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OSCILLATION OF SOLUTIONS TO NEUTRAL NONLINEAR IMPULSIVE HYPERBOLIC EQUATIONS WITH SEVERAL DELAYS

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ABSTRACT. In this article, we study oscillatory properties of solutions to neutral nonlinear impulsive hyperbolic partial differential equations with several delays. We establish sufficient conditions for oscillation of all solutions.

1. INTRODUCTION

The theory of partial differential equations can be applied to many fields, such as to biology, population growth, engineering, generic repression, control theory and climate model. In the last few years, the fundamental theory of partial differential equations with deviating argument has undergone intensive development. The qualitative theory of this class of equations, however, is still in an initial stage of development. Many srudies have been done under the assumption that the state variables and system parameters change continuously. However, one may easily visualize situations in nature where abrupt change such as shock and disasters may occur. These phenomena are short-time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is, in the form of impulses.

In 1991, the first paper [4] on this class of equations was published. However, on oscillation theory of impulsive partial differential equations only a few of papers have been published. Recently, Bainov, Minchev, Liu and Luo [1, 2, 6, 7, 8, 9, 10, 11] investigated the oscillation of solutions of impulsive partial differential equations with or without deviating argument. But there is a scarcity in the study of oscillation theory of nonlinear impulsive hyperbolic equations of neutral type with several delays.

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In this article, we discuss oscillatory properties of solutions for the nonlinear impulsive hyperbolic equation of neutral type with several delays.

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\Big[u(t,x)+\sum_{i=1}^{m} g_i u(t-\tau_i,x)\Big] \\
&= a(t)h(u)\Delta u - q(t,x)f(u(t,x)) + \sum_{r=1}^{l} a_r(t)h_r(u(t-\sigma_r,x))\Delta u(t-\sigma_r,x) \quad (1.1)\\
&\quad -\sum_{j=1}^{n} q_j(t,x)f_j(u(t-\rho_j,x)), \quad t\neq t_k,\ (t,x)\in\mathbb{R}^+\times\Omega = G,
\end{aligned}$$

$$u(t_k^+,x) = h_k(t_k,x,u(t_k,x)), \quad k=1,2,\dots, \quad (1.2)$$

$$u_t(t_k^+,x) = p_k(t_k,x,u_t(t_k,x)), \quad k=1,2,\dots, \quad (1.3)$$

with the boundary conditions

$$u=0, \quad (t,x)\in\mathbb{R}^+\times\partial\Omega, \quad (1.4)$$

$$\frac{\partial u}{\partial n} + \varphi(t,x)u = 0, \quad (t,x)\in\mathbb{R}^+\times\partial\Omega, \quad (1.5)$$

and the initial condition

$$u(t,x) = \Phi(t,x),\ \frac{\partial u(t,x)}{\partial t} = \Psi(t,x), \quad (t,x)\in[-\delta,0]\times\Omega.$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and n is a unit exterior normal vector of $\partial\Omega$, $\delta=\max\{\tau_i,\sigma_r,\rho_j\}$, $\Phi(t,x)\in C^2([-\delta,0]\times\Omega,\mathbb{R})$, $\Psi(t,x)\in C^1([-\delta,0]\times\Omega,\mathbb{R})$.

This article is organized as follows. in Section 2, we study the oscillatory properties of solutions for problems (1.1), (1.4). In Section 3, we discuss oscillatory properties of solutions for problems (1.1), (1.5).

We will use the following conditions:

(H1) $a(t), a_i(t)\in PC(\mathbb{R}^+,\mathbb{R}^+)$, τ_i,σ_r,ρ_j are positive constants, $q(t,x), q_j(t,x)$ are functions in $C(\mathbb{R}^+\times\bar\Omega,(0,\infty))$, g_i is a non-negative constant, $\sum_{i=1}^m g_i < 1$, $i=1,2,\dots,m$, $j=1,2,\dots,n$, $r=1,2,\dots,l$; where PC denote the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\dots$, and left continuous at $t=t_k$, $k=1,2,\dots$.

(H2) $h(u), h_r(u)\in C^1(\mathbb{R},\mathbb{R})$, $f(u), f_r(u)\in C(\mathbb{R},\mathbb{R})$; $f(u)/u\geq C$ a positive constant, $f_j(u)/u\geq C_j$ a positive constant, $u\neq 0$; $h(0)=0$, $h_r(0)=0$, $uh'(u)\geq 0$, $uh_r'(u)\geq 0$, $\varphi(t,x)\in C(\mathbb{R}^+\times\partial\Omega,\mathbb{R})$, $h(u)\varphi(t,x)\geq 0$, $h_r(u)\varphi(t-\sigma_r,x)\geq 0$, $r=1,2,\dots,l$, $0<t_1<t_2<\dots<t_k<\dots$, $\lim_{k\to\infty} t_k=\infty$.

(H3) $u(t,x)$ and their derivatives $u_t(t,x)$ are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\dots$, and left continuous at $t=t_k$, $u(t_k,x)=u(t_k^-,x)$, $u_t(t_k,x)=u_t(t_k^-,x)$, $k=1,2,\dots$.

(H4) $h_k(t_k,x,u(t_k,x)), p_k(t_k,x,u_t(t_k,x))\in PC(\mathbb{R}^+\times\bar\Omega\times\mathbb{R},\mathbb{R})$, $k=1,2,\dots$, and there exist positive constants $\overline{a}_k,\underline{a}_k,\bar b_k,\underline{b}_k$ and $\bar b_k\leq\underline{a}_k$ such that for $k=1,2,\dots$,

$$\underline{a}_k \leq \frac{h_k(t_k,x,\eta)}{\eta}\leq\overline{a}_k, \quad \underline{b}_k\leq\frac{p_k(t_k,x,\phi)}{\phi}\leq\bar b_k.$$

Let us construct the sequence $\{\bar{t}_k\} = \{t_k\} \cup \{t_{ki}\} \cup \{t_{kr}\} \cup \{t_{kj}\}$, where $t_{ki} = t_k + \tau_i,\ t_{kr} = t_k + \sigma_r,\ t_{kj} = t_k + \rho_j$ and $\bar{t}_k < \bar{t}_{k+1},\ i = 1, 2, \dots, m$, $r = 1, 2, \dots, l,\ j = 1, 2, \dots, n,\ k = 1, 2, \dots$.

Definition 1.1. By a solution of problems (1.1), (1.4) ((1.5)) with initial condition, we mean that any function $u(t, x)$ for which the following conditions are valid:

(1) If $-\delta \le t \le 0$, then $u(t, x) = \Phi(t, x)$, $\frac{\partial u(t,x)}{\partial t} = \Psi(t, x)$.
(2) If $0 \le t \le \bar{t}_1 = t_1$, then $u(t, x)$ coincides with the solution of the problems (1.1)–(1.3) and (1.4) ((1.5)) with initial condition.
(3) If $\bar{t}_k < t \le \bar{t}_{k+1}$, $\bar{t}_k \in \{t_k\} \setminus (\{t_{ki}\} \cup \{t_{kr}\} \cup \{t_{kj}\})$, then $u(t, x)$ coincides with the solution of the problems (1.1)–(1.3) and (1.4) ((1.5)).
(4) If $\bar{t}_k < t \le \bar{t}_{k+1}$, $\bar{t}_k \in \{t_{ki}\} \cup \{t_{kr}\} \cup \{t_{kj}\}$, then $u(t, x)$ satisfies (1.4) ((1.5)) and coincides with the solution of the problem

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\Big[u(t^+, x) + \sum_{i=1}^{m} g_i u((t-\tau_i)^+, x)\Big] \\
&= a(t)h(u(t^+, x))\Delta u(t^+, x) - q(t, x)f(u(t^+, x)) \\
&\quad + \sum_{r=1}^{l} a_r(t)h_r(u((t-\sigma_r)^+, x))\Delta u((t-\sigma_r)^+, x) \\
&- \sum_{j=1}^{n} q_j(t, x)f_j(u((t-\rho_j)^+, x)), \quad t \ne t_k,\ (t, x) \in \mathbb{R}^+ \times \Omega = G,
\end{aligned}$$

$$\begin{gathered}
u(\bar{t}_k^+, x) = u(\bar{t}_k, x), \quad u_t(\bar{t}_k^+, x) = u_t(\bar{t}_k, x), \\
\text{for} \bar{t}_k \in (\{t_{ki}\} \cup \{t_{kr}\} \cup \{t_{kj}\}) \setminus \{t_k\},
\end{gathered}$$

or

$$\begin{gathered}
u(\bar{t}_k^+, x) = h_{k_i}(\bar{t}_k, x, u(\bar{t}_k, x)), \quad u_t(\bar{t}_k^+, x) = p_{k_i}(\bar{t}_k, x, u_t(\bar{t}_k, x)), \\
\text{for } \bar{t}_k \in (\{t_{ki}\} \cup \{t_{kr}\} \cup \{t_{kj}\}) \cap \{t_k\},
\end{gathered}$$

where the number k_i is determined by the equality $\bar{t}_k = t_{k_i}$.

We introduce the notation:

$$\begin{gathered}
\Gamma_k = \{(t, x) : t \in (t_k, t_{k+1}), x \in \Omega\}, \quad \Gamma = \cup_{k=0}^{\infty} \Gamma_k, \\
\bar{\Gamma}_k = \{(t, x) : t \in (t_k, t_{k+1}), x \in \bar{\Omega}\}, \quad \bar{\Gamma} = \cup_{k=0}^{\infty} \bar{\Gamma}_k, \\
v(t) = \int_{\Omega} u(t, x)\, dx,\ q(t) = \min_{x \in \bar{\Omega}} q(t, x), \quad q_j(t) = \min_{x \in \bar{\Omega}} q_j(t, x).
\end{gathered}$$

Definition 1.2. The solution $u \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ of problems (1.1), (1.4) ((1.5)) is called non-oscillatory in the domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

2. Oscillation properties for (1.1), (1.4)

For the main result of this article, we need following lemmas.

Lemma 2.1. *Let $u \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ be a positive solution of* (1.1), (1.4) *in* G, *then function* $w(t)$ *satisfies the impulsive differential inequality*

$$w''(t) + C\Big(1 - \sum_{i=1}^{m} g_i\Big) q(t) w(t) + \sum_{j=1}^{n} C_j \Big(1 - \sum_{i=1}^{m} g_i\Big) q_j(t) w(t - \rho_j) \le 0, \quad t \ne t_k, \tag{2.1}$$

$$\underline{a}_k \le \frac{w(t_k^+)}{w(t_k)} \le \bar{a}_k, \quad k = 1, 2, \dots, \tag{2.2}$$

$$\underline{b}_k \le \frac{w'(t_k^+)}{w'(t_k)} \le \bar{b}_k, \quad k = 1, 2, \dots, \tag{2.3}$$

where $w(t) = v(t) + \sum_{i=1}^{m} g_i v(t - \tau_i)$.

Proof. Let $u(t, x)$ be a positive solution of the problem (1.1), (1.4) in G. Without loss of generality, we may assume that there exists a $T > 0$, $t_0 > T$ such that $u(t, x) > 0$, $u(t - \tau_i, x) > 0$, $i = 1, 2, \dots, m$, $u(t - \sigma_r, x) > 0$, $r = 1, 2, \dots, l$, $u(t - \rho_j, x) > 0$, $j = 1, 2, \dots n$, for any $(t, x) \in [t_0, \infty) \times \Omega$.

For $t \ge t_0$, $t \ne t_k$, $k = 1, 2, \dots$, integrating (1.1) with respect to x over Ω yields

$$\begin{aligned}
&\frac{d^2}{dt^2}\Big[\int_\Omega u(t,x)\,dx + \sum_{i=1}^{m} g_i \int_\Omega u(t - \tau_i, x)\,dx\Big] \\
&= a(t) \int_\Omega h(u)\Delta u\,dx - \int_\Omega q(t,x) f(u(t,x))\,dx \\
&\quad + \sum_{r=1}^{l} a_r(t) \int_\Omega h_r(u(t - \sigma_r, x))\Delta u(t - \sigma_r, x)\,dx \\
&\quad - \sum_{j=1}^{n} \int_\Omega q_j(t,x) f_j(u(t - \rho_j, x))\,dx.
\end{aligned}$$

By Green's formula and the boundary condition, we have

$$\begin{aligned}
\int_\Omega h(u)\Delta u\,dx &= \int_{\partial\Omega} h(u)\frac{\partial u}{\partial n} ds - \int_\Omega h'(u)|\mathrm{grad} u|^2\,dx \\
&= -\int_\Omega h'(u)|\mathrm{grad} u|^2\,dx \le 0,
\end{aligned}$$

$$\int_\Omega h_r(u(t - \sigma_r, x))\Delta u(t - \sigma_r, x)\,dx \le 0.$$

From condition (H2), we can easily obtain

$$\begin{gathered}
\int_\Omega q(t,x) f(u(t,x))\,dx \ge C q(t) \int_\Omega u(t,x)\,dx, \\
\int_\Omega q_j(t,x) f_j(u(t - \rho_j, x))\,dx \ge C_j q_j(t) \int_\Omega u(t - \rho_j, x)\,dx.
\end{gathered}$$

From the above it follows that

$$\frac{d^2}{dt^2}\Big[v(t) + \sum_{i=1}^{m} g_i v(t - \tau_i)\Big] + C q(t) v(t) + \sum_{j=1}^{n} C_j q_j(t) v(t - \rho_j) \le 0.$$

Set $w(t) = v(t) + \sum_{i=1}^m g_i v(t-\tau_i)$, we have $w(t) \geq v(t)$, then we can obtain

$$w''(t) + Cq(t)v(t) + \sum_{j=1}^n C_j q_j(t) v(t-\rho_j) \leq 0, \quad t \neq t_k.$$

From this inequality, we have $w''(t) < 0$, and $w'(t) > 0$, $w(t) > 0$ for $t \geq t_0$. In fact, if $w'(t) < 0$, there exists $t_1 \geq t_0$ such that $w'(t_1) < 0$. Hence we have

$$w(t) - w(t_1) = \int_{t_1}^t w'(s)ds \leq \int_{t_1}^t w'(t_1)ds = w'(t_1)(t-t_1),$$
$$\lim_{t\to+\infty} w(t) = -\infty.$$

This is a contradiction, so $w'(t) > 0$. Because

$$\begin{aligned}
v(t) &= w(t) - \sum_{i=1}^m g_i v(t-\tau_i) \\
&= w(t) - \sum_{i=1}^m g_i \Big[w(t-\tau_i) - \sum_{i=1}^m g_i v(t-2\tau_i) \Big] \\
&= w(t) - \sum_{i=1}^m g_i w(t-\tau_i) + \sum_{i=1}^m g_i \Big[\sum_{i=1}^m g_i v(t-2\tau_i) \Big] \\
&\geq \Big(1 - \sum_{i=1}^m g_i \Big) w(t),
\end{aligned}$$

$$v(t-\rho_j) \geq \Big(1 - \sum_{i=1}^m g_i \Big) w(t-\rho_j).$$

Hence, we obtain

$$w''(t) + C\Big(1 - \sum_{i=1}^m g_i\Big) q(t) w(t) + \sum_{j=1}^n C_j \Big(1 - \sum_{i=1}^m g_i\Big) q_j(t) w(t-\rho_j) \leq 0, \quad t \neq t_k.$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \dots$, from (1.2), (1.3) and condition (H4), we obtain

$$\underline{a}_k \leq \frac{u(t_k^+, x)}{u(t_k, x)} \leq \bar{a}_k, \tag{2.4}$$

$$\underline{b}_k \leq \frac{u_t(t_k^+, x)}{u_t(t_k, x)} \leq \bar{b}_k. \tag{2.5}$$

According to the $v(t) = \int_\Omega u(t,x)\,dx$, we obtain

$$\underline{a}_k \leq \frac{v(t_k^+)}{v(t_k)} \leq \bar{a}_k, \quad \underline{b}_k \leq \frac{v'(t_k^+)}{v'(t_k)} \leq \bar{b}_k.$$

Because $w(t) = v(t) + \sum_{i=1}^m g_i v(t-\tau_i)$, we finally have

$$\underline{a}_k \leq \frac{w(t_k^+)}{w(t_k)} \leq \bar{a}_k, \quad \underline{b}_k \leq \frac{w'(t_k^+)}{w'(t_k)} \leq \bar{b}_k.$$

Hence we obtain that $w(t)$ is a solution of impulsive differential inequality (2.1)–(2.3). This completes the proof. □

Lemma 2.2 ([5, Theorem 1.4.1]). *Assume that*

(A1) *the sequence* $\{t_k\}$ *satisfies* $0 < t_0 < t_1 < t_2 < \dots$, $\lim_{k\to\infty} t_k = \infty$*;*

(A2) *$m(t) \in PC^1[\mathbb{R}^+, \mathbb{R}]$ is left continuous at t_k for $k = 1, 2, \dots$;*
(A3) *for $k = 1, 2, \dots$ and $t \geq t_0$,*
$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k,$$
$$m(t_k^+) \leq d_k m(t_k) + e_k,$$
where $p(t), q(t) \in C(\mathbb{R}^+, \mathbb{R})$, $d_k \geq 0$ and e_k are constants. PC denote the class of piecewise continuous function from $\mathbb{R}^+$ to $\mathbb{R}$, with discontinuities of the first kind only at $t = t_k$, $k = 1, 2, \dots$.

Then
$$m(t) \leq m(t_0) \prod_{t_0<t_k<t} d_k \exp\Big(\int_{t_0}^t p(s)ds\Big) + \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\Big(\int_s^t p(r)dr\Big) q(s)ds + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j \exp\Big(\int_{t_k}^t p(s)ds\Big) e_k.$$

Lemma 2.3 ([11])**.** *Let $w(t)$ be an eventually positive (negative) solution of the differential inequality (2.1)–(2.3). Assume that there exists $T \geq t_0$ such that $w(t) > 0$ ($w(t) < 0$) for $t \geq T$. If*
$$\lim_{t\to+\infty} \int_{t_0}^t \prod_{t_0<t_k<s} \frac{b_k}{a_k} ds = +\infty \tag{2.6}$$
hold, then $w'(t) \geq 0$ ($w'(t) \leq 0$) for $t \in [T, t_l] \cup \big(\cup_{k=l}^{+\infty} (t_k, t_{k+1}]\big)$, where $l = \min\{k : t_k \geq T\}$.

The following theorem is the first main result of this article.

Theorem 2.4. *If condition (2.6) and the following condition holds,*
$$\lim_{t\to+\infty} \int_{t_0}^t \prod_{t_0<t_k<s} \frac{a_k}{b_k} F(s)ds = +\infty, \tag{2.7}$$
where
$$F(s) = C\Big(1 - \sum_{i=1}^m g_i\Big) q(s) + \sum_{j=1}^n C_j\Big(1 - \sum_{i=1}^m g_i\Big) q_j(s) \exp(-\delta z(t_0)).$$
Then each solution of (1.1)–(1.3), (1.4) oscillates in G.

Proof. Let $u(t,x)$ be a non-oscillatory solution of (1.1), (1.4). Without loss of generality, we can assume that there exists $T > 0$, $t_0 \geq T$, such that $u(t,x) > 0$, $u(t-\tau_i, x) > 0$, $i = 1, 2, \dots, m$, $u(t-\sigma_r, x) > 0$, $r = 1, 2, \dots, l$, $u(t-\rho_j, x) > 0$, $j = 1, 2, \dots, n$ for any $(t,x) \in [t_0, \infty) \times \Omega$. From Lemma 2.1, we know that $w(t)$ is a solution of (2.1)–(2.3).

For $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \dots$, define
$$z(t) = \frac{w'(t)}{w(t)}, \quad t \geq t_0. \tag{2.8}$$
From Lemma 2.3, we have $z(t) \geq 0$, $t \geq t_0$, $w'(t) - z(t)w(t) = 0$. We may assume that $w(t_0) = 1$, thus in view of (2.1)–(2.3) we have that for $t \geq t_0$,
$$w(t) = \exp\Big(\int_{t_0}^t z(s)ds\Big), \tag{2.9}$$

$$w'(t) = z(t)\exp\Big(\int_{t_0}^{t} z(s)ds\Big), \tag{2.10}$$

$$w''(t) = z^2(t)\exp\Big(\int_{t_0}^{t} z(s)ds\Big) + z'(t)\exp\Big(\int_{t_0}^{t} z(s)ds\Big). \tag{2.11}$$

We substitute (2.9)–(2.11) into (2.1) and obtain

$$\begin{aligned}
&z^2(t)\exp\Big(\int_{t_0}^{t} z(s)ds\Big) + z'(t)\exp\Big(\int_{t_0}^{t} z(s)ds\Big)\\
&+ C\Big(1-\sum_{i=1}^{m} g_i\Big)q(t)\exp\Big(\int_{t_0}^{t} z(s)ds\Big)\\
&+ \sum_{j=1}^{n} C_j\Big(1-\sum_{i=1}^{m} g_i\Big)q_i(t)\exp\Big(\int_{t_0}^{t-\rho_j} z(s)ds\Big) \le 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&z^2(t) + z'(t) + C\Big(1-\sum_{i=1}^{m} g_i\Big)q(t)\\
&+ \sum_{j=1}^{n} C_j\Big(1-\sum_{i=1}^{m} g_i\Big)q_j(t)\exp\Big(-\int_{t-\rho_j}^{t} z(s)ds\Big) \le 0, \quad t \ne t_k,
\end{aligned}$$

then we have

$$\begin{aligned}
&z'(t) + C\Big(1-\sum_{i=1}^{m} g_i\Big)q(t)\\
&+ \sum_{j=1}^{n} C_j\Big(1-\sum_{i=1}^{m} g_i\Big)q_j(t)\exp\Big(-\int_{t-\rho_j}^{t} z(s)ds\Big) \le 0, \quad t \ne t_k.
\end{aligned}$$

From above inequality and condition $\overline{b}_k \le \underline{a}_k$, we know that $z(t)$ is non-increasing, then $z(t) \le z(t_0)$, for $t \ge t_0$, we obtain

$$z'(t) + C\Big(1-\sum_{i=1}^{m} g_i\Big)q(t) + \sum_{j=1}^{n} C_j\Big(1-\sum_{i=1}^{m} g_i\Big)q_j(t)\exp(-\delta z(t_0)) \le 0, \quad t \ne t_k.$$

From (2.2), (2.3) and (2.8), we obtain

$$z(t_k^+) = \frac{w'(t_k^+)}{w(t_k^+)} \le \frac{\overline{b}_k w'(t_k)}{\underline{a}_k w(t_k)} = \frac{\overline{b}_k}{\underline{a}_k} z(t_k),$$

so we can easily obtain

$$\begin{gathered}
z'(t) \le -C\Big(1-\sum_{i=1}^{m} g_i\Big)q(t) - \sum_{j=1}^{n} C_j\Big(1-\sum_{i=1}^{m} g_i\Big)q_j(t)\exp(-\delta z(t_0)) \quad t \ne t_k,\\
z(t_k^+) \le \frac{\overline{b}_k}{\underline{a}_k} z(t_k).
\end{gathered}$$

Let

$$-F(t) = -C\Big(1-\sum_{i=1}^{m} g_i\Big)q(t) - \sum_{j=1}^{n} C_j\Big(1-\sum_{i=1}^{m} g_i\Big)q_j(t)\exp(-\delta z(t_0)).$$

Then according to Lemma 2.2, we have

$$\begin{aligned} z(t) &\le z(t_0)\prod_{t_0<t_k<t}\frac{\bar{b}_k}{\underline{a}_k}+\int_{t_0}^{t}\prod_{s<t_k<t}\frac{\bar{b}_k}{\underline{a}_k}\,(-F(s))\,ds \\ &= \prod_{t_0<t_k<t}\frac{\bar{b}_k}{\underline{a}_k}\Big[z(t_0)-\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{\underline{a}_k}{\bar{b}_k}F(s)ds\Big]<0. \end{aligned}$$

Since $z(t)\ge 0$, this is a contradiction. The proof is complete. □

3. Oscillation properties of the problem (1.1), (1.5)

For the second main theorem, we need following lemma.

Lemma 3.1. *Let $u\in C^2(\Gamma)\cap C^1(\bar{\Gamma})$ be a positive solution of (1.1), (1.5) in G, then function $w(t)$ satisfies the impulsive differential inequality*

$$w''(t)+C\Big(1-\sum_{i=1}^{m}g_i\Big)q(t)w(t)+\sum_{j=1}^{n}C_j\Big(1-\sum_{i=1}^{m}g_i\Big)q_j(t)w(t-\rho_j)\le 0,\quad t\ne t_k, \tag{3.1}$$

$$\underline{a}_k\le\frac{w(t_k^+)}{w(t_k)}\le\bar{a}_k,\quad k=1,2,\dots, \tag{3.2}$$

$$\underline{b}_k\le\frac{w'(t_k^+)}{w'(t_k)}\le\bar{b}_k,\quad k=1,2,\dots, \tag{3.3}$$

where $w(t)=v(t)+\sum_{i=1}^{m}g_iv(t-\tau_i)$.

Proof. Let $u(t,x)$ be a positive solution of the problem (1.1), (1.5) in G. Without loss of generality, we may assume that there exists a $T>0$, $t_0>T$ such that $u(t,x)>0$, $u(t-\tau_i,x)>0$, $i=1,2,\dots,m$, $u(t-\sigma_r,x)>0$, $r=1,2,\dots,l$, $u(t-\rho_j,x)>0$, $j=1,2,\dots n$, for any $(t,x)\in[t_0,\infty)\times\Omega$. For $t\ge t_0$, $t\ne t_k$, $k=1,2,\dots$, integrating (1.1) with respect to x over Ω yields

$$\begin{aligned} &\frac{d^2}{dt^2}\Big[\int_\Omega u(t,x)\,dx+\sum_{i=1}^{m}g_i\int_\Omega u(t-\tau_i,x)\,dx\Big] \\ &=a(t)\int_\Omega h(u)\Delta u\,dx-\int_\Omega q(t,x)f(u(t,x))\,dx \\ &\quad+\sum_{r=1}^{l}a_r(t)\int_\Omega h_r(u(t-\sigma_r,x))\Delta u(t-\sigma_r,x)\,dx \\ &\quad-\sum_{j=1}^{n}\int_\Omega q_j(t,x)f_j(u(t-\rho_j,x))\,dx. \end{aligned}$$

By Green's formula and the boundary condition, we have

$$\begin{aligned} \int_\Omega h(u)\Delta u\,dx &= \int_{\partial\Omega}h(u)\frac{\partial u}{\partial n}ds-\int_\Omega h'(u)|\mathrm{grad}u|^2\,dx \\ &=-\int_{\partial\Omega}h(u)\varphi(t,x)u\,ds-\int_\Omega h'(u)|\mathrm{grad}u|^2\,dx \\ &\le-\int_\Omega h'(u)|\mathrm{grad}u|^2\,dx\le 0, \end{aligned}$$

$$\int_\Omega h_r(u(t-\sigma_r,x))\Delta u(t-\sigma_r,x)\,dx \le 0.$$

The rest of the proof is similar to the one in Lemma 2.1. We omit it. □

The following theorem is the second main result of this article.

Theorem 3.2. *If conditions* (2.6) *and* (2.7) *hold, then each solution of* (1.1)–(1.3), (1.5) *oscillates in* G.

The proof of the above theorem is similar to that of Theorem 2.4. We omit it.

4. Examples

Example 4.1. Consider the equation

$$\begin{gathered}
\frac{\partial^2}{\partial t^2}\Big[u(t,x)+\frac12 u(t-\frac{\pi}{2},x)\Big] = u^2\Delta u - ue^{u^2} + e^t u^2(t-\frac{\pi}{2},x)\Delta u(t-\frac{\pi}{2},x) \\
- (x^2+1)e^t u(t-\frac{3\pi}{2},x)e^{u^2(t-\frac{3\pi}{2},x)}, \\
t>1,\quad t\ne 2^k,\quad (t,x)\in\mathbb{R}^+\times\Omega=G, \\
u((2^k)^+,x) = (4+\sin 2^k\cos x)u(2^k,x),\quad k=1,2,\dots, \\
u_t((2^k)^+,x) = (2+\sin 2^k\cos x)u_t(2^k,x),\quad k=1,2,\dots,
\end{gathered}$$

with the boundary condition

$$u=0,\quad (t,x)\in\mathbb{R}^+\times\partial\Omega,$$

where $a(t)=1$, $a_1(t)=e^t$, $\tau_1=\frac{\pi}{2}$, $\sigma_1=\frac{\pi}{2}$, $\rho_1=\frac{3\pi}{2}$, $h(u)=u^2$, $h_1(u)=u^2$, $f(u)=ue^{u^2}$, $f_1(u)=ue^{u^2}$, $q(t,x)=1$, $q_1(t,x)=(x^2+1)e^t$, $g_1=\frac12$, $t_k=2^k$. It is easy to verify that the condition (H1)–(H4) and the conditions of Theorem 2.4 are satisfied. Hence the all solutions of above problem oscillate.

Example 4.2. Consider the equation

$$\begin{gathered}
\frac{\partial^2}{\partial t^2}\Big[u(t,x)+\frac12 u(t-\frac{\pi}{2},x)\Big] \\
= u^2\Delta u - ue^{u^2} + e^t u^2(t-\frac{\pi}{2},x)\Delta u(t-\frac{\pi}{2},x) - (x^2+1)e^t u(t-\frac{3\pi}{2},x)e^{u^2(t-\frac{3\pi}{2},x)}, \\
t>1,\quad t\ne 3k,\quad (t,x)\in\mathbb{R}^+\times\Omega=G, \\
u((3k)^+,x) = (4+\sin 3k\cos x)u(3k,x),\quad k=1,2,\dots, \\
u_t((3k)^+,x) = (2+\sin 3k\cos x)u_t(3k,x),\quad k=1,2,\dots,
\end{gathered}$$

with the boundary condition

$$\frac{\partial u}{\partial n} + t^2x^2u = 0,\quad (t,x)\in\mathbb{R}^+\times\partial\Omega,$$

where $a(t)=1$, $a_1(t)=e^t$, $\tau_1=\frac{\pi}{2}$, $\sigma_1=\frac{\pi}{2}$, $\rho_1=\frac{3\pi}{2}$, $h(u)=u^2$, $h_1(u)=u^2$, $f(u)=ue^{u^2}$, $f_1(u)=ue^{u^2}$, $q(t,x)=1$, $q_1(t,x)=(x^2+1)e^t$, $g_1=\frac12$, $t_k=3k$, $\varphi(t,x)=t^2x^2$. It is easy to verify that the condition H) and condition of Theorem 3.2 are satisfied. Hence the all solutions of the above problem oscillate.

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