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POSITIVE BLOWUP SOLUTIONS FOR SOME FRACTIONAL SYSTEMS IN BOUNDED DOMAINS

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Abstract. Using some potential theory tools and the Schauder fixed point theorem, we prove the existence of a positive continuous weak solution for the fractional system

$$(-\Delta)^{\alpha/2}u + p(x)u^\sigma v^r = 0, \quad (-\Delta)^{\alpha/2}v + q(x)u^s v^\beta = 0$$

in a bounded $C^{1,1}$-domain D in $\mathbb{R}^n$ $(n \geq 3)$, subject to Dirichlet conditions, where $0 < \alpha < 2$, $\sigma, \beta \geq 1$, $s, r \geq 0$. The potential functions p, q are nonnegative and required to satisfy some adequate hypotheses related to the Kato class $K_\alpha(D)$. We also investigate the global behavior of such solution.

1. Introduction and statement of main results

Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $(n \geq 3)$ and $0 < \alpha < 2$. This paper is devoted to the study of the following system involving the fractional Laplacian

$$\begin{gathered}
(-\Delta)^{\alpha/2}u + p(x)u^\sigma v^r = 0 \quad \text{in } D,\\
(-\Delta)^{\alpha/2}v + q(x)u^s v^\beta = 0 \quad \text{in } D,\\
\lim_{x \to z \in \partial D} \frac{u(x)}{M_D^\alpha 1(x)} = \varphi(z),\\
\lim_{x \to z \in \partial D} \frac{v(x)}{M_D^\alpha 1(x)} = \psi(z),
\end{gathered} \tag{1.1}$$

where $\sigma, \beta \geq 1$, $r, s \geq 0$, the functions φ and ψ are positive continuous on ∂D and the nonnegative potential functions p, q are required to satisfy some adequate hypotheses related to the Kato class $K_\alpha(D)$ (see Definition 1.1 below). The function $M_D^\alpha 1(x)$ is defined on D by

$$M_D^\alpha 1(x) = \int_{\partial D} M_D^\alpha(x, z)\nu(dz). \tag{1.2}$$

Here, ν is an appropriate measure on ∂D which will be defined later in (1.7) and $M_D^\alpha(x, z)$ is the Martin kernel of the killed symmetric α-stable process $X^D = (X_t^D)_{t>0}$ in D associated to $(-\Delta)^{\alpha/2}$.

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For the reader convenience, we recall the definition of the fractional Laplacian $-(-\Delta)^{\alpha/2}$ which is a nonlocal operator and can be defined by the formula

$$-(-\Delta)^{\alpha/2}u(x) = c_{n\,,\,\alpha} \lim_{\epsilon \searrow 0} \int_{(|x-y|>\epsilon)} \frac{u(y)-u(x)}{|x-y|^{n+\alpha}},$$

where $c_{n,\alpha}$ is a dimensional constant that depends on n and α (see [5, 4, 14] for more details).

Fractional Laplacian is of interest in many branches of sciences such as physics, biologists, queuing theory, operation research, mathematical finance and risk estimation. The fractional powers of the Laplacian in all of $\mathbb{R}^n$ are useful to describe anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, and American options in finance, see [3, 18, 19].

In the classical case (i.e. $\alpha = 2$), there is a large amount of literature dealing with the existence, nonexistence and qualitative analysis of positive solutions for problems related to (1.1); see for example, the papers of Cirstea and Radulescu [13], Ghanmi et al [16], Ghergu and Radulescu [17], Lair and Wood [20], [21], Mu et al [24] and references therein. In these works various existence results of positive bounded solutions or positive blow-up solutions (called also large solutions) have been established and a precise global behavior is given. We note also that several methods have been used to treat these systems such as sub and super-solutions method, variational method and topological methods. These results have been extended recently by Alsaedi et al in [2] for $n \geq 3$ and by Alsaedi in [1] for $n = 2$, in the case $\alpha = 2$, $\sigma, \beta \geq 1$, $s > 0$, $r > 0$, where the authors established the existence of a positive continuous bounded solution for (1.1) in the case $n \geq 3$ and positive continuous solution having logarithm growth at infinity in an exterior domain of $\mathbb{R}^2$. Recently, there has been intensive interest in studying the fractional Laplacian $(-\Delta)^{\alpha/2}$, the development of its potential theory and the global behavior of its Green function G_D^α, see [8, 9, 10, 11].

In this article, we will exploit these potential theory tools and the properties of the Kato class $K_\alpha(D)$, defined and studied in [7], to study the existence of positive continuous solutions (in the sense of distributions) for (1.1). More precisely, we aim first at proving the existence and uniqueness of a positive continuous solution (in the sense of distributions) for the scalar equation

$$\begin{gathered}
(-\Delta)^{\alpha/2}u + p_0(x)u^\gamma = 0 \quad \text{in } D, \\
u > 0 \quad \text{in } D, \\
\lim_{x\to z\in\partial D} \frac{u(x)}{M_D^\alpha 1(x)} = \varphi(z),
\end{gathered} \tag{1.3}$$

where $\gamma \geq 1$ and p_0 is a nonnegative Borel measurable function in D satisfying the condition

(H1) The function $x \to (\delta(x))^{(\frac{\alpha}{2}-1)(\gamma-1)}p_0(x) \in K_\alpha(D)$,

where $\delta(x)$ denotes the Euclidian distance from x to the boundary of D and the class $K_\alpha(D)$ is defined by means of the Green's function G_D^α of $(-\Delta)^{\alpha/2}$ as follows.

Definition 1.1 ([7])**.** A Borel measurable function φ in D belongs to the Kato class $K_\alpha(D)$ if

$$\lim_{r\to 0}\Big(\sup_{x\in D}\int_{(|x-y|\leq r)\cap D}\big(\frac{\delta(y)}{\delta(x)}\big)^{\alpha/2}G_D^\alpha(x,y)|\varphi(y)|dy\Big)=0.$$

It has been shown in [7], that the function

$$x\to(\delta(x))^{-\lambda}\text{ belongs }K_\alpha(D)\text{ if and only if }\lambda<\alpha. \tag{1.4}$$

For more examples of functions belonging to $K_\alpha(D)$, we refer to [7]. Note that for the classical case (i.e. $\alpha=2$), the class $K_2(D)$ was introduced and studied in [23].

Using (1.4), hypothesis (H1) is satisfied if p_0 verifies the following condition: There exists a constant $C>0$, such that for each $x\in D$,

$$p_0(x)\leq\frac{C}{(\delta(x))^\tau},\quad\text{with }\tau+(1-\frac{\alpha}{2})(\gamma-1)<\alpha.$$

To state our existence result for (1.1), we denote by $M_D^\alpha\varphi$ (see [7]), the unique positive continuous solution of

$$\begin{gathered}(-\Delta)^{\alpha/2}u=0\quad\text{in }D,\ (\text{in the sense of distributions})\\ \lim_{x\to z\in\partial D}\frac{u(x)}{M_D^\alpha 1(x)}=\varphi(z)\,.\end{gathered}\tag{1.5}$$

We recall also that in [9], the authors have proved the existence of a constant $C>0$ such that for each $x\in D$,

$$\frac{1}{C}(\delta(x))^{\frac{\alpha}{2}-1}\leq M_D^\alpha 1(x)\leq C(\delta(x))^{\frac{\alpha}{2}-1}. \tag{1.6}$$

Using some potential theory tools and an approximating sequence, we establish the following result.

Theorem 1.2. *Under hypothesis* (H1)*, problem* (1.3) *has a unique positive continuous solution satisfying for each $x\in D$*

$$c_0M_D^\alpha\varphi(x)\leq u(x)\leq M_D^\alpha\varphi(x),$$

where the constant $c_0\in(0,1]$.

Next we exploit the result of Theorem 1.2 to prove the existence of a positive continuous solution (u,v) to the system ((1.1). To this end, we assume the following hypothesis:

(H2) The functions p,q are nonnegative Borel measurable functions such that

$$x\mapsto(\delta(x))^{(\frac{\alpha}{2}-1)(\sigma+r-1)}p(x)\in K_\alpha(D),\quad x\mapsto(\delta(x))^{(\frac{\alpha}{2}-1)(\beta+s-1)}q(x)\in K_\alpha(D).$$

Then by using the Schauder's fixed point theorem, we prove the following result.

Theorem 1.3. *Under assumption* (H2)*, system* (1.1) *has a positive continuous solution (u,v) satisfying: for each $x\in D$,*

$$c_1M_D^\alpha\varphi(x)\leq u(x)\leq M_D^\alpha\varphi(x)\text{ and }c_2M_D^\alpha\psi(x)\leq v(x)\leq M_D^\alpha\psi(x),$$

where c_1,c_2 constants in $(0,1]$.

We note that contrary to the classical case $\alpha = 2$ and $n \geq 3$, in our situation the solution blows up on the boundary of D.

The content of this paper is organized as follows. In Section 2, we collect some properties of functions belonging to the Kato class $K_\alpha(D)$, which are useful to establish our results. Our main results are proved in Section 3.

As usual, let $B^+(D)$ be the set of nonnegative Borel measurable functions in D. We denote by $C_0(D)$ the set of continuous functions in D vanishing continuously on ∂D. Note that $C_0(D)$ is a Banach space with respect to the uniform norm $\|u\|_\infty = \sup\limits_{x\in D}|u(x)|$. When two positive functions f and g are defined on a set S, we write $f \approx g$ if the two sided inequality $\frac{1}{C}g \leq f \leq Cg$ holds on S.

Let G_D be the Green function of the Dirichlet Laplacian in D. The Martin kernel $M_D(.,.)$ of the killed Brownian motion is defined by

$$M_D(x,z) = \lim_{D\ni y\to z} \frac{G_D(x,y)}{G_D(x,z)} \quad \text{for } x \in D \text{ and } z \in \partial D.$$

Similarly, the Martin Kernel of the killed process X^D is defined by

$$M_D^\alpha(x,z) = \lim_{D\ni y\to z} \frac{G_D^\alpha(x,y)}{G_D^\alpha(x,z)} \quad \text{for } x \in D \text{ and } z \in \partial D.$$

Using, the Hergoltz theorem, there exists a positive measure ν in ∂D such that

$$1 = \int_{\partial D} M_D(x,z)\,\nu(dz). \tag{1.7}$$

This measure ν is used in (1.2) to define $M_D^\alpha 1$. We define the potential kernel G_D^α of X^D by

$$G_D^\alpha f(x) := \int_D G_D^\alpha(x,y)f(y)dy, \text{ for } f \in B^+(D) \text{ and } x \in D. \tag{1.8}$$

Finally, let us recall some potential theory tools that will be needed in section 3 and we refer to [7, 12, 22] for more details. For $q \in B^+(D)$, we define the kernel V_q on $B^+(D)$ by

$$V_q f(x) := \int_0^\infty E^x(e^{-\int_0^t q(X_s^D)ds} f(X_t^D))dt, \quad x \in D, \tag{1.9}$$

with $V_0 := V = G_D^\alpha$ and E^x stands for the expectation with respect to the symmetric α-stable process X^D starting from x. If q satisfies $Vq < \infty$, we have the following resolvent equation

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{1.10}$$

It follows that for each each measurable function u in D such that $V(q|u|) < \infty$, we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{1.11}$$

2. The Kato class $K_\alpha(D)$

Proposition 2.1. *[7] Let q be a function in $K_\alpha(D)$, then we have*

(i) $a_\alpha(q) := \sup_{x,y\in D} \int_D \frac{G_D^\alpha(x,z)G_D^\alpha(z,y)}{G_D^\alpha(x,y)}|q(z)|dz < \infty$.

(ii) *Let h be a positive α-superharmonic function with respect to X^D. Then, for all $x \in D$ we have*

$$\int_D G_D^\alpha(x,y)h(y)|q(y)|dy \le a_\alpha(q)h(x). \tag{2.1}$$

Furthermore, for each $x_0 \in \overline{D}$, we have

$$\lim_{r\to 0}\Big(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D} G_D^\alpha(x,y)h(y)|q(y)|dy\Big) = 0. \tag{2.2}$$

(iii) *The function $x \to (\delta(x))^{\alpha-1}q(x)$ is in $L^1(D)$.*

The next two Lemmas will play a special role.

Lemma 2.2 ([7]). *Let q be a nonnegative function in $K_\alpha(D)$ and h be a positive finite α-superharmonic function with respect to X^D. Then for all $x \in D$, such that $0 < h(x) < \infty$, we have*

$$\exp(-a_\alpha(q))h(x) \le h(x) - V_q(qh)(x) \le h(x).$$

Lemma 2.3. *Let q be a nonnegative function in $K_\alpha(D)$, then the family of functions*

$$\Lambda_q = \big\{\frac{1}{M_D^\alpha\varphi(x)}\int_D G_D^\alpha(x,y)M_D^\alpha\varphi(y)f(y)dy,\ |f|\le q\big\}$$

is uniformly bounded and equicontinuous in $\overline{D}$. Consequently Λ_q is relatively compact in $C_0(D)$.

Proof. Taking $h \equiv M_D^\alpha\varphi$ in (2.1), we deduce that for $|f| \le q$ and $x \in D$, we have

$$|\int_D \frac{G_D^\alpha(x,y)}{M_D^\alpha\varphi(x)}M_D^\alpha\varphi(y)f(y)dy| \le \int_D \frac{G_D^\alpha(x,y)}{M_D^\alpha\varphi(x)}M_D^\alpha\varphi(y)q(y)dy \le a_\alpha(q) < \infty. \tag{2.3}$$

So the family Λ_q is uniformly bounded.

Next we aim at proving that the family Λ_q is equicontinuous in $\overline{D}$. First, we recall the following interesting sharp estimates on G_D^α, which is proved in [8]:

$$G_D^\alpha(x,y) \approx |x-y|^{\alpha-n}\min\Big(1,\frac{(\delta(x)\delta(y))^{\alpha/2}}{|x-y|^\alpha}\Big). \tag{2.4}$$

Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. By (2.2), there exists $r > 0$ such that

$$\sup_{z\in D}\frac{1}{M_D^\alpha\varphi(z)}\int_{B(x_0,2r)\cap D} G_D^\alpha(z,y)M_D^\alpha\varphi(y)q(y)dy \le \frac{\varepsilon}{2}.$$

If $x_0 \in D$ and $x, x' \in B(x_0,r)\cap D$, then for $|f| \le q$, we have

$$\begin{aligned}
&\Big|\int_D \frac{G_D^\alpha(x,y)}{M_D^\alpha\varphi(x)}M_D^\alpha\varphi(y)f(y)dy - \int_D \frac{G_D^\alpha(x',y)}{M_D^\alpha\varphi(x')}M_D^\alpha\varphi(y)f(y)dy\Big| \\
&\le \int_D \Big|\frac{G_D^\alpha(x,y)}{M_D^\alpha\varphi(x)} - \frac{G_D^\alpha(x',y)}{M_D^\alpha\varphi(x')}\Big|M_D^\alpha\varphi(y)q(y)dy \\
&\le 2\sup_{z\in D}\int_{B(x_0,2r)\cap D}\frac{1}{M_D^\alpha\varphi(z)}G_D^\alpha(z,y)M_D^\alpha\varphi(y)q(y)dy \\
&\quad + \int_{(|x_0-y|\ge 2r)\cap D}\Big|\frac{G_D^\alpha(x,y)}{M_D^\alpha\varphi(x)} - \frac{G_D^\alpha(x',y)}{M_D^\alpha\varphi(x')}\Big|M_D^\alpha\varphi(y)q(y)dy \\
&\le \varepsilon + \int_{(|x_0-y|\ge 2r)\cap D}\Big|\frac{G_D^\alpha(x,y)}{M_D^\alpha\varphi(x)} - \frac{G_D^\alpha(x',y)}{M_D^\alpha\varphi(x')}\Big|M_D^\alpha\varphi(y)q(y)dy.
\end{aligned}$$

On the other hand, for every $y \in B^c(x_0, 2r) \cap D$ and $x, x' \in B(x_0, r) \cap D$, by using (2.4) and the fact that $M_D^\alpha \varphi(z) \approx (\delta(z))^{\frac{\alpha}{2}-1}$, we have

$$\Big|\frac{1}{M_D^\alpha \varphi(x)} G_D^\alpha(x,y) - \frac{1}{M_D^\alpha \varphi(x')} G_D^\alpha(x',y)\Big| M_D^\alpha \varphi(y) \le C(\delta(y))^{\alpha-1}.$$

Now since $x \to \frac{1}{M_D^\alpha \varphi(x)} G_D^\alpha(x,y)$ is continuous outside the diagonal and $q \in K_\alpha(D)$, we deduce by the dominated convergence theorem and Proposition 2.1 (iii), that

$$\int_{(|x_0-y|\ge 2r)\cap D} \Big|\frac{G_D^\alpha(x,y)}{M_D^\alpha \varphi(x)} - \frac{G_D^\alpha(x',y)}{M_D^\alpha \varphi(x')}\Big| M_D^\alpha \varphi(y) q(y) dy \to 0 \quad \text{as } |x - x'| \to 0.$$

If $x_0 \in \partial D$ and $x \in B(x_0, r) \cap D$, then we have

$$\Big|\int_D \frac{G_D^\alpha(x,y)}{M_D^\alpha \varphi(x)} M_D^\alpha \varphi(y) f(y) dy\Big| \le \frac{\varepsilon}{2} + \int_{(|x_0-y|\ge 2r)\cap D} \frac{G_D^\alpha(x,y)}{M_D^\alpha \varphi(x)} M_D^\alpha \varphi(y) q(y) dy.$$

Now, since $\frac{G_D^\alpha(x,y)}{M_D^\alpha \varphi(x)} \to 0$ as $|x - x_0| \to 0$, for $|x_0 - y| \ge 2r$, then by same argument as above, we obtain

$$\int_{(|x_0-y|\ge 2r)\cap D} \frac{G_D^\alpha(x,y)}{M_D^\alpha \varphi(x)} M_D^\alpha \varphi(y) q(y) dy \to 0 \quad \text{as } |x - x_0| \to 0.$$

Consequently, by Ascoli's theorem, we deduce that Λ_q is relatively compact in $C_0(D)$. $\square$

3. Proofs of Theorems 1.2 and 1.3

The next Lemma will be used for uniqueness.

Lemma 3.1 ([7, Lemma 4])**.** *Let $h \in B^+(D)$ and υ be a nonnegative α-superharmonic function on D with respect to X^D. Let z be a Borel measurable function in D such that $V(h|z|) < \infty$ and $\upsilon = z + V(hz)$. Then z satisfies*

$$0 \le z \le \upsilon.$$

Proof of Theorem 1.2. Let φ be a positive continuous function on ∂D. We recall that on D we have

$$M_D^\alpha \varphi(x) \approx M_D^\alpha 1(x) \approx (\delta(x))^{\frac{\alpha}{2}-1}.$$

Let $\widetilde{p_0} = \gamma(M_D^\alpha \varphi)^{\gamma-1} p_0$ and put $c_0 = e^{-a_\alpha(\widetilde{p_0})}$, where $a_\alpha(\widetilde{p_0})$ is given by Proposition 2.1(i). Since by (H1), $\widetilde{p_0} \in K_\alpha(D)$, it follows from Proposition 2.1 that $V(\widetilde{p_0}) \le a_\alpha(\widetilde{p_0}) < \infty$. Define the nonemty closed bounded convex Λ by

$$\Lambda = \{\omega \in B^+(D) : \ c_0 \le \omega \le 1\}.$$

Let T be the operator defined on Λ by

$$T\omega := 1 - \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0} M_D^\alpha \varphi) + \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0}\omega M_D^\alpha \varphi - p_0(\omega M_D^\alpha \varphi)^\gamma).$$

We claim that T maps Λ to itself. Indeed, for each $\omega \in \Lambda$ we have

$$T\omega \le 1 - \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(p_0(\omega M_D^\alpha \varphi)^\gamma) \le 1.$$

On the other hand, since the function $\widetilde{p_0}\omega M_D^\alpha \varphi - p_0(\omega M_D^\alpha \varphi)^\gamma \ge 0$, we deduce by Lemma 2.2 with $h = M_D^\alpha \varphi$, that $T\omega \ge 1 - \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0} M_D^\alpha \varphi) \ge c_0$. Hence $T\Lambda \subset \Lambda$. Next, we aim at proving that T is nondecreasing on Λ. To this end, we let ω_1, $\omega_2 \in \Lambda$ such that $\omega_1 \leq \omega_2$. Using the fact that the function $t \to \gamma t - t^\gamma$ is nondecreasing on $[0,1]$, we deduce that

$$\begin{aligned}
&T\omega_2 - T\omega_1\\
&= \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0}\omega_2 M_D^\alpha \varphi - p_0(\omega_2 M_D^\alpha \varphi)^\gamma) - \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0}\omega_1 M_D^\alpha \varphi - p_0(\omega_1 M_D^\alpha \varphi)^\gamma)\\
&= \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(p_0(M_D^\alpha \varphi)^\gamma[(\gamma\omega_2 - \omega_2^\gamma) - (\gamma\omega_1 - \omega_1^\gamma)]) \geq 0.
\end{aligned}$$

Next we define the sequence $(\omega_k)_{k\geq 0}$ by

$$\begin{gathered}
\omega_0 = 1 - \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0} M_D^\alpha \varphi),\\
\omega_{k+1} = T\omega_k.
\end{gathered}$$

Clearly $\omega_0 \in \Lambda$ and $\omega_1 = T\omega_0 \geq \omega_0$. Thus, from the monotonicity of T, we deduce that

$$c_0 \leq \omega_0 \leq \omega_1 \leq \dots \leq \omega_k \leq 1.$$

So, the sequence $(\omega_k)_{k\geq 0}$ converges to a measurable function $\omega \in \Lambda$. Therefore by applying the monotone convergence theorem, we obtain

$$\omega = 1 - \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0} M_D^\alpha \varphi) + \frac{1}{M_D^\alpha \varphi} V_{\widetilde{p_0}}(\widetilde{p_0}\omega M_D^\alpha \varphi - p_0(\omega M_D^\alpha \varphi)^\gamma)$$

Put $u = \omega M_D^\alpha \varphi$. Then we have

$$u = M_D^\alpha \varphi - V_{\widetilde{p_0}}(\widetilde{p_0} M_D^\alpha \varphi) + V_{\widetilde{p_0}}(\widetilde{p_0} u - p_0 u^\gamma) \tag{3.1}$$

or equivalently

$$u - V_{\widetilde{p_0}}(\widetilde{p_0} u) = M_D^\alpha \varphi - V_{\widetilde{p_0}}(\widetilde{p_0} M_D^\alpha \varphi) - V_{\widetilde{p_0}}(p_0 u^\gamma). \tag{3.2}$$

Observe that by Proposition 2.1 (ii), we have $V(\widetilde{p_0} u) < \infty$. So applying the operator $(I + V(\widetilde{p_0}.))$ on both sides of (3.2), we deduce by using (1.10) and (1.11) that

$$u = M_D^\alpha \varphi - V(p_0 u^\gamma).$$

Now using (H1) and similar argument as in the proof of Lemma 2.3, we prove that $x \mapsto \frac{1}{M_D^\alpha \varphi} V(p_0 u^\gamma) \in C_0(D)$. So u is a continuous function in D and u is a solution of (1.3). It remains to prove the uniqueness of such a solution. Let u be a continuous solution of (1.3). Since the function $x \mapsto \frac{u(x)}{M_D^\alpha 1(x)}$ is continuous and positive in D such that $\lim_{x\to z\in\partial D} \frac{u(x)}{M_D^\alpha 1(x)} = \varphi(z)$, it follows that $u(x) \approx M_D^\alpha 1(x) \approx M_D^\alpha \varphi(x)$. Then by using this fact and Lemma 2.3, we have

$$\begin{gathered}
(-\Delta)^{\alpha/2}(u + V(p_0 u^\gamma)) = 0 \quad \text{in } D,\\
\lim_{x\to z\in\partial D} \frac{(u + V(p_0 u^\gamma))(x)}{M_D^\alpha 1(x)} = \varphi(z).
\end{gathered}$$

So from the uniqueness of problem (1.5) (see [7]]), we deduce that

$$u + V(p_0 u^\gamma) = M_D^\alpha \varphi \text{ in } D.$$

It follows that if u and v are two continuous solution of (1.3), then $z = v - u$ satisfies

$$z + V(p_0 h z) = 0 \quad \text{in } D,$$

where h is the nonnegative measurable function defined in D by

$$h(x)=\begin{cases}\frac{v^\gamma-u^\gamma}{v-u}, & \text{if } u(x)\neq v(x),\\ 0, & \text{if } u(x)=v(x).\end{cases}$$

Since $V(p_0h|z|)<\infty$, we deduce by Lemma 3.1 that $z=0$, and so $u=v$. $\square$

Proof of Theorem 1.3. Let $\widetilde{p}=\sigma(M_D^\alpha\varphi)^{\sigma-1}(M_D^\alpha\psi)^r p$, $\widetilde{q}=\beta(M_D^\alpha\psi)^{\beta-1}(M_D^\alpha\varphi)^s q$. Then by hypothesis $(\mathbf{H}_2)$, $\widetilde{p}$ and $\widetilde{q}$ are $K_\alpha(D)$.

Put $c_1=e^{-a_\alpha(\widetilde{p})}$, $c_2=e^{-a_\alpha(\widetilde{q})}$. Note that by Proposition 2.1, we have $a_\alpha(\widetilde{p})<\infty$ and $a_\alpha(\widetilde{q})<\infty$. Consider the nonemty closed convex set Γ defined by

$$\Gamma=\{(y,z)\in C(\overline{D})\times C(\overline{D}):\quad c_1\le y\le 1 \text{ and } c_2\le z\le 1\}.$$

Let T be the operator defined on the set Γ by $T(y,z):=(\omega,\theta)$, such that $(\widetilde{u}=\omega M_D^\alpha\varphi,\widetilde{v}=\theta M_D^\alpha\psi)$ is the unique positive continuous solution of the problem

$$\begin{gathered}(-\Delta)^{\alpha/2}\widetilde{u}+((M_D^\alpha\psi)^r z^r p)(x)\widetilde{u}^\sigma=0 \quad \text{in } D,\\ (-\Delta)^{\alpha/2}\widetilde{v}+((M_D^\alpha\varphi)^s y^s q)(x)\widetilde{v}^\beta=0 \quad \text{in } D,\\ \lim_{x\to z\in\partial D}\frac{\widetilde{u}(x)}{M_D^\alpha 1(x)}=\varphi(z),\\ \lim_{x\to z\in\partial D}\frac{\widetilde{v}(x)}{M_D^\alpha 1(x)}=\psi(z),\end{gathered}$$

According to Theorem 1.2, we have

$$\begin{gathered}\omega=1-\frac{1}{M_D^\alpha\varphi}V(z^r\omega^\sigma(M_D^\alpha\psi)^r(M_D^\alpha\varphi)^\sigma p),\\ \theta=1-\frac{1}{M_D^\alpha\psi}V(y^s\omega^\beta(M_D^\alpha\varphi)^s(M_D^\alpha\psi)^\beta q).\end{gathered}$$

Moreover we have $c_1\le\omega\le 1$ and $c_2\le\theta\le 1$ and by Lemma 2.3, $T(\Gamma)$ is equicontinuous on $\overline{D}$. Since $T(\Gamma)$ is also bounded, then we deduce that $T(\Gamma)$ is relatively compact in $C(\overline{D})\times C(\overline{D})$. This implies in particular that $T(\Gamma)\subset\Gamma$.

Next, we shall prove the continuity of the operator T in Γ in the supremum norm. Let $(y_k,z_k)_k$ be a sequence in Γ which converges uniformly to a function (y,z) in Γ. Put $(\omega_k,\theta_k)=T(y_k,z_k)$ and $(\omega,\theta)=T(y,z)$. Then we have

$$\begin{aligned}|\omega_k-\omega|&=\Big|\frac{1}{M_D^\alpha\varphi}V(z^r\omega^\sigma(M_D^\alpha\psi)^r(M_D^\alpha\varphi)^\sigma p)-\frac{1}{M_D^\alpha\varphi}V(z_k^r\omega_k^\sigma(M_D^\alpha\psi)^r(M_D^\alpha\varphi)^\sigma p)\Big|\\ &\le\frac{1}{\sigma M_D^\alpha\varphi}V(|z^r\omega^\sigma-z_k^r\omega_k^\sigma|(M_D^\alpha\varphi)\widetilde{p}).\end{aligned}$$

Using the fact that $|z^r\omega^\sigma-z_k^r\omega_k^\sigma|\le 2$ and that $\widetilde{p}\in K_\alpha(D)$, we deduce by Proposition 2.1 and the dominated convergence theorem, that $\omega_k\to\omega$ as $k\to\infty$. Similarly we prove that $\theta_k\to\theta$ as $k\to\infty$. So $T(y_k,z_k)\to T(y,z)$ as $k\to\infty$. Since $T(\Gamma)$ is relatively compact in $C(\overline{D})\times C(\overline{D})$, we deduce that

$$\|T(y_k,z_k)-T(y,z)\|_\infty\to 0 \quad \text{as } k\to\infty.$$

Now the Schauder fixed point theorem implies that there exists $(y,z)\in\Gamma$ such that $T(y,z)=(y,z)$. Which is equivalent to

$$\begin{gathered}u=M_D^\alpha\varphi-V(pu^\sigma v^r),\\ v=M_D^\alpha\psi-V(qu^s v^\beta),\end{gathered}$$

where $(u,v)=(yM_D^\alpha\varphi, zM_D^\alpha\psi)$. The pair (u,v) is a required solution of (1.1) in the sense of distributions. This completes the proof. $\square$

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