EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS

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Abstract. Using a version of the generalized mountain pass theorem, we obtain the existence of nontrivial solutions for a class of superquadratic elliptic systems.

1. Introduction and statement of results

Consider the elliptic system

\begin{align*}
-\Delta u &= H_v(u,v,x), \quad \text{in } \Omega, \\
-\Delta v &= H_u(u,v,x), \quad \text{in } \Omega, \\
u &= 0, \quad v &= 0, \quad \text{on } \partial \Omega, 
\end{align*}

(1.1)

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), with smooth boundary \( \partial \Omega \), and \( H_u \) denotes the partial derivative of \( H \) with respect to \( u \).

The system (1.1) has been already studied in the recent works [1, 2, 4, 5, 6, 7, 8, 10, 11] and the reference therein. Using the generalized mountain pass theorem in its infinite dimensional setting, Benci and Rabinowitz [1] studied a special case of the system

\begin{align*}
-\Delta w &= H_w(w,z,x), \\
\Delta z &= H_z(w,z,x),
\end{align*}

(1.2)

which is equivalent to system (1.1).

In Clément, De Figueiredo and Mitidieri [4] discussed the existence of a positive solution for the system below subjected to Dirichlet boundary conditions:

\begin{align*}
-\Delta u &= f(v), \quad -\Delta v = g(u), \quad \text{in } \Omega.
\end{align*}

(1.3)

In this case, the Hamiltonian is \( H(u,v) = F(v) + G(u) \), where \( F(t) = \int_0^t f(s)ds \), and similarly \( G \) is a primitive of \( g \). The approach in [4] for system (1.3) was via a Topological argument, using a theorem of Krasnoselski on Fixed Point Index for compact mappings in cones in Banach spaces.

Using a variational approach through a version of the generalized mountain pass theorem, De Figueiredo and Felmer [8] obtained the existence of nontrivial solutions for system (1.1), which extends the results in [1] and [4]. Felmer and Wang [10] proved the existence of infinitely many strong solutions for the elliptic system (1.1).

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De Figueiredo and Ding [7] studied the existence and multiplicity of solutions of the elliptic system (1.2). For more details on semilinear elliptic systems of the Hamiltonian types, we refer the reader to [6] and the references therein.

We say that \((u, v)\) is a strong solution of (1.1) if
\[ u \in W^{2,p/(p-1)}(\Omega) \cap W^{1,0}_0(\Omega), \quad v \in W^{2,q/(q-1)}(\Omega) \cap W^{1,0}_0(\Omega), \]
and \((u, v)\) satisfies
\[ -\Delta u = H_v(u, v, x) \quad \text{and} \quad -\Delta v = H_u(u, v, x) \quad \text{a.e. in } \Omega. \]

In this article, motivated by [8], we study the existence of strong solutions for the elliptic system (1.1). This kind of Hamiltonian was studied recently by Chen and Tang [3] in the context of Hamiltonian systems.

Here and in the sequel, we assume that
\[ p \geq \alpha > p - 1 > 0 \quad \text{and} \quad q \geq \beta > q - 1 > 0 \]
such that
\[
\begin{align*}
(i) & \quad \frac{1}{\alpha} + \frac{1}{\beta} < 1, \\
(ii) & \quad \{2 - \left(\frac{1}{p} + \frac{1}{q}\right)\} \max\left\{\frac{p}{\alpha}, \frac{q}{\beta}\right\} < 1 + \frac{2}{N}, \\
(iii) & \quad \frac{p-1}{p} \frac{q}{\beta} < 1, \quad \frac{q-1}{q} \frac{p}{\alpha} < 1.
\end{align*}
\]

We will always assume \(N \geq 3\). If \(N = 2\) or \(N = 1\), we need less restrictive assumptions. Furthermore, in the case \(N \geq 5\), we also impose
\[
(iv) \quad \left(1 - \frac{1}{p}\right) \max\left\{\frac{p}{\alpha}, \frac{q}{\beta}\right\} < \frac{N+4}{2N}, \quad \left(1 - \frac{1}{q}\right) \max\left\{\frac{p}{\alpha}, \frac{q}{\beta}\right\} < \frac{N+4}{2N}.
\]

Our main results are the following theorems.

**Theorem 1.1.** Suppose that \(H\) satisfies:
\[
\begin{align*}
(H0) & \quad H : \mathbb{R}^2 \times \overline{\Omega} \to \mathbb{R} \text{ is of class } C^1, \\
(H1) & \quad H(u, v, x) \geq 0 \text{ for all } (u, v, x) \in \mathbb{R}^2 \times \overline{\Omega}, \\
(H2) & \quad \text{There exists } c_0 > 0 \text{ such that} \\
& \quad \frac{1}{\alpha} H_u(u, v, x) \cdot u + \frac{1}{\beta} H_v(u, v, x) \cdot v \geq H(u, v, x) > 0 \quad \text{for all } (u, v) \in \mathbb{R}^2, \quad |(u, v)| \geq c_0 \text{ and } x \in \overline{\Omega}, \\
(H3) & \quad \lim_{|(u, v)| \to 0} \frac{H(u, v, x)}{|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha}} = 0 \quad \text{uniformly for } x \in \overline{\Omega}; \\
(H4) & \quad \text{There exists } c_1 > 0 \text{ such that} \\
& \quad |H_u(u, v, x)| \leq c_1(|u|^{p-1} + |v|^{(p-1)q/p + 1}), \\
& \quad |H_v(u, v, x)| \leq c_1(|v|^{q-1} + |u|^{(q-1)p/q + 1}) \quad \text{for all } (u, v) \in \mathbb{R}^2 \text{ and } x \in \overline{\Omega}.
\end{align*}
\]

Then problem (1.1) possesses at least one nontrivial strong solution.

**Remark 1.2.** For Hamiltonian systems, the corresponding superquadratic condition (H2) is due to Felmer [9]. The hypothesis (H3) was introduced in [3].

**Theorem 1.3.** Suppose that \(H\) satisfies (H1)–(H4) and \(H^0 : \mathbb{R}^2 \times \overline{\Omega} \to \mathbb{R} \text{ is of class } C^1\).
(H5) \( H_u(u,v,x) \geq 0, H_v(u,v,x) \geq 0 \) for all \((u,v) \in \mathbb{R}^2, u \geq 0, v \geq 0, x \in \Omega; \)

(H6) \( H_u(u,v,x) = 0 \) when \( u = 0 \), \( H_v(u,v,x) = 0 \) when \( v = 0 \).

Then (1.1) possesses at least one positive solution \((u,v)\) with \( u(x) > 0, v(x) > 0 \) if \( x \in \Omega \).

**Remark 1.4.** It is easy to show that our Theorems 1.1 and 1.3 generalize Theorems 0.1 and 0.3 in [8]. There are functions \( H \) satisfying our Theorems and not satisfying the corresponding results in [8]. In fact, for \( \alpha > 1, \beta > 1 \) satisfying \( 1/\alpha + 1/\beta < 1 \), let

\[
H(u,v,x) = a_1(|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha})^{\gamma_1} + a_2(|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha})^{\gamma_2},
\]

where \( a_1 > 0, a_2 > 0, 1 < \gamma_1 < \alpha\beta/(\alpha + \beta) < \gamma_2 \). Choose \( \gamma_2 = (\alpha\beta + 1)/(\alpha + \beta) \), \( p = \alpha + 1/\beta, q = \beta + 1/\alpha \), then \( H \) satisfies our Theorems and does not satisfy the corresponding results in [8].

**Remark 1.5.** If \( H(u,v) = |u|^p/p + |v|^q/q \) then one could use a fourth-order approach and then assumption (iv) would not be necessary (see [2, 5]). We do not know if (iv) can be avoided for general Hamiltonians.

2. **Proof of main results**

To set up our problem variationally, we shall have to utilize fractional Sobolev spaces. For more details and references we cite [8]. Consider the spaces \( E^s \), which are obtained as the domains of fractional powers of the operator

\[
-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega) \rightarrow L^2(\Omega),
\]

where \( \Delta \) denotes the Laplacian and \( H^2(\Omega), H^1_0(\Omega) \) are the usual Sobolev spaces. Namely \( E^s = D((-\Delta)^s/2) \) for \( 0 \leq s \leq 2 \), and the corresponding operator is denoted by

\[
A^s : E^s \rightarrow L^2(\Omega).
\]

The spaces \( E^s \) are Hilbert spaces with inner product

\[
(u,v)_{E^s} = \int_\Omega A^suA^sv\,dx.
\]

Its associated norm is denoted by \( \|u\|_{E^s} \). In \( E^s \), we find the Poincaré's inequality for the operator \( A^s \)

\[
\|A^su\|_{L^2(\Omega)} \geq \lambda_1^{s/2}\|u\|_{L^2(\Omega)} \quad \text{for all} \quad u \in E^s,
\]

where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \).

Next, we define the spaces on which we set up the problem. For numbers \( s > 0 \) and \( t > 0 \) with \( s + t = 2 \), we define the Hilbert space \( E = E^s \times E^t \) and the bilinear form \( B : E \times E \rightarrow \mathbb{R} \) by the formula

\[
B((u,v),(\phi,\psi)) = \int_\Omega (A^suA^t\phi + A^s\phi A^t v)\,dx.
\]

The bilinear form \( B \) is continuous and symmetric. There exists a selfadjoint bounded linear operator \( L : E \rightarrow E \) such that

\[
B(z,\eta) = (Lz,\eta)_E
\]
for all \( z, \eta \in E \). Here \((\cdot, \cdot)_E\) denotes the natural inner product in \( E \) induced by \( E^s \) and \( E^t \). We can also define the quadratic form \( Q : E \rightarrow \mathbb{R} \) associated to \( B \) and \( L \) as

\[
Q(z) = \frac{1}{2}(Lz, z)_E = \int_{\Omega} A^s u A^t v \, dx
\]

(2.1)

for all \( z = (u, v) \in E \). The operator \( L \) defined above can be written as [8, Proposition 1.1]

\[
L(u, v) = ((A^s)^{-1} A^t v, (A^t)^{-1} A^s u).
\]

(2.2)

We define the subspaces

\[
E^+ = \{(u, A^{-t} A^s u) | u \in E^s \}, \quad E^- = \{(u, -A^{-t} A^s u) | u \in E^s \},
\]

(2.3)

which give a natural splitting \( E = E^+ \oplus E^- \). The spaces \( E^+ \) and \( E^- \) are the positive and negative eigenspaces of \( L \), they are consequently orthogonal with respect to the bilinear form \( B \); that is,

\[
B(z^+, z^-) = 0, \quad \forall z^+ \in E^+, \forall z^- \in E^-.
\]

We also find that

\[
\frac{1}{2} \|z\|^2_E = Q(z^+) - Q(z^-),
\]

(2.4)

where \( z = z^+ + z^- \), \( z^\pm \in E^\pm \).

Now we will choose the numbers \( s \) and \( t \) defining the orders of the Sobolev spaces involved. From inequality (ii), we see the existence of \( s, t \in \mathbb{R}, s + t = 2 \) such that

\[
(1 - \frac{1}{p}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{1}{2} + \frac{s}{N}
\]

(2.5)

and

\[
(1 - \frac{1}{q}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{1}{2} + \frac{t}{N}.
\]

(2.6)

By (iii) and (iv), if \( N \geq 5 \), we can choose \( s > 0 \) and \( t > 0 \). Since \( p/\alpha \geq 1 \) and \( q/\beta \geq 1 \), we obtain from (2.5) and (2.6) that

\[
\frac{1}{p} > \frac{1}{2} - \frac{s}{N}, \quad \frac{1}{q} > \frac{1}{2} - \frac{t}{N}.
\]

(2.7)

These last inequalities and Sobolev Embedding Theorem give the compact inclusions (see [8, Theorem 1.1])

\[
E^s \hookrightarrow L^p(\Omega), \quad E^t \hookrightarrow L^q(\Omega).
\]

Now we can define a functional \( \Phi : E \rightarrow \mathbb{R} \) as

\[
\Phi(z) = Q(z) - H(z) = \int_{\Omega} A^s u A^t v \, dx - \int_{\Omega} H(u, v, x) \, dx
\]

(2.8)

for \( z = (u, v) \in E \). The functional \( \Phi \) is of class \( C^1 \). The functional

\[
H(u, v) = \int_{\Omega} H(u(x), v(x), x) \, dx
\]

is of class \( C^1 \) and its derivative is given by

\[
H'(u, v)(\phi, \psi) = \int_{\Omega} H_u(u, v, x) \phi + H_v(u, v, x) \psi \, dx
\]

for all \( (u, v), (\phi, \psi) \in E \). Moreover \( H' : E \rightarrow E \) is a compact operator (see [8]).
For details and proof of the aspects discussed so far, we refer the reader to [8]. In particular, see in [8] that critical points of $\Phi$ correspond to the strong solutions of (1.1).

For our proofs, we introduce the following abstract critical point theorem due to Felmer [9]. We consider a Hilbert space $E$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We assume that $E$ has a splitting $E = X \oplus Y$, where the subspaces $X$ and $Y$ are not necessarily orthogonal and both of them can be infinite dimensional. Let $\Phi : E \to \mathbb{R}$ be a functional having the structure

$$
\Phi(z) = \frac{1}{2} \langle Lz, z \rangle + \mathcal{H}(z).
$$

(I1) $L : E \to E$ is a linear, bounded, selfadjoint operator.

(I2) $\mathcal{H}'$ is compact.

(I3) There are two linear bounded, invertible operators $B_1, B_2 : E \to E$ satisfying: If $\omega \in \mathbb{R}_0^+$, the linear operator

$$
\hat{B}(\omega) = P_X B_1^{-1} \exp(\omega L) B_2 : X \to X
$$

is invertible.

Here $P_X$ denotes the projection of $E$ onto $X$ induced by the splitting $E = X \oplus Y$, and $\mathbb{R}_0^+$ is a set of nonnegative real numbers.

Let $\rho > 0$ and define

$$
S = \{ B_1 z : \| z \| = \rho, z \in Y \}. \tag{2.9}
$$

For $z_+ \in Y, z_+ \neq 0$, $\sigma > \rho/\| B_1^{-1} B_2 z_+ \|$ and $M > \rho$, we define

$$
Q = \{ B_2 (\tau z_+ + z) : 0 \leq \tau \leq \sigma, \| z \| \leq M, z \in X \}. \tag{2.10}
$$

We define $\partial Q$ as the boundary of $Q$ relative to the subspace

$$
\{ B_2 (\tau z_+ + z) | \tau \in \mathbb{R}, z \in X \}.
$$

Let us consider the class of functions

$$
\Gamma = \{ h \in C(E \times [0,1], E) : h \text{ satisfies the following three conditions} \}
$$

(1) $h(z,t) = \exp(\omega(z,t)L)z + K(z,t)$, where $\omega : E \times [0,1] \to \mathbb{R}_0^+$ is continuous and transforms bounded sets into bounded sets, and $K : E \times [0,1] \to E$ is compact.

(2) $h(z,t) = z$ for all $z \in \partial Q$ and all $t \in [0,1]$.

(3) $h(z,0) = z$ for all $z \in Q$.

**Theorem 2.1** ([9]). Let $\Phi : E \to \mathbb{R}$ be a $C^1$ functional satisfying the Palais-Smale condition and (I1)--(I3). Furthermore assume that there is a constant $\delta > 0$ such that

(IS) $\Phi(z) \geq \delta$ for all $z \in S$,

(IQ) $\Phi(z) \leq 0$ for all $z \in \partial Q$.

Then $\Phi$ possesses a critical point with critical value $d \geq \delta$ characterized by

$$
d = \inf_{h \in \Gamma} \sup_{z \in Q} \Phi(h(z,1)).
$$

Here, we define the operators $B_1, B_2$ and the splitting $E = E^s \times E^t = E^- \oplus E^+$. Let $X = E^-$ and $Y = E^+$. We define $B_1 : E \to E$ by

$$
B_1(u,v) = (\rho^{\beta-1}u, \rho^{\alpha-1}v) \tag{2.11}
$$
and $B_2 : E \to E$ by
\[ B_2(u, v) = (\sigma^{\beta-1}u, \sigma^{\alpha-1}v). \] (2.12)

Certainly $B_1$ and $B_2$ are bounded linear operators and both of them are invertible. From (2.9) and (2.11), we obtain
\[ S = \{(\rho^{\beta-1}u, \rho^{\alpha-1}v) : \|(u, v)\| = \rho, (u, v) \in E^+\}. \] (2.13)

By (2.10) and (2.12), we have
\[ Q = \{\tau(\sigma^{\beta-1}u_+, \sigma^{\alpha-1}v_+) + (\sigma^{\beta-1}u, \sigma^{\alpha-1}v) : 0 \leq \tau \leq \sigma, 0 \leq \|(u, v)\| \leq M, (u, v) \in E^-\}, \] (2.14)

where $z_+ = (u_+, v_+) \in E^+$ with $u_+$ some fixed eigenvector of $-\Delta$. In what follows, we note that $z_+$ is an eigenvector of $L$ associated to a positive eigenvalue (i.e. to 1). We assume $\|z_+\|_E = 1$. We denote by $\partial Q$ the boundary of $Q$ relative to the subspace
\[ \{\tau(\sigma^{\beta-1}u_+, \sigma^{\alpha-1}v_+) + (\sigma^{\beta-1}u, \sigma^{\alpha-1}v) : \tau \in \mathbb{R}, (u, v) \in E^-\}. \]

Now, we can give the proof of our Theorems.

**Proof of Theorem 1.1.** The proof is divided into several steps.

**Step 1:** $\Phi$ satisfies the Palais-Smale condition. See [8, Proposition 2.1].

**Step 2:** We claim that $\Phi$ satisfies (I1)–(I3). From (2.1) and (2.8), we have
\[ \Phi(z) = Q(z) - \int_{\Omega} H(u, v, x)dx \]
\[ = \frac{1}{2}(Lz, z)_E - \int_{\Omega} H(u, v, x)dx. \]

Taking $H(z) = \int_{\Omega} H(z, x)dx$, we obtain
\[ (\Phi'(z), \eta) = (Lz, \eta) - (H'(z), \eta), \]
where $z = (u, v)$ and $\eta = (\phi, \psi)$. So, $\Phi' = L - \mathcal{H}'$, where $L$ is a linear bounded self-adjoint operator. And, from the growth hypothesis (H4), $\mathcal{H}'$ is a compact operator. Thus, $\Phi$ satisfies (I1) and (I2). From (2.2), one has
\[ L(u, v) = ((A^*)^{-1}A^t v, (A^t)^{-1}A^t u). \]

It is well known that
\[ \exp(\omega L) = 1 + \omega L + \frac{1}{2!}\omega^2 L^2 + \frac{1}{3!}\omega^3 L^3 + \frac{1}{4!}\omega^4 L^4 + \ldots, \]
\[ \cosh(\omega L) = 1 + \frac{1}{2!}\omega^2 L^2 + \frac{1}{4!}\omega^4 L^4 + \ldots, \]
\[ \sinh(\omega L) = \omega L + \frac{1}{3!}\omega^3 L^3 + \frac{1}{5!}\omega^5 L^5 + \ldots. \]

Hence, for $\omega \in \mathbb{R}$, the operator $\exp(\omega L) : E \to E$ is given by
\[ \exp(\omega L)(u, v) = \cosh(\omega)(u, v) + \sinh(\omega)(A^{-s}A^t v, A^{-t}A^s u). \] (2.15)

We can give an explicit formula for $B(u, v)$. For $z \in E^-$, one has $z = (u, -A^tA^s u)$ with $u \in E^s$. From (2.11), (2.12) and (2.15), one sees
\[ B_1^{-1}\exp(\omega L)B_2 z = (\xi u, \eta A^{-t}A^s u). \]
If we assume

\[ \xi = \frac{\cosh(\omega)\sigma^{\beta-1} - \sinh(\omega)\sigma^{\alpha-1}}{\rho^{\beta-1}}, \quad \eta = \frac{-\cosh(\omega)\sigma^{\alpha-1} + \sinh(\omega)\sigma^{\beta-1}}{\rho^{\alpha-1}}. \]

Since the orthogonal projections \( P^\pm : E \to E^\pm \) are given by the formula (see [3])

\[ P^\pm(u, v) = \frac{1}{2}(u \pm A^{-s}A^tv, v \pm A^{-t}A^su). \]

Using the formula for the projection into \( E^{-} \), we obtain

\[ \tilde{B}(\omega)z = P^-(\xi u, \eta A^{-t}A^su) \]
\[ = \frac{1}{2}(\xi u - \eta, \eta A^{-t}A^su - \xi A^{-t}A^su) \]
\[ = \frac{1}{2}(\xi - \eta, u, -(\xi - \eta)A^{-t}A^su) \]
\[ = \frac{\theta}{2}(u, -A^{-t}A^su), \]

where

\[ \theta = \left\{ \frac{\sigma^{\beta-1}}{\rho^{\beta-1}} + \frac{\sigma^{\alpha-1}}{\rho^{\alpha-1}} \cosh(\omega) - \frac{\sigma^{\alpha-1}}{\rho^{\beta-1}} + \frac{\sigma^{\beta-1}}{\rho^{\alpha-1}} \sinh(\omega) \right\}. \]

If we assume \( \sigma > 1 \) and \( \rho < 1 \), it is easy to see that \( \theta \) is positive. In fact

\[ \left( \frac{\sigma^{\beta-1}}{\rho^{\beta-1}} + \frac{\sigma^{\alpha-1}}{\rho^{\alpha-1}} \right) - \left( \frac{\sigma^{\alpha-1}}{\rho^{\beta-1}} + \frac{\sigma^{\beta-1}}{\rho^{\alpha-1}} \right) = \frac{(\rho^{\beta-1} - \rho^{\alpha-1})(\sigma^{\alpha-1} - \sigma^{\beta-1})}{\rho^{\alpha+\beta-2}} \]

is positive so that \( \theta > 0 \) independently of the value of \( \omega \in \mathbb{R} \). It implies that \( \tilde{B}(\omega) \) is invertible.

**Step 3:** We claim that (IS) is satisfied, that is, there exist \( \rho > 0 \) and \( \delta > 0 \) such that \( \Phi(z) \geq \delta, \forall z \in S \), where \( S \) is defined by \( \text{(2.13)} \).

From hypothesis (H3) and (H4), for each \( \epsilon > 0 \), we have

\[ H(u, v, x) \leq \varepsilon(|u|^{1+\alpha/\beta} + |v|^{1+\beta/\alpha}) + c_2(|u|^p + |v|^q), \quad (2.16) \]

where \( c_2 = c_2(\epsilon) > 0 \). Let \( \tilde{z} = (u, v) \in E^+ \) and take \( z = (\rho^{\beta-1}u, \rho^{\alpha-1}v) \) for some \( \rho > 0 \). Then, by \( (2.16) \), one has

\[ \int_\Omega H(u, v, x)dx \]
\[ \leq \varepsilon \left( \rho^{(\beta-1)(1+\alpha/\beta)} \int_\Omega |u|^{1+\alpha/\beta}dx + \rho^{(\alpha-1)(1+\beta/\alpha)} \int_\Omega |v|^{1+\beta/\alpha}dx \right) \]
\[ + c_2 \left( \rho^{(\beta-1)p} \int_\Omega |u|^pdx + \rho^{(\alpha-1)q} \int_\Omega |v|^qdx \right). \]

Since \( \alpha \leq p, \beta \leq q \), by (i) and \( (2.7) \), one sees that

\[ \frac{1}{1+\alpha/\beta} = \frac{\beta}{\alpha + \beta} > \frac{1}{\rho} > \frac{1}{2} = \frac{s}{N} \]

and

\[ \frac{1}{1+\beta/\alpha} = \frac{\alpha}{\alpha + \beta} > \frac{1}{q} > \frac{1}{2} = \frac{t}{N}. \]

Hence, Sobolev Embedding Theorem gives the compact inclusions (see [S] Theorem 1.1)

\[ E^s \hookrightarrow L^{1+\alpha/\beta}(\Omega), \quad E^t \hookrightarrow L^{1+\beta/\alpha}(\Omega). \]
By (2.17), there exist two positive constants \(c_3\) and \(c_4\) such that
\[
\int_{\Omega} H(u, v, x) \, dx \leq \varepsilon c_3 (\rho^{(\beta-1)(1+\alpha/\beta)} \| \tilde{z} \|_{E}^{1+\alpha/\beta} + \rho^{(\alpha-1)(1+\beta/\alpha)} \| \tilde{z} \|_{E}^{1+\beta/\alpha}) + c_4 (\rho^{(\beta-1)p} \| \tilde{z} \|_{E}^{p} + \rho^{(\alpha-1)q} \| \tilde{z} \|_{E}^{q}).
\]
(2.18)

As \((u, v) \in E^+\), then \(v = A^{-t} A^* u\) and \(u = A^{-s} A^t v\). We obtain
\[
Q(z) = \int_{\Omega} \rho^{|\alpha-1| A^* u \rho^{\alpha-1} A^t v} \, dx = \rho^{\alpha-2} \int_{\Omega} A^* u A^t v \, dx.
\]
(2.19)

It follows from (2.4) and (2.19) that
\[
Q(z) = \frac{1}{2} \rho^{\alpha+\beta-2} \| \tilde{z} \|^2_{E}.
\]
(2.20)

If we consider \(\rho = \| \tilde{z} \|_{E}\), from (2.18) and (2.20), we obtain
\[
\Phi(z) \geq \frac{1}{2} \rho^{\alpha+\beta - \varepsilon c_3 (\rho^{\beta+\alpha} + \rho^{\alpha+\beta}) - c_4 (\rho^{\beta p} + \rho^{\alpha q})}
\]
\[
= \frac{1}{2} - 2 \varepsilon c_3) \rho^{\beta+\alpha} - c_4 (\rho^{\beta p} + \rho^{\alpha q}).
\]
(2.21)

Since \(\frac{1}{\alpha} + \frac{1}{\beta} < 1, \alpha \leq p\) and \(\beta \leq q\), one has \(\beta + \alpha < \alpha q\) and \(\beta + \alpha < \beta p\). Taking \(\varepsilon = 1/(8c_3)\), if \(\rho\) is small enough, by (2.21), there exists \(\delta > 0\) such that
\[
\Phi(z) \geq \delta > 0, \quad \text{if} \quad \| \tilde{z} \|_{E} = \rho
\]
and this inequality holds for \(z \in S\), according to the definition of \(S\).

Step 4. We claim that (IQ) is satisfied, that is, there are constants \(\sigma > 0\) and \(M > 0\) such that \(\Phi(z) \leq 0\) for all \(z \in \partial Q\), where \(Q\) is defined by (2.14). For \(\tau \in \mathbb{R}^+\), \((u, v) \in E^-\), we take
\[
z = \tau (\sigma^{-1} \tau^{\alpha-1} u, \sigma^{\alpha-1} v) + (\sigma^{-1} \tau^{\alpha-1} u, \sigma^{\alpha-1} v).
\]
(2.22)

From (2.3), by the definitions of \(E^+\) and \(E^-\), one has
\[
v_{+} = A^{-t} A^* u, \quad v = -A^{-t} A^* u.
\]
(2.23)

Then, from (2.22) and (2.23) we obtain
\[
Q(z) = \int_{\Omega} (\tau\sigma^{\beta-1} A^* u_{+} + \sigma^{\beta-1} A^* u)(\tau\sigma^{\alpha-1} A^* u_{+} - \sigma^{\alpha-1} A^* u) \, dx
\]
\[
= \sigma^{\alpha+\beta-2} \int_{\Omega} (\tau A^* u_{+} + A^* u)(\tau A^* u_{+} - A^* u) \, dx
\]
\[
= \frac{1}{2} \sigma^{\alpha+\beta-2} (\tau^2 - \| (u, v) \|_{E}^2).
\]
(2.24)

By hypothesis (H1), we see that for \(\tau = 0\),
\[
\Phi(z) \leq 0.
\]
(2.25)

It follows from (H2) that there are constants \(c_5 > 0\) and \(c_6 > 0\) such that
\[
H(u, v, x) \geq c_5 (|u|^\alpha + |v|^\beta) - c_6.
\]

So, we have
\[
\int_{\Omega} H(z, x) \, dx \geq c_5 \int_{\Omega} (\sigma^{\alpha(\beta-1)} |\tau u_{+} + u|^\alpha + \sigma^{\beta(\alpha-1)} |\tau v_{+} + v|^\beta) \, dx - c_6 |\Omega|.
\]
(2.26)
Now, every $u$ can be decomposed as $u = \gamma u_+ + \hat{u}$, where $\hat{u}$ is orthogonal to $u^+$ in $L^2(\Omega)$, and $\gamma \in \mathbb{R}$. We obtain from Hölder’s inequality that

$$(\tau + \gamma) \int_{\Omega} |u^+|^2 dx = \int_{\Omega} (\tau u^+ + u) u^+ dx \leq \|\tau u^+ + u\|_{L^\alpha(\Omega)} \|u^+\|_{L^{\alpha'}(\Omega)}.$$  

Hence, for some constant $c_7 > 0$, we get

$$\tau + \gamma \leq c_7 \|\tau u^+ + u\|_{L^\alpha(\Omega)}.$$  

(2.27)

Similarly, we obtain

$$\tau - \gamma \leq c_7 \|\tau v^+ + v\|_{L^\beta(\Omega)}.$$  

(2.28)

If $\gamma \geq 0$, we get from (2.24), (2.26) and (2.27) that

$$\Phi(z) \leq \frac{1}{2} \sigma^{\alpha + \beta - 2} \tau^2 - c_8 \tau^\alpha \sigma^{\alpha (\beta - 1)} + c_6 |\Omega|.$$  

(2.29)

Choosing $\tau = \sigma$, and taking $\sigma$ large enough it follows from $1/\alpha + 1/\beta < 1$, (2.29), and (2.30) that

$$\Phi(z) \leq 0.$$  

(2.31)

Finally, we choose $M$. Given $\tau \in (0, \sigma)$, we deduce from (2.24) and (2.26) that

$$\Phi(z) \leq \frac{1}{2} \sigma^{\alpha + \beta - 2} \tau^2 - c_8 \tau^\beta \sigma^{\beta (\alpha - 1)} + c_6 |\Omega|.$$  

(2.30)

Choosing $\tau = \sigma$, and taking $\sigma$ large enough it follows from $1/\alpha + 1/\beta < 1$, (2.29), and (2.30) that

$$\Phi(z) \leq 0.$$  

(2.31)

Thus, from (2.25), (2.31) and (2.32), we have

$$\Phi(z) \leq 0, \quad \forall z \in \partial Q.$$  

Hence, the hypothesis of Theorem 2.1 is satisfied. Thus, there exists $z \in E$ such that $\Phi'(z) = 0$; i.e., $z$ is an $(s,t)$-weak solution of (1.1). Next, [8, Theorem 1.2] gives that $z = (u,v)$ is such that $u \in W^{2,p/(p-1)}(\Omega) \cap W^{1,p/(p-1)}_0(\Omega)$ and $v \in W^{2,q/(q-1)}(\Omega) \cap W^{1,q/(q-1)}_0(\Omega)$. That is, $(u,v)$ is a strong solution of (1.1).

Moreover, $(0,0)$ is a solution of (1.1). Since $\Phi(z) \geq \delta > 0$ and $\Phi(0,0) = 0$, it implies that $(u,v)$ is not trivial. □

Proof of Theorem 1.3: Here, we define the functional $\hat{\Phi} : E \to \mathbb{R}$ as

$$\hat{\Phi}(z) = \mathcal{Q}(z) - \int_{\Omega} \hat{H}(z,x) dx,$$

where

$$\hat{H}(u,v,x) = \begin{cases} 
H(u,v,x), & \text{if } u \geq 0, v \geq 0, \\
H(0,v,x), & \text{if } u \leq 0, v \geq 0, \\
H(u,0,x), & \text{if } u \geq 0, v \leq 0, \\
0, & \text{if } u \leq 0, v \leq 0.
\end{cases}$$

Hence, the hypothesis of Theorem 2.1 is satisfied. Thus, there exists $z \in E$ such that $\hat{\Phi}'(z) = 0$; i.e., $z$ is an $(s,t)$-weak solution of (1.1). Next, [8, Theorem 1.2] gives that $z = (u,v)$ is such that $u \in W^{2,p/(p-1)}(\Omega) \cap W^{1,p/(p-1)}_0(\Omega)$ and $v \in W^{2,q/(q-1)}(\Omega) \cap W^{1,q/(q-1)}_0(\Omega)$. That is, $(u,v)$ is a strong solution of (1.1).

Moreover, $(0,0)$ is a solution of (1.1). Since $\hat{\Phi}(z) \geq \delta > 0$ and $\hat{\Phi}(0,0) = 0$, it implies that $(u,v)$ is not trivial. □
From (H6), $\hat{H}$ is of class $C^{1,\sigma}$. And, $\hat{H}$ satisfies (H1), (H3) and (H4). Moreover, (H2) is satisfied in a restricted form. Obviously, the critical points of $\hat{\Phi}$ correspond to the strong solutions of

$$
-\Delta u = \hat{H}_v(u, v, x), \text{ in } \Omega,
$$

$$
-\Delta v = \hat{H}_u(u, v, x), \text{ in } \Omega,
$$

$$
u = 0, \quad v = 0, \quad \text{on } \partial \Omega.
$$

Since $\hat{H}_u(u, v, x) \geq 0$ and $\hat{H}_v(u, v, x) \geq 0$, by the maximum principle, we obtain that $u > 0$ and $v > 0$ in $\Omega$. As the proof of Theorem 1.1 we can get that hypotheses of Theorem 2.1 still hold. Hence, (1.1) possesses at least one positive solution $(u, v)$ with $u(x) > 0, v(x) > 0$ if $x \in \Omega$. 

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