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THE BARRIER STRIP TECHNIQUE FOR A BOUNDARY VALUE PROBLEM WITH P-LAPLACIAN

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Dedicated to Professor Jean Mawhin on his 70th birthday

Abstract. We study the solvability of the boundary value problem

$$(\phi_p(x'))' = f(t,x,x'), \quad x(0) = A,\ x'(1) = B,$$

where $\phi_p(s) = s|s|^{p-2}$, using the barrier strip type arguments. We establish the existence of $C^2[0,1]$-solutions, restricting our considerations to $p \in (1,2]$. The existence of positive monotone solutions is also considered.

1. Introduction

In this article, we study the existence of C^2-solutions to the boundary-value problem (BVP)

$$\begin{gathered}(\phi_p(x'))' = f(t,x,x'), \quad t \in (0,1), \\ x(0) = A, \quad x'(1) = B, \quad B > 0\end{gathered} \tag{1.1}$$

where $\phi_p(s) = s|s|^{p-2}$, $p \in (1,2]$, and the scalar function $f(t,x,y)$ is defined for $(t,x,y) \in [0,1] \times D_x \times D_y$, $D_x, D_y \subseteq R$, and continuous on a suitable subset of its domain.

Various boundary-value problems for (1.1) have been studied in the general case $p > 1$, and the obtained results guarantee C^1-solutions.

Guo and Tian [4] discussed the existence of positive solutions of the differential equation $\phi_p(x'))' + q(t)f(t,x) = 0$, $t \in (0,1)$, satisfying either $x(0) = x'(1) = 0$ or $x(0) = x(1) = 0$, where $p > 1$, $f : [0,1] \times [0,\infty) \to [-M,\infty)$ and $q : (0,1) \to [0,\infty)$ are continuous.

The solvability of BVPs for the equation

$$-(\phi(x'))' = q(x'(t))f(t,x(t),x'(t)),$$

with nonlinear functional boundary conditions, and for the equation

$$(\phi(x'))' = f(t,x(t),x(\tau(t)),x'(t)),$$

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with homogeneous Neumann boundary conditions, has been studied in Cabada and Pouso [2] and Liu [6], respectively. In these works $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism, and f is a Carathéodory function; in [2], the right side is discontinuous in the x' argument.

Much attention has been paid to singular problems with p-Laplacian. Lü and Zhong [7] consider the BVP

$$
\begin{gathered}
(\phi(x'))' + f(t, x(t)) = 0, \quad t \in (0, 1), \\
x(0) = x(1) = 0
\end{gathered}
\tag{1.2}
$$

where $\phi_p(s) = s|s|^{p-2}$, $p > 1$, and $f : C((0, 1) \times [0, \infty), [0, \infty))$ may be singular at the ends of the interval. Similar problems, singular not only at $t = 0$ and $t = 1$ but also at $x = 0$, are studied in Agarwal et al [1] and Jiang et al [5]; the main nonlinearity in [5] depends on x'.

Staněk [8] showed that the equation

$$
(\phi(x'))' + \mu f(t, x, x') = 0, \quad t \in (0, T), \tag{1.3}
$$

where the parameter μ is positive, has a solution satisfying boundary conditions of the form (1.2). Here $\phi \in C(\mathbb{R})$ is an odd and increasing function, and $f \in C([0, T] \times (0, \infty) \times (\mathbb{R} \setminus \{0\}))$ is singular at $x = 0$. Staněk [9] study the solvability of a BVP for (1.3) (in the case $\mu < 0$) with the boundary conditions

$$
x(0) - \alpha x'(0) = A, \quad x(T) + \beta x'(0) + \gamma x'(T) = A, \quad \alpha, A > 0,\ \beta, \gamma \ge 0 .
$$

Now $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism, and $f(t, x, y)$ satisfies the Carathéodory conditions on $[0, T] \times D$, $D = (0, (1 + \beta/\alpha)A] \times (\mathbb{R} \setminus \{0\})$, is singular at $x = 0$ and may be singular at $y = 0$.

Note that in the most of the cited papers, the obtained results guarantee positive solutions. As a rule, they are established under the assumption that the considered problems admit lower and upper solutions or that growth type conditions are satisfied.

To prove our existence result, we use the Topological transversality theorem [3]. For its application, the needed a priori bounds follow from the assumption:

(R1) There are constants $L_i, F_i, i = 1, 2$, and a sufficiently small $\sigma > 0$ such that

$$
\begin{gathered}
F_1 > 0, \quad L_2 - \sigma \ge L_1 \ge B \ge F_1 \ge F_2 + \sigma, \\
[A - \sigma, \quad L + \sigma] \subseteq D_x, \quad [F_2, L_2] \subseteq D_y,
\end{gathered}
$$

where $L = L_1 + A$,

$$
\begin{gathered}
f(t, x, y) \in C\big([0, 1] \times [A - \sigma, L + \sigma] \times [F_1 - \sigma, L_1 + \sigma]\big), \\
f(t, x, y) \ge 0 \quad \text{for } (t, x, y) \in [0, 1] \times D_x \times [L_1, L_2], nonumber \qquad (1.4) \\
f(t, x, y) \le 0 \quad \text{for } (t, x, y) \in [0, 1] \times D_A \times [F_2, F_1] \qquad (1.5)
\end{gathered}
$$

where $D_A = (-\infty, L] \cap D_x$.

Let us recall that the strips $[0, 1] \times [L_1, L_2]$ and $[0, 1] \times [F_2, F_1]$ are called barrier strips since they keep the values of x' between themselves.

2. Fixed point theorem

The proofs of the following theorems can be found in Granas et al [3]. To state them, we need standard topological notions.

Let Y be a convex subset of a Banach space E and $U \subset Y$ be open in Y. Let $L_{\partial U}(\overline{U}, Y)$ be the set of compact maps from $\overline{U}$ to Y which are fixed point free on ∂U; here, as usual, $\overline{U}$ and ∂U are the closure of U and boundary of U in Y.

A map F in $L_{\partial U}(\overline{U}, Y)$ is essential if every map G in $L_{\partial U}(\overline{U}, Y)$ such that $G/\partial U = F/\partial U$ has a fixed point in U. It is clear, in particular, every essential map has a fixed point in U.

Theorem 2.1 (Topological transversality theorem)**.** *Let Y be a convex subset of a Banach space E and $U \subset Y$ be open. Suppose:*

- (i) *$F, G : \overline{U} \to Y$ are compact maps.*
- (ii) *$G \in L_{\partial U}(\overline{U}, Y)$ is essential.*
- (iii) *$H(x, \lambda), \lambda \in [0, 1]$, is a compact homotopy joining F and G; i.e., $H(x, 1) = F(x)$ and $H(x, 0) = G(x)$.*
- (iv) *$H(x, \lambda), \lambda \in [0, 1]$, is fixed point free on ∂U.*

Then $H(x, \lambda)$, $\lambda \in [0, 1]$, has at least one fixed point in U and in particular there is a $x_0 \in U$ such that $x_0 = F(x_0)$.

Theorem 2.2. *Let $l \in U$ be fixed and $F \in L_{\partial U}(\overline{U}, Y)$ be the constant map $F(x) = l$ for $x \in \overline{U}$. Then F is essential.*

3. Auxiliary results

For $\lambda \in [0, 1]$ consider the family of BVPs

$$\begin{gathered} (\phi_p(x'))' = \lambda f(t, x, x'), \quad t \in (0, 1), \\ x(0) = A, \quad x'(1) = B, \end{gathered} \tag{3.1}$$

where $f : [0, 1] \times D_x \times D_y \to \mathbb{R}$, $D_x, D_y \subseteq \mathbb{R}$. Since

$$\phi_p(s) = s|s|^{p-2} = \begin{cases} s^{p-1}, & s \geq 0 \\ -(-s)^{p-1}, & s < 0, \end{cases}$$

we obtain

$$\phi_p'(s) = (p-1)|s|^{p-2} = \begin{cases} (p-1)s^{p-2}, & s \geq 0 \\ (p-1)(-s)^{p-2}, & s < 0 \end{cases}$$

and $(\phi_p(x'(t)))' = (p-1)|x'(t)|^{p-2}x''(t)$, if $x''(t)$ exists. So, we can write (3.1) in the form

$$\begin{gathered} (p-1)|x'(t)|^{p-2}x''(t) = \lambda f(t, x, x'),\ t \in (0, 1), \\ x(0) = A, \quad x'(1) = B, \end{gathered} \tag{3.2}$$

Our first auxiliary result gives a priori bounds for the $C^2[0, 1]$-solutions of the family (3.1) (as well as of (3.2)).

Lemma 3.1. *Let* (R1) *hold and $x \in C^2[0, 1]$ be a solution to family* (3.1) *for each fixed $p \in (1, 2]$. Then*

$$A \leq x(t) \leq L,\ F_1 \leq x'(t) \leq L_1, \quad m_p \leq x''(t) \leq M_p \quad \textit{for } t \in [0, 1],$$

where $m_p = m(p-1)^{-1}L_1^{2-p}$, $M_p = M(p-1)^{-1}L_1^{2-p}$, $m = \min\{f(t, x, y) : (t, x, y) \in [0, 1] \times [A, L] \times [F_1, L_1]\}$ and $M = \max\{f(t, x, y) : (t, x, y) \in [0, 1] \times [A, L] \times [F_1, L_1]\}$.

Proof. Let us assume on the contrary that

$$x'(t) \le L_1 \quad \text{for } t \in [0,1] \tag{3.3}$$

is not true. Then $x'(1) = B \le L_1$ and $x' \in C[0,1]$ imply that the set

$$S_+ = \{t \in [0,1] : L_1 < x'(t) \le L_2\}$$

is not empty. Then, there exists an interval $[\alpha, \beta] \subset S_+$ with the property

$$x'(\alpha) > x'(\beta). \tag{3.4}$$

This inequality and the continuity of $x'(t)$ guarantee the existence of a $\gamma \in [\alpha, \beta]$ such that

$$x''(\gamma) < 0.$$

On the other hand, as $x(t)$ is a $C^2[0,1]$-solution of (3.1), we have

$$(\gamma, x(\gamma), x'(\gamma)) \in [0,1] \times D_x \times D_y.$$

More precisely, $(\gamma, x(\gamma), x'(\gamma)) \in S_+ \times D_x \times (L_1, L_2]$, which allows to use (R1) to obtain

$$0 > (p-1)|x'(\gamma)|^{p-2} x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) \ge 0,$$

a contradiction. Thus (3.3) is true.

Now, by the mean value theorem, for each $t \in (0,1]$ there exists $\xi \in (0,t)$ such that $x(t) - x(0) = x'(\xi)t$, which yields

$$x(t) \le L \quad \text{for } t \in [0,1].$$

Next, suppose that the set

$$S_- = \{t \in [0,1] : F_2 \le x'(t) < F_1\}$$

is not empty. Following the reasoning giving (3.3) and using (1.5), we reach again a contradiction from which we conclude that

$$0 < F_1 \le x'(t) \quad \text{for } t \in [0,1],$$
$$A \le x(t) \quad \text{for } t \in [0,1].$$

To establish the bounds for $x''(t)$, we observe that from the assumptions

$$f(t,x,y) \ge 0 \quad \text{for } (t,x,y) \in [0,1] \times D_x \times [L_1, L_2]$$

and $[A - \sigma, L + \sigma] \subseteq D_x$ we have, in particular,

$$f(t,x,L_1) \ge 0 \quad \text{for } t \in [0,1] \times [A, L],$$

which implies $M \ge 0$. Similarly, (1.5) implies

$$f(t,x,F_1) \le 0 \quad \text{for } t \in [0,1] \times [A, L],$$

from where it follows $m \le 0$.

Further, using $0 < x'(t) \le L_1,\ t \in [0,1]$, and $-(p-2) \ge 0$, we get

$$0 < (x'(t))^{-(p-2)} \le L_1^{-(p-2)}, \quad t \in [0,1].$$

Then

$$0 < \frac{1}{(p-1)(x'(t))^{p-2}} \le \frac{1}{(p-1)L_1^{p-2}}.$$

Now, multiplying both sides of this inequality by $\lambda M \ge 0$ and $\lambda m \le 0$, we obtain respectively

$$\frac{\lambda M}{(p-1)(x'(t))^{p-2}} \le \frac{\lambda M}{(p-1)L_1^{p-2}} \le \frac{M}{(p-1)L_1^{p-2}} = M_p,\ t \in [0,1],$$

and

$$\frac{\lambda m}{(p-1)(x'(t))^{p-2}} \geq \frac{\lambda m}{(p-1)L_1^{p-2}} \geq \frac{m}{(p-1)L_1^{p-2}} = m_p, \quad t \in [0,1].$$

On the other hand, by the obtained a priori bounds for $x(t)$ and $x'(t)$, for each $t \in [0,1]$ we have $x(t) \in [A, L]$ and $x'(t) \in [F_1, L_1]$. Consequently,

$$m \leq f(t, x(t), x'(t)) \leq M \quad \text{for } t \in [0,1]$$

and multiplying by $\lambda(p-1)^{-1}|x'(t)|^{2-p} = \lambda(p-1)^{-1}(x'(t))^{2-p} \geq 0$, $\lambda, t \in [0,1]$, we reach

$$\frac{\lambda m}{(p-1)|x'(t)|^{p-2}} \leq \frac{\lambda f(t, x(t), x'(t))}{(p-1)|x'(t)|^{p-2}} \leq \frac{\lambda M}{(p-1)|x'(t)|^{p-2}}$$

for $\lambda, t \in [0,1]$ which combined with the obtained above yields the bounds for $x''(t)$. $\square$

Now, we introduce the set $C^2_{BC}[0,1] = \{x \in C^2[0,1] : x(0) = A, x'(1) = B\}$ and the operator $V : C^2_{BC}[0,1] \to C[0,1]$, defined by $Vx = (\phi_p(x'))' = (p-1)|x'(t)|^{p-2}x''(t)$, and the operator $W : C[0,1] \to C^2_{BC}[0,1]$, defined by

$$(Wy)(t) = A + \int_0^t \phi_q\Big(\int_1^s y(v)dv + \phi_p(B)\Big)ds,$$

where $\phi_q(s) = s|s|^{q-2}$, with $p^{-1} + q^{-1} = 1, p \in (1,2]$, is the inverse of the function $\phi_p(s)$.

The following lemmas give some useful properties of W.

Lemma 3.2. *The operator W is well defined for each $p \in (1,2]$.*

Proof. It is clear that for each $y \in C[0,1]$ the functions

$$h(t) = \int_1^t y(v)dv + \phi_p(B)$$

and $h'(t) = y(t)$ are continuous for $t \in [0,1]$. Then

$$(\phi_q(h(t)))' = (q-1)|h(t)|^{q-2}h'(t)$$

is also continuous for $t \in [0,1]$ since $q - 2 = \frac{2-p}{p-1} \geq 0$ for $p \in (1,2]$. Thus, $(Wy)''(t) = (\phi_q(h(t)))'$ is in $C[0,1]$. Finally, it is easy to check that

$$(Wy)(0) = A, \quad (Wy)'(1) = B,$$

which means $(Wy)(t) \in C^2_{BC}[0,1]$. $\square$

Lemma 3.3. *The operator W is continuous for each $p \in (1,2]$.*

Proof. Let $y_n, y \in C[0,1]$, $n \in N$, be such that $\|y_n - y\| \to 0$ as $n \to \infty$. According to Lemma 3.2, $Wy_n, Wy \in C^2_{BC}[0,1]$. We have to show that

$$\|(Wy_n)(t) - (Wy)(t)\|_{C^2_{BC}[0,1]} = \|(Wy_n)(t) - (Wy)(t)\|_{C^2[0,1]} \to 0;$$

i.e.,

$$\begin{aligned} &\lim_{n\to\infty} \|(Wy_n)^{(i)}(t) - (Wy)^{(i)}(t)\|_{C[0,1]} \\ &= \lim_{n\to\infty} \max_{t\in[0,1]} |(Wy_n)^{(i)}(t) - (Wy)^{(i)}(t)| = 0, \quad i = 0,1,2. \end{aligned} \tag{3.5}$$

In other words, we have to show that the sequences $\{(Wy_n)^{(i)}\}$ converge uniformly on $[0,1]$ to $(Wy)^{(i)}$, $i=0,1,2$, respectively. To this end, we see firstly that

$$\begin{aligned}
&\lim_{n\to\infty}\max_{t\in[0,1]}|(Wy_n)''(t)-(Wy)''(t)|\\
&=\lim_{n\to\infty}\max_{t\in[0,1]}\Big|\frac{d}{dt}\phi_q\Big(\int_1^t y_n(v)dv+\phi_p(B)\Big)-\frac{d}{dt}\phi_q\Big(\int_1^t y(v)dv+\phi_p(B)\Big)\Big|\\
&=(q-1)\lim_{n\to\infty}\max_{t\in[0,1]}\Big|\Big|\int_1^t y_n(v)dv+\phi_p(B)\Big|^{q-2}y_n(t)-\Big|\int_1^t y(v)dv+\phi_p(B)\Big|^{q-2}y(t)\Big|\\
&=0
\end{aligned}$$

and so $\{(Wy_n)''\}, n\in N$, converges uniformly on $[0,1]$ to $(Wy)''$. By the continuity and the uniform convergence of the functions $\{(Wy_n)''\}, n\in N$, it follows that the sequence $\{\int_0^t(Wy_n)''(v)dv\}, n\in N$, converges uniformly to $\int_0^t(Wy)''(v)dv$ on $[0,1]$. Then, the sequence $\{(Wy_n)'\}, n\in N$, converges uniformly to $(Wy)'$ on $[0,1]$ which means that $\{Wy_n\}, n\in N$, converges uniformly to Wy on $[0,1]$ and so (3.5) holds. □

Lemma 3.4. *The operator W is the inverse operator of V.*

Proof. It is clear, each function $x\in C^2_{BC}[0,1]$ has a unique image $Vx\in C[0,1]$. Also, each function $y\in C[0,1]$ has a unique inverse image $x\in C^2_{BC}[0,1]$ of the form

$$x(t)=A+\int_0^t\phi_q\Big(\int_1^s y(v)dv+\phi_p(B)\Big)ds,$$

which is the solution of the BVP

$$\begin{gathered}
(\phi_p(x'))'=y(t),\quad t\in(0,1),\\
x(0)=A,\quad x'(1)=B.
\end{gathered}$$

So, the operator V is one-to-one. Further, to show that W is an invertible map, let $Vx=y$; i.e., $(\phi_p(x'))'=y$. Then

$$\begin{aligned}
W(Vx)=Wy&=A+\int_0^t\phi_q\big(\int_1^s y(v)dv+\phi_p(B)\big)ds\\
&=A+\int_0^t\phi_q\big(\int_1^s(\phi_p(x'(v)))'dv+\phi_p(B)\big)ds\\
&=A+\int_0^t\phi_q\big(\phi_p(x'(s))-\phi_p(x'(1))+\phi_p(B)\big)ds\\
&=A+\int_0^t\phi_q\big(\phi_p(x'(s))\big)ds\\
&=A+\int_0^t x'(s)ds=A+x(t)-x(0)=x(t).
\end{aligned}$$

□

4. Main result

We state our existence result as follows.

Theorem 4.1. *Let* (R1) *hold. Then for each $p\in(1,2]$, BVP* (1.1) *has at least one solution in $C^2[0,1]$.*

Proof. We will prove the assertion for an arbitrary fixed $p \in (1,2]$. At first, preparing the application of Theorem 2.1, we introduce the set

$$U = \big\{x \in C^2_{BC}[0,1] : A - \sigma < x < L + \sigma, F_1 - \sigma < x' < L_1 + \sigma,\\ m_p - \sigma < x'' < M_p + \sigma\big\}.$$

According to Lemma 3.1, all $C^2[0,1]$-solutions of family (3.1) (or (3.2)) are interior points of U. Next, consider the maps

$$j : C^2_{BC}[0,1] \to C^1[0,1], \quad \text{defined by } jx = x,$$

and

$$\Phi : C^1[0,1] \to C[0,1], \quad \text{defined by } (\Phi x)(t) = f(t, x(t), x'(t))$$

for $t \in [0,1]$ and $x(t) \in j(\overline{U})$.

Now, using the map W, introduce the homotopy

$$H_\lambda : \overline{U} \times [0,1] \to C^2_{BC}[0,1],$$

defined by $H(x,\lambda) \equiv H_\lambda(x) \equiv \lambda W\Phi j(x) + (1-\lambda)l$, where $l = Bt + A$ is the unique solution of

$$(\phi_p(x'))' = 0,\ t \in (0,1),\\ x(0) = A,\ x'(1) = B.$$

Since the map j is completely continuous, Φ is continuous because f is continuous, and W is continuous by Lemma 3.2, the homotopy is compact. For its fixed points we have

$$\lambda W\Phi j(x) + (1-\lambda)l = x$$

and

$$Vx = \lambda \Phi j(x).$$

The last means that the fixed points of H_λ coincide with the $C^2[0,1]$-solutions of (3.1) which are not in ∂U, by Lemma 3.1. Besides, $H_0(x)$ maps each function $x \in \overline{U}$ in l; i.e., it is a constant map and so is essential, by Theorem 2.2.

So, we can apply Theorem 2.1. It infers that the map $H_1(x)$ has a fixed point in U. It is easy to see that it is a $C^2[0,1]$-solution of the BVPs of families (3.1) and (3.2) obtained for $\lambda = 1$ and, what is the same, of (1.1). $\square$

The next result guarantees important properties of the established solutions.

Theorem 4.2. *Let $A > 0$ ($A = 0$) and* (R1) *hold. Then for each $p \in (1,2]$ BVP* (1.1) *has at least one positive (nonnegative) increasing solution in $C^2[0,1]$.*

Proof. By Theorem 4.1, BVP (1.1) has a solution $x(t) \in C^2[0,1]$ for each $p \in (1,2]$ and by Lemma 3.1 it is such that

$$x(t) \ge A \quad \text{and} \quad x'(t) \ge F_1 > 0 \quad \text{for } t \in [0,1],$$

from where the assertion follows immediately. $\square$

Example 4.3. Consider the BVP

$$(\phi_p(x'))' = P_n(x'), \quad t \in (0,1),\\ x(0) = 0, \quad x'(1) = B, \quad B > 0,$$

where $p \in (1,2]$, and the polynomial $P_n(y), n \ge 2$, has two simple zeros p_1 and p_2 such that $p_2 > B > p_1 > 0$.

Clearly, there is a sufficiently small $\delta > 0$ such that

$$p_2 - \delta > B > p_1 + \delta, \quad p_1 - \delta > 0$$

and $P_n(y) \neq 0$ for $t \in (p_1 - \delta, p_1) \cup (p_1, p_1 + \delta) \cup (p_2 - \delta, p_2) \cup (p_2, p_2 + \delta)$.

So, in the case $P_n(y) < 0$ for $t \in (p_1 - \delta, p_1)$ and $P_n(y) > 0$ for $t \in (p_2, p_2 + \delta)$, we can choose $F_2 = p_1 - \delta$, $F_1 = p_1$, $L_1 = p_2$ and $L_2 = p_2 + \delta$ to see that (R1) holds and so the considered problem has a nonnegative increasing solution in $C^2[0, 1]$, by Theorem 4.2.

The same conclusion follows similarly in the rest three cases for the sign of $P_n(y)$ near p_1 and p_2.

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