| \). Then \( Y \) and \( Z \) are Banach spaces when they endowed with the norm \( \| \cdot \|_0 \).
We define \( L : \text{Dom} L \subset Y \to Z \) and \( N : Y \to Z \) by setting \( Ly = \dot{y} \) and \( Ny = Fy = (F_1y, \ldots, F_ny) \), where

\[
F_iy(t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) \varphi_j(y_j(t)) - \sum_{j=1}^n \sum_{k=1}^m b_{ijk}(t) \varphi_j(y_j(t - \tau_{ijk}(t)))
- \sum_{j=1}^n \sum_{k=1}^m \int_{-\infty}^{t} c_{ijk}(t, s) \varphi_j(y_j(s)) ds - d_i(t)(V_i\varphi_i(y_i))(t) \tag{3.5}
- \sum_{k=1}^m e_{ik}(t)(V_i\varphi_i(y_i))(t - \sigma_{ik}(t)) - \int_{-\infty}^{t} f_i(t, s)(V_i\varphi_i(y_i))(s) ds.
\]

Further, we define continuous projectors \( P : Y \to Y \) and \( Q : Z \to Z \) as follows

\[
P_y = \frac{1}{\omega} \int_{0}^{\omega} y(s) ds, \quad Qz = \frac{1}{\omega} \int_{0}^{\omega} z(s) ds.
\]

We easily see that \( \text{Im} L = \{ z \in Z : \int_{0}^{\omega} z(s) ds = 0 \} \) and \( \ker L = \mathbb{R}^n \). So, \( \text{Im} L \) is closed in \( Z \) and \( \dim \ker L = n = \text{codim} \text{Im} L \). Hence, \( L \) is a Fredholm mapping of index zero. Clearly that \( \text{Im} P = \ker L, \text{Im} L = \ker Q = \text{Im}(I - Q) \). Furthermore, the generalized inverse (to \( L \)) \( K_P : \text{Im} L \to \ker P \cap \text{Dom} L \) has the form

\[
K_Pz(t) = \int_{0}^{t} z(s) ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} z(s) ds dt.
\]

We know that

\[
QN y(t) = \frac{1}{\omega} \int_{0}^{\omega} Fy(t) dt.
\]

Thus,

\[
K_P(I - Q)N y(t) = \left( K_PN - K_PQN \right) y(t)
= \int_{0}^{t} Fy(s) ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} Fy(s) ds dt + \left( \frac{1}{2} - \frac{t}{\omega} \right) \int_{0}^{\omega} Fy(s) ds.
\]

It is easy to see that \( QN \) and \( K_P(I - Q)N \) are continuous. Furthermore, it can be verified that \( K_P(I - Q)N(\hat{\Omega}) \) is compact for any open bounded set \( \Omega \subset Y \) by using Arzela-Ascoli theorem and \( QN(\hat{\Omega}) \) is bounded. Therefore, \( N \) is \( L \)-compact on \( \hat{\Omega} \) for any open bounded subset \( \Omega \subset Y \). Now we are in a position to search for an appropriate open bounded subset \( \Omega \) for the application of the continuation theorem (Lemma \[2.1\]) to system (3.4).

Corresponding to the operator equation \( Ly = \lambda Ny \) (\( \lambda \in (0, 1) \)), we have

\[
\dot{y}_i(t) = \lambda \left[ r_i(t) - \sum_{j=1}^n a_{ij}(t) \varphi_j(y_j(t)) - \sum_{j=1}^n \sum_{k=1}^m b_{ijk}(t) \varphi_j(y_j(t - \tau_{ijk}(t)))
- \sum_{j=1}^n \sum_{k=1}^m \int_{-\infty}^{t} c_{ijk}(t, s) \varphi_j(y_j(s)) ds - d_i(t)(V_i\varphi_i(y_i))(t) \right.
- \sum_{k=1}^m e_{ik}(t)(V_i\varphi_i(y_i))(t - \sigma_{ik}(t)) - \int_{-\infty}^{t} f_i(t, s)(V_i\varphi_i(y_i))(s) ds \right].
\]
Integrating (3.6) from 0 to \( \omega \) and simplifying, we obtain

\[
\bar{r}_i \omega = \sum_{j=1}^{n} \int_0^\omega a_{ij}(t) \varphi_j(y_j(t)) dt + \sum_{j=1}^{n} \sum_{k=1}^{m} \int_0^\omega b_{ijk}(t) \varphi_j(y_j(t - \tau_{ijk}(t))) dt + \sum_{j=1}^{n} \sum_{k=1}^{m} \int_0^\omega c_{ijk}(t) \varphi_j(y_j(s)) ds \, dt + \int_0^\omega d_i(t)(V_i \varphi_i(y_i))(t) dt + \sum_{k=1}^{m} \int_0^\omega e_{ik}(t)(V_i \varphi_i(y_i))(t - \sigma_{ik}(t)) dt + \int_0^\omega f_i(t, s)(V_i \varphi_i(y_i))(s) ds \, dt.
\]

(3.7)

Let

\[
y_i(\eta) = \max_{t \in [0, \omega]} y_i(t), \quad y_i(\theta_i) = \min_{t \in [0, \omega]} y_i(t), \quad \eta, \theta_i \in [0, \omega], \quad i = 1, 2, \ldots, n.
\]

(3.8)

It is easy to see from (1.3), (2.2) and (3.8) that

\[
\int_0^\omega d_i(t)(V_i \varphi_i(y_i))(t) dt \geq \bar{P}_i \omega \varphi_i(y_i(\theta_i)),
\]

(3.9)

\[
\sum_{k=1}^{m} \int_0^\omega e_{ik}(t)(V_i \varphi_i(y_i))(t - \sigma_{ik}(t)) dt \geq \bar{Q}_i \omega \varphi_i(y_i(\theta_i)),
\]

and

\[
\int_0^\omega f_i(t, s)(V_i \varphi_i(y_i))(s) ds \, dt \geq \bar{R}_i \omega \varphi_i(y_i(\theta_i))
\]

(3.10)

It follows from (1.3), (3.7), (3.8) and (3.9) that

\[
\bar{r}_i \geq \left( a_{ii} + \sum_{k=1}^{m} b_{ik} + \sum_{k=1}^{m} c_{iik} + \bar{P}_i + \bar{Q}_i + \bar{R}_i \right) \varphi_i(y_i(\theta_i)) = A_i \varphi_i(y_i(\theta_i)),
\]

(3.11)

for \( i = 1, 2, \ldots, n \). Thus, by (3.2) and Remark 2.6 we have

\[
y_i(\theta_i) \leq b_i \left( \frac{\bar{r}_i}{A_i} \right), \quad i = 1, 2, \ldots, n.
\]

(3.12)

From (3.6) and (3.7), we know that

\[
\int_0^\omega |\dot{y}_i(t)| dt \leq (\bar{r}_i + \|\bar{r}_i\|) \omega, \quad i = 1, 2, \ldots, n,
\]

(3.13)

and thus, by (3.12),

\[
y_i(t) \leq y_i(\theta_i) + \int_0^\omega |\dot{y}_i(t)| dt \leq h_i \left( \frac{\bar{r}_i}{A_i} + (\bar{r}_i + \|\bar{r}_i\|) \omega \right), \quad t \in [0, \omega],
\]

(3.14)
for \( i = 1, 2, \ldots, n \). It is easy to see from \((3.7), (3.8), (3.10)\) and \((3.14)\) that
\[
(a_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}_{ijk} + \bar{P}_{i} + \bar{Q}_{i} + \bar{R}_{i})\bar{\varphi}_{i}(y_{i}(\eta_{i}))
\]
\[
\geq \bar{r}_{i} - \sum_{j \neq i}^{n} \left( \bar{a}_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}_{ijk} + \bar{P}_{j} + \bar{Q}_{j} + \bar{R}_{j} \right)\varphi_{j}(y_{j}(\eta_{j})),
\]
for \( i = 1, 2, \ldots, n \). Thus, by \((3.3)\) and \((3.14)\), we have
\[
\varphi_{i}(y_{i}(\eta_{i})) \geq \frac{\bar{r}_{i} - \sum_{j \neq i}^{n} \left( \bar{a}_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}_{ijk} + \bar{P}_{j} + \bar{Q}_{j} + \bar{R}_{j} \right)\varphi_{j}(B_{j})}{\bar{a}_{ii} + \sum_{k=1}^{m} \bar{b}_{iik} + \sum_{k=1}^{m} \bar{c}_{iik} + \bar{P}_{i} + \bar{Q}_{i} + \bar{R}_{i}} =: c_{i},
\]
for \( i = 1, 2, \ldots, n \); or
\[
y_{i}(\eta_{i}) \geq h_{i}(c_{i}), \quad i = 1, 2, \ldots, n,
\]
\((3.16)\)

From \((3.13)\) and \((3.16)\), it follows that
\[
y(t) \geq y_{i}(\eta_{i}) - \int_{0}^{t} |\dot{y}_{i}(t)| dt \geq h_{i}(c_{i}) - (\bar{r}_{i} + |\bar{r}_{i}|)\omega =: D_{i}, \; t \in [0, \omega],
\]
\((3.17)\)

for \( i = 1, 2, \ldots, n \). From \((3.12)\) and \((3.16)\) we see that
\[
\|y\| \leq M := \sum_{i=1}^{n} (|B_{i}| + |D_{i}|).
\]
\((3.18)\)

By \((3.3)\), Lemma 2.3 implies that the following system of algebraic equations
\[
\bar{r}_{i} = \sum_{j=1}^{n} \left[ \bar{a}_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}_{ijk} \right] \varphi_{j}(y_{j}) + (\bar{P}_{i} + \bar{Q}_{i} + \bar{R}_{i})\varphi_{i}(y_{i}),
\]
\((3.19)\)

for \( i = 1, 2, \ldots, n \), has a unique solution \( y^{*} = (y_{1}^{*}, \ldots, y_{n}^{*}) \in \mathbb{R}^{n} \). Let \( S = \|y^{*}\|_{0} + M \).

Evidently, \( S \) is independent of the choice of \( \lambda \). Let \( \Omega := \{ y \in Y : \|y\|_{1} < S \} \). It is clear that \( \Omega \) satisfies the requirement (i) in Lemma 2.1. Moreover, \( QN_{Y} \neq 0 \) for any \( y \in \partial\Omega \cap \ker L \). Let us take \( J = I \), where \( I \) is the identity operator on \( \mathbb{R}^{n} \). By straightforward computation, we have
\[
\deg(JQN, \Omega \cap \ker L, 0) = \text{sgn} \left\{ (-1)^{n} \left[ \det(w_{ij}) \right] \varphi_{1}(y_{1}^{*}) \ldots \varphi_{n}(y_{n}^{*}) \right\} \neq 0,
\]
where \( w_{ii} = A_{i} \),
\[
w_{ij} = a_{ij} + \sum_{k=1}^{m} \bar{b}_{ijk} + \sum_{k=1}^{m} \bar{c}_{ijk}, \quad i, j = 1, 2, \ldots, n.
\]

By Lemma 2.1 we conclude that the equation \( Ly = Ny \) has at least one solution in \( Y \). Therefore, system \((1.1)\) has at least one positive \( \omega \)-periodic solution. The proof is complete. \( \Box \)
Theorem 3.2. Assume that (H1)–(H5), (3.1) and (3.3) hold. If there exist positive constants \( \nu_1, \nu_2, \ldots, \nu_n \) such that

\[
\min_{t \in [0, \omega]} \left\{ \nu_i \left[ a_{ii}(t) - \beta_i(t) - \sum_{k=1}^{m} \frac{p_{ik}(\xi_{ik}(t))}{1 - \gamma_{ik}(\xi_{ik}(t))} - \int_0^{+\infty} v_i(t + \tau, t) d\tau \right] \\
- \sum_{j \neq i}^{n} \nu_j a_{ji}(t) - \sum_{j=1}^{n} \nu_j \sum_{k=1}^{m} \frac{b_{ijk}(\mu_{ijk}(t))}{1 - \tau_{ijk}(\mu_{ijk}(t))} \\
- \sum_{j=1}^{n} \nu_j \sum_{k=1}^{m} \int_{0}^{+\infty} c_{ijk}(t + \tau, t) d\tau \right\} > 0,
\]

(3.20)

for \( i = 1, 2, \ldots, n \), then system (1.1) has a unique positive \( \omega \)-periodic solution which is globally asymptotically stable.

Proof. From (3.20), we conclude that \( \bar{a}_{ii} > 0 \) for \( i = 1, 2, \ldots, n \), which means that (3.2) holds. By Theorem 3.1 system (1.1) has a positive \( \omega \)-periodic solution \((\bar{x}(t), \bar{u}(t))\). Let \((x(t), u(t))\) be any other positive solution of (1.1) with initial condition (1.2). Consider the Liapunov functional \( V(t) = V(\bar{x}(t), \bar{u}(t)), (x(t), u(t)) \) defined by

\[
V(t) = \sum_{i=1}^{n} \nu_i \left\{ V_{i}^{(1)}(t) + V_{i}^{(2)}(t) + \sum_{j=1}^{n} \sum_{k=1}^{m} \left[ V_{ijk}^{(3)}(t) + V_{ijk}^{(4)}(t) \right] \\
+ \sum_{k=1}^{m} \left[ V_{ik}^{(5)}(t) + V_{ik}^{(6)}(t) \right] + V_{i}^{(7)}(t) + V_{i}^{(8)}(t) \right\},
\]

(3.21)

where, for \( i, j, k = 1, 2, \ldots, n, k = 1, 2, \ldots, m \), we have

\[
V_{i}^{(1)}(t) = \left| \int_{\bar{x}_{i}(t)}^{x_{i}(t)} \frac{ds}{g_i(s)} \right|, \quad V_{i}^{(2)}(t) = |u_{i}(t) - \bar{u}_{i}(t)|,
\]

\[
V_{ijk}^{(3)}(t) = \int_{t-\tau_{ijk}(s)}^{t} \frac{b_{ijk}(\mu_{ijk}(s))}{1 - \tau_{ijk}(\mu_{ijk}(s))} |x_{j}(s) - \bar{x}_{j}(s)| ds,
\]

\[
V_{ijk}^{(4)}(t) = \int_{0}^{+\infty} \int_{t-\tau}^{t} c_{ijk}(s + \tau, s) |x_{j}(s) - \bar{x}_{j}(s)| ds d\tau,
\]

\[
V_{ik}^{(5)}(t) = \int_{t-\sigma_{ik}(s)}^{t} \frac{e_{ik}(\zeta_{ik}(s))}{1 - \sigma_{ik}(\zeta_{ik}(s))} |u_{i}(s) - \bar{u}_{i}(s)| ds,
\]

\[
V_{ik}^{(6)}(t) = \int_{t-\gamma_{ik}(t)}^{t} \frac{p_{ik}(\xi_{ik}(s))}{1 - \gamma_{ik}(\xi_{ik}(s))} |x_{i}(s) - \bar{x}_{i}(s)| ds,
\]

\[
V_{i}^{(7)}(t) = \int_{0}^{+\infty} \int_{t-\tau}^{t} f_{i}(s + \tau, s) |u_{i}(s) - \bar{u}_{i}(s)| ds d\tau,
\]

\[
V_{i}^{(8)}(t) = \int_{0}^{+\infty} \int_{t-\tau}^{t} v_{i}(s + \tau, s) |u_{i}(s) - \bar{u}_{i}(s)| ds d\tau
\]
Clearly $V(t)$ is continuous on $[0, +\infty)$. Calculating the upper right derivative of $V_i^{(1)}(t), \ldots, V_i^{(8)}(t)$ along the solutions of system (1.1) for $t > 0$, we obtain

$$D^+ V_i^{(1)} = \left[ \frac{\dot{x}_i(t)}{g_i(x_i(t))} - \frac{\dot{x}_i(t)}{g_i(\bar{x}_i(t))} \right] \text{sgn}[x_i(t) - \bar{x}_i(t)]$$

$$= \text{sgn}[x_i(t) - \bar{x}_i(t)] \left[ - \sum_{j=1}^n a_{ij}(x_j(t) - \bar{x}_j(t)) + \sum_{j=k}^m b_{ijk}(x_j(t - \tau_{ijk}(t)) - \bar{x}_j(t - \tau_{ijk}(t))) \right.\left. - \sum_{j=1}^n \sum_{k=1}^m \int_t^{-\infty} c_{ijk}(t, \tau)[x_j(\tau) - \bar{x}_j(\tau)]d\tau - \int_t^{-\infty} f_i(t, \tau)[u_i(\tau) - \bar{u}_i(\tau)]d\tau \right]$$

and thus,

$$D^+ V_i^{(1)} \leq -a_{ii}(t)|x_i(t) - \bar{x}_i(t)| + \sum_{j \neq i}^n a_{ij}(t)|x_j(t) - \bar{x}_j(t)|$$

$$+ \sum_{j=1}^n \sum_{k=1}^m b_{ijk}(t)|x_j(t - \tau_{ijk}(t)) - \bar{x}_j(t - \tau_{ijk}(t))|$$

$$+ \sum_{j=1}^n \sum_{k=1}^m \int_t^{-\infty} c_{ijk}(t, \tau)|x_j(\tau) - \bar{x}_j(\tau)|d\tau - \int_t^{-\infty} f_i(t, \tau)|u_i(\tau) - \bar{u}_i(\tau)|d\tau$$

$$+ \int_{-\infty}^t f_i(t, \tau)|u_i(\tau) - \bar{u}_i(\tau)|d\tau.$$ (3.22)

$$D^+ V_i^{(2)} = [\dot{u}_i(t) - \dot{\bar{u}}_i(t)] \text{sgn}[u_i(t) - \bar{u}_i(t)]$$

$$= \text{sgn}[u_i(t) - \bar{u}_i(t)] \left[ -\alpha_i(t)(u_i(t) - \bar{u}_i(t)) + \beta_i(t)(x_i(t) - \bar{x}_i(t)) \right]$$

$$+ \sum_{k=1}^m p_{ik}(t)[x_i(t - \gamma_{ik}(t)) - \bar{x}_i(t - \gamma_{ik}(t))]$$

$$+ \int_{-\infty}^t v_i(t, \tau)(x_i(\tau) - \bar{x}_i(\tau))d\tau$$

$$\leq -\alpha_i(t)|u_i(t) - \bar{u}_i(t)| + \beta_i(t)|x_i(t) - \bar{x}_i(t)|$$

$$+ \sum_{k=1}^m p_{ik}(t)|x_i(t - \gamma_{ik}(t)) - \bar{x}_i(t - \gamma_{ik}(t))|$$

$$+ \int_{-\infty}^t v_i(t, \tau)|x_i(\tau) - \bar{x}_i(\tau)|d\tau.$$ (3.23)
\[ \dot{V}_{ijk}^{(3)} = \frac{b_{ijk} \mu_{ijk}(t)}{1 - \tau_{ijk}(\mu_{ijk}(t))} |x_j(t) - \tilde{x}_j(t)| - b_{ijk}(t) |x_j(t - \tau_{ijk}(t)) - \tilde{x}_j(t - \tau_{ijk}(t))|, \]  
(3.24)

\[ \dot{V}_{ijk}^{(4)} = \int_0^{+\infty} c_{ijk}(t + \tau, t) |x_j(t) - \tilde{x}_j(t)| d\tau - \int_0^{+\infty} c_{ijk}(t, t - \tau) |x_j(t - \tau) - \tilde{x}_j(t - \tau)| d\tau; \]  
(3.25)

\[ \dot{V}_{ik}^{(5)} = \frac{e_{ik}(\tilde{\zeta}_{ik}(t))}{1 - \tilde{\sigma}_{ik}(\tilde{\zeta}_{ik}(t))} \left| u_i(t) - \tilde{u}_i(t) \right| - e_{ik}(t) |u_i(t - \sigma_{ik}(t)) - \tilde{u}_i(t - \sigma_{ik}(t))|; \]  
(3.26)

\[ \dot{V}_{ik}^{(6)} = \frac{p_{ik}(\tilde{\xi}_{ik}(t))}{1 - \tilde{\gamma}_{ik}(\tilde{\xi}_{ik}(t))} \left| x_i(t) - \tilde{x}_i(t) \right| - p_{ik}(t) |x_i(t - \gamma_{ik}(t)) - \tilde{x}_i(t - \gamma_{ik}(t))|; \]  
(3.27)

\[ \dot{V}_i^{(7)} = \int_0^{+\infty} f_i(t + \tau, t) |u_i(t) - \tilde{u}_i(t)| d\tau - \int_0^{+\infty} f_i(t, t - \tau) |u_i(t - \tau) - \tilde{u}_i(t - \tau)| d\tau; \]  
(3.28)

\[ \dot{V}_i^{(8)} = \int_0^{+\infty} v_i(t + \tau, t) |x_i(t) - \tilde{x}_i(t)| d\tau - \int_0^{+\infty} v_i(t, t - \tau) |x_i(t - \tau) - \tilde{x}_i(t - \tau)| d\tau. \]  
(3.29)

From (3.21) - (3.29) it follows that

\[ D^+ V \leq \sum_{i=1}^n \nu_i \left\{ \left[ -\alpha_i(t) + \beta_i(t) + \sum_{k=1}^m \frac{p_{ik}(\xi_{ik}(t))}{1 - \gamma_{ik}(\xi_{ik}(t))} \right] \right. \]

\[ + \int_0^{+\infty} v_i(t + \tau, t) d\tau \left| x_i(t) - \tilde{x}_i(t) \right| \]

\[ + \sum_{j \neq i} \alpha_{ij}(t) |x_j(t) - \tilde{x}_j(t)| + \sum_{j=1}^n \sum_{k=1}^m \frac{b_{ijk}(\mu_{ijk}(t))}{1 - \tau_{ijk}(\mu_{ijk}(t))} |x_j(t) - \tilde{x}_j(t)| \]

\[ + \sum_{j=1}^n \sum_{k=1}^m \left[ \int_0^{+\infty} c_{ijk}(t + \tau, t) d\tau \left| x_j(t) - \tilde{x}_j(t) \right| \right] + \sum_{i=1}^n \nu_i \left\{ -\alpha_i(t) + d_i(t) \right. \]

\[ + \sum_{k=1}^m \frac{e_{ik}(\tilde{\zeta}_{ik}(t))}{1 - \tilde{\sigma}_{ik}(\tilde{\zeta}_{ik}(t))} + \int_0^{+\infty} f_i(t + \tau, t) d\tau \left| u_i(t) - \tilde{u}_i(t) \right| \]

\[ = \sum_{i=1}^n \nu_i \left[ -\alpha_i(t) + \beta_i(t) + \sum_{k=1}^m \frac{p_{ik}(\xi_{ik}(t))}{1 - \gamma_{ik}(\xi_{ik}(t))} \right] + \int_0^{+\infty} v_i(t + \tau, t) d\tau \]

\[ + \sum_{j \neq i} \nu_j a_{ij}(t) + \sum_{j=1}^n \nu_j \sum_{k=1}^m \frac{b_{ijk}(\mu_{ijk}(t))}{1 - \tau_{ijk}(\mu_{ijk}(t))} \]

\[ + \sum_{j=1}^n \nu_j \sum_{k=1}^m \left[ \int_0^{+\infty} c_{ijk}(t + \tau, t) d\tau \right] |x_j(t) - \tilde{x}_j(t)| \]
where $\alpha_i(t) = \sum_{k=1}^{m} e_{ik}(\zeta_{ik}(t))$.

By (3.20), there exists a positive number $\delta$ such that

$$
\min_{t \in [0, \omega]} \left\{ \nu_i \left[ a_{ii}(t) - \beta_i(t) - \sum_{k=1}^{m} \frac{p_{ik}(\xi_{ik}(t))}{1 - \gamma_{ik}(\xi_{ik}(t))} - \int_{0}^{+\infty} v_i(t + \tau, t) d\tau \right] \right.
$$

$$
- \sum_{j \neq i} \nu_j a_{ji}(t) - \sum_{j=1}^{n} \nu_j \sum_{k=1}^{m} \frac{b_{jik}(\mu_{jik}(t))}{1 - \tau_{jik}(\mu_{jik}(t))}
$$

$$
- \sum_{j=1}^{n} \nu_j \sum_{k=1}^{m} \int_{0}^{+\infty} c_{jik}(t + \tau, t) d\tau > \delta,
$$

for $i = 1, 2, \ldots, n$. In the view of (3.30) and (3.31) we have

$$
D^+ V(t) \leq -\delta \sum_{i=1}^{n} \left\{ |x_i(t) - \tilde{x}_i(t)| + |u_i(t) - \tilde{u}_i(t)| \right\}, \quad t \geq 0.
$$

Integrating both sides of (3.32) from 0 to $t$, we obtain

$$
V(t) - V(0) \leq -\delta \int_{0}^{t} \sum_{i=1}^{n} \left\{ |x_i(s) - \tilde{x}_i(s)| + |u_i(s) - \tilde{u}_i(s)| \right\} ds, \quad t \geq 0.
$$

Thus,\[\int_{0}^{t} \sum_{i=1}^{n} \left\{ |x_i(s) - \tilde{x}_i(s)| + |u_i(s) - \tilde{u}_i(s)| \right\} ds \leq \frac{V(0)}{\delta}, \quad t \geq 0,\]which implies\[\int_{0}^{+\infty} \sum_{i=1}^{n} \left\{ |x_i(s) - \tilde{x}_i(s)| + |u_i(s) - \tilde{u}_i(s)| \right\} ds \leq \frac{V(0)}{\delta}.\]Since $\nu_i \left\{ \int_{\tilde{x}_i(t)}^{x_i(t)} \frac{ds}{g_i(s)} + |u_i(t) - \tilde{u}_i(t)| \right\} \leq V(t) \leq V(0), \quad t \geq 0,$ it follows from (H3) that $x_i(t)$ and $u_i(t)$ are bounded on $[0, +\infty)$, and hence from (1.1) we can conclude that $\dot{x}_i(t)$ and $\dot{u}_i(t)$ are also bounded on $[0, +\infty)$. This implies that $x_i(t)$ and $u_i(t)$ are uniformly continuous on $[0, +\infty)$. Therefore, $\sum_{i=1}^{n} |x_i(t) - \tilde{x}_i(t)| + |u_i(t) - \tilde{u}_i(t)|$ is uniformly continuous on $[0, +\infty)$. Thus, (3.33) implies that

$$
\lim_{t \to +\infty} \sum_{i=1}^{n} \left\{ |x_i(t) - \tilde{x}_i(t)| + |u_i(t) - \tilde{u}_i(t)| \right\} = 0
$$

and $(\tilde{x}(t), \tilde{u}(t))$ is the unique positive $\omega$-periodic solution of system (1.1). The proof is complete. $\square$
As an example we consider system (1.1) with

\[ n = 2, \quad m = 1, \quad r_1(t) = r_2(t) = \frac{\ln 5}{2}, \quad \alpha_1(t) = \alpha_2(t) = 6, \]

\[ a_{11}(t) = a_{22}(t) = 21 + \sin 2\pi t, \quad a_{12}(t) = a_{21}(t) = 1 + \sin 2\pi t, \]

\[ b_{111}(t) = b_{121}(t) = b_{211}(t) = b_{221}(t) = 1 + 2\sin \pi t, \]

\[ c_{111}(t, s) = c_{121}(t, s) = c_{211}(t, s) = c_{221}(t, s) = v_1(t, s) = v_2(t, s) = \exp\{-(t - s)\}, \]

\[ e_{11}(t) = e_{21}(t) = 1 + \cos 2\pi t, \quad d_1(t) = d_2(t) = 1 - \sin 2\pi t, \]

\[ \beta_1(t) = \beta_2(t) = 1 + \cos 2\pi t, \quad p_{111}(t) = p_{211}(t) = 1 - \cos 2\pi t, \]

\[ \tau_{111}(t) = \tau_{121}(t) = \tau_{211}(t) = \tau_{221}(t) \equiv \tau^* > 0, \]

\[ \sigma_{11}(t) = \sigma_{21}(t) \equiv \sigma^* > 0, \quad \gamma_{11}(t) = \gamma_{21}(t) \equiv \gamma^* > 0, \quad g_1(v) = g_2(v) = v. \]

It is easy to see that (H1)–(H5) hold. By straightforward computation, we have

\[ c_{ij1}(t) = f_{i1}^*(t) = v^*_i(t) = 1, \quad \bar{P}_i = \bar{Q}_i = \bar{R}_i = \frac{3}{5}, \]

\[ A_i = 24.8, \quad B_i = \ln \frac{\ln 5}{49.6} + \ln 5, \quad \varphi_i(B_i) = \frac{5 \ln 5}{49.6}, \quad i, j = 1, 2. \]

Let \( \nu_1 = \nu_2 = 1 \). We can easily see that

\[ \int_0^1 \left[ \beta_i(s) + p_{i1}(s) + v^*_i(s) \right] ds = 3 > 0, \]

\[ \sum_{j \neq i}^2 \left( a_{ij} + \bar{b}_{ij1} + c_{ij1}^* + \bar{P}_j + \bar{Q}_j + \bar{R}_j \right) \varphi_j(B_j) = \frac{30}{31} \times \frac{\ln 5}{2} < \frac{\ln 5}{2} = \bar{r}_i, \]

\[ \min_{t \in [0, 2\pi]} \left\{ \nu_i \left[ a_{ii}(t) - \beta_i(t) - \frac{p_{i1}(\xi_{i1}(t))}{1 - \gamma_{i1}(\xi_{i1}(t))} - \int_0^{+\infty} v_i(t + \tau, t) d\tau \right] - \sum_{j \neq i}^{2} \nu_j a_{ji}(t) \right\} \]

\[ - \sum_{j = 1}^2 \nu_j \frac{b_{ji1}(\mu_{ji1}(t))}{1 - \gamma_{ji1}(\mu_{ji1}(t))} - \sum_{j = 1}^2 \nu_j \int_0^{+\infty} c_{ji1}(t + \tau, t) d\tau \}

\[ = \min_{t \in [0, 2\pi]} \left[ 14 - \cos 2\pi (t + \gamma^*) - 2\sin 2\pi (t + \tau^*) \right] > 0, \]

\[ \min_{t \in [0, 2\pi]} \left\{ \alpha_i(t) - d_i(t) - \frac{e_{i1}(\xi_{i1}(t))}{1 - \sigma_{i1}(\xi_{i1}(t))} - \int_0^{+\infty} f_i(t + \tau, t) d\tau \right\} \]

\[ = \min_{t \in [0, 2\pi]} \left[ 3 + \sin 2\pi t - \cos 2\pi (t + \sigma^*) \right] > 0, \quad i = 1, 2 \]

Therefore, conditions (3.1), (3.3) and (3.30) hold. Hence, by Theorem 3.2 the system has a unique positive 1-periodic solution which is globally asymptotically stable.

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