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BOUNDARY STABILIZATION OF MEMORY-TYPE THERMOELASTIC SYSTEMS

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Abstract. In this article we consider an n-dimentional thermoelastic system with a viscoelastic damping localized on a part of the boundary. We establish an explicit and general decay rate result that allows a larger class of relaxation functions and generalizes previous results existing in the literature.

1. Introduction

In this article we are concerned with the problem

$$\begin{gathered}
u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0, \quad \text{in } \Omega\times(0,\infty) \\
b\theta_t - h\Delta\theta + \beta\operatorname{div} u_t = 0, \quad \text{in } \Omega\times(0,\infty) \\
u = 0, \quad \text{on } \Gamma_0\times(0,\infty) \\
u(x,t) = -\int_0^t g(t-s)\Big(\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v\Big)(s)ds, \quad \text{on } \Gamma_1\times(0,\infty) \\
\theta = 0, \quad \text{on } \partial\Omega\times(0,\infty) \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad x\in\Omega,
\end{gathered} \tag{1.1}$$

which is a thermoelastic system subjected to the effect of a viscoelastic damping acting on a part of the boundary. Here Ω is a bounded domain of $\mathbb{R}^n$ $(n\geq 2)$ with a smooth boundary $\partial\Omega = \Gamma_0\cup\Gamma_1$, v is the unit outward normal to $\partial\Omega$, $u = u(x,t)\in\mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temprature, and the relaxation function g is a positive differentiable function. The coefficients $b, h, \beta, \mu, \lambda$ are positive constants, where μ, λ are Lame moduli. In this work, we study the decay properties of the solutions of (1.1) for functions g of more general type.

Over the past few decades, there has been a lot of work on local existence, global existence, well-posedeness, and asymptotic behavior of solutions to some initial-boundary value problems in both one-dimensional and multi-dimensional thermoelasticity. In the absence of the viscoelastic term, it is well-known (see [2, 4, 10]) that the one dimensional linear thermoelastic system associated with various types of boundary conditions decays to zero exponentially. Irmscher and Racke [4] obtained explicit sharp exponential decay rates for solutions of the system

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of classical thermoelasticity in one dimension. They also considered the model of thermoelasticity with second sound and compared the results of both models with respect to the asymptotic behavior of solutions. Also, Rivera and Qin [13, 18] established the global existence, uniqueness and exponential stability of solutions to equations of one-dimensional nonlinear thermoelasticity with thermal memory subject to Dirichlet-Dirichlet or Dirichlet-Neumann boundary conditions.

In the multi-dimensional case the situation is much different. It was shown that the dissipation given by heat conduction is not strong enough to produce uniform rate of decay to the solution as in the one-dimensional case. We have the pioneering work of Dafermos [3], in which he proved an asymptotic stability result; but no rate of decay has been given. The uniform rate of decay for the solution in two or three dimensional space was obtained by Jiang, Rivera and Racke [7] in special situation like radial symmetry. Lebeau and Zuazua [8] proved that the decay rate is never uniform when the domain is convex. Thus, to solve this problem, additional damping mechanisms are necessary. In this aspect, Pereira and Menzala [17] introduced a linear internal damping effective in the whole domain, and established the uniform decay rate. A similar result was obtained by Liu [9] for a linear boundary velocity feedback acting on the elastic component of the system, and by Liu and Zuazua [11] for a nonlinear boundary feedback. Oliveira and Charao [16] improved the result in [17] by including a weak localized dissipative term effective only in a neighborhood of part of the boundary and proved an exponential decay result when the damping term is linear and a polynomial decay result for a nonlinear damping term. Recently, Mustafa [15] treated weak frictional damping of more general type and established an explicit and general decay result. For more literature on the subject, we refer the reader to books by Jiang and Racke [6] and Zheng [19].

Regarding viscoelastic damping, we mention that viscoelastic materials are those with properties that are intermediate between elasticity and viscosity. As a result of this behavior, some of the energy stored in a viscoelastic system is recovered upon removal of the load, and the remainder is dissipated in the form of heat causing a damping for the system. This type of material possesses a characteristic which can be referred to as a memory effect. That is, the material response not only does depend on the current state, but also on all past occurrences, and in a general sense, the material has a memory keeping all past states. As a conclusion, this memory effect is expressed by an integral term from the initial time 0 up to the time t with kernel usually called the relaxation function. Rivera and Racke [14] considered magneto-thermoelastic model with a boundary condition of memory type. If g is the relaxation function and k is the resolvent kernel of $-g'/g(0)$, they showed that the energy of the solution decays exponentially (polynomially) when k and $(-k')$ decay exponentially (polynomially). Messaoudi and Al-Shehri [12] considered a wider class of kernels k that are not necessarily decaying exponentially or polynomially and proved a more general energy decay result.

Our aim in this work is to investigate (1.1) for resolvent kernels of general-type decay and obtain a more general and explicit energy decay formula, from which the usual exponential and polynomial decay rates are only special cases of our result. The proof is based on the multiplier method and makes use of some properties of convex functions including the use of the general Young's inequality and Jensen's inequality. The paper is organized as follows. In section 2, we present some notation and material needed for our work. Some technical lemmas and the proof of our main result will be given in section 3.

2. Preliminaries

We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant. In the sequel we assume that system (1.1) has a unique solution

$$u \in C(\mathbb{R}_+; H^2(\Omega)^n \cap V^n) \cap C^1(\mathbb{R}_+; V^n) \cap C^2(\mathbb{R}_+; L^2(\Omega)^n),$$
$$\theta \in C(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)).$$

where $V = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0\}$. This result can be proved, for initial data in suitable function spaces, using standard arguments such as the Galerkin method.

First we state the following hypothesis

(A1) Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed and disjoint, with $\operatorname{meas}(\Gamma_0) > 0$, v is the unit outward normal to $\partial\Omega$, and there exists a fixed point $x_0 \in \mathbb{R}^n$ such that, for $m(x) = x - x_0$, $m \cdot v \le 0$ on Γ_0 and $m \cdot v > 0$ on Γ_1.

We remark that (A1) implies that there exist constants δ_0 and R such that

$$m \cdot v \ge \delta_0 > 0 \text{ on } \Gamma_1 \quad \text{and} \quad |m(x)| \le R \quad \text{for all } x \in \Omega. \tag{2.1}$$

We denote by k the resolvent kernel of $(-g'/g(0))$ which satisfies

$$k(t) + \frac{1}{g(0)}(g' * k)(t) = -\frac{1}{g(0)}g'(t), \quad t \ge 0$$

where * denotes the convolution product

$$(u * v)(t) = \int_0^t u(t-s)v(s)ds.$$

By differentiating the equation

$$u(x,t) = -\int_0^t g(t-s)\Big(\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v\Big)(s)ds$$

and taking $\alpha = \frac{1}{g(0)}$, we obtain

$$\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v = -\alpha\Big[u_t + g' * \Big(\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v\Big)\Big]$$

on $\Gamma_1 \times (0,\infty)$. Using the Volterra's inverse operator, we obtain

$$\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v = -\alpha[u_t + k * u_t], \quad \text{on } \Gamma_1 \times (0,\infty)$$

which gives, assuming throughout the paper that $u_0 \equiv 0$,

$$\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v = -\alpha[u_t + k(0)u + k' * u], \quad \text{on } \Gamma_1 \times (0,\infty). \tag{2.2}$$

Therefore, we use (2.2) instead of the boundary condition on $\Gamma_1 \times (0,\infty)$ in (1.1) and also consider the following assumption on k,

(A2) $k : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^2 function such that
$$k(0) > 0, \quad \lim_{t\to\infty} k(t) = 0, \quad k'(t) \le 0$$
and there exists a positive function $H \in C^1(\mathbb{R}_+)$, with $H(0) = 0$, and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, such that
$$k''(t) \ge H(-k'(t)), \quad \forall t > 0.$$

Now, we introduce the energy functional
$$\begin{aligned} E(t) := \frac{1}{2}\int_\Omega \Big(|u_t|^2 + \mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + b\theta^2\Big)dx \\ + \frac{\alpha}{2}k(t)\int_{\Gamma_1} |u|^2\, d\Gamma - \frac{\alpha}{2}\int_{\Gamma_1} (k' \circ u)(t)\, d\Gamma \end{aligned}$$
where $|\nabla u|^2 = \sum_{i=1}^n |\nabla u_i|^2$ and
$$(f \circ w)(t) = \int_0^t f(t-s)|w(t) - w(s)|^2 ds.$$
Our main stability result is the following.

Theorem 2.1. *Assume that* (A1) *and* (A2) *hold. Then there exist positive constants* k_1, k_2, k_3 *and* ε_0 *such that the solution of* (1.1) *satisfies*
$$E(t) \le k_3 H_1^{-1}(k_1 t + k_2) \quad \forall t \ge 0, \tag{2.3}$$
where
$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\varepsilon_0 s)}ds \qquad \text{and} \quad H_0(t) = H(D(t))$$
provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and
$$\int_0^{+\infty} \frac{-k'(s)}{H_0^{-1}(k''(s))} ds < +\infty. \tag{2.4}$$
Moreover, if $\int_0^1 H_1(t)dt < +\infty$ for some choice of D, then we have the improved estimate
$$E(t) \le k_3 G^{-1}(k_1 t + k_2) \quad \text{where} \quad G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)} ds. \tag{2.5}$$

In particular, this last estimate is valid for the special case $H(t) = ct^p$, for $1 \le p < \frac{3}{2}$.

Remarks. 1. Using the properties of H, one can show that the function H_1 is strictly decreasing and convex on $(0, 1]$, with $\lim_{t\to 0} H_1(t) = +\infty$. Therefore, Theorem 2.1 ensures
$$\lim_{t\to\infty} E(t) = 0.$$

2. Our main result is obtained under very general hypotheses on the resolvent kernel k that allow to deal with a much larger class of functions k that guarantee the uniform stability of (1.1) with an explicit formula for the decay rates of the energy.

3. The usual exponential and polynomial decay rate estimates, already proved for k satisfying $k'' \ge d(-k')^p$, $1 \le p < 3/2$, are special cases of our result. We will provide a "simpler" proof for these special cases.

4. The condition $k'' \geq d(-k')^p$, $1 \leq p < 3/2$ assumes $(-k'(t)) \leq \omega e^{-dt}$ when $p = 1$ and $(-k'(t)) \leq \omega/t^{\frac{1}{p-1}}$ when $1 < p < 3/2$. Our result allows resolvent kernels whose derivatives are not necessarily of exponential or polynomial decay. For instance, if

$$k'(t) = -\exp(-t^q)$$

for $0 < q < 1$, then $k''(t) = H(-k'(t))$ where, for $t \in (0, r]$, $r < 1$,

$$H(t) = \frac{qt}{[\ln(1/t)]^{\frac{1}{q}-1}}$$

which satisfies hypothesis (A2). Also, by taking $D(t) = t^\alpha$, (2.4) is satisfied for any $\alpha > 1$. Therefore, we can use Theorem 2.1 and do some calculations (see the appendix) to deduce that the energy decays at the same rate of $(-k'(t))$, that is

$$E(t) \leq c\exp(-\omega t^q).$$

5. The well-known Jensen's inequality will be of essential use in establishing our main result. If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and j are integrable functions on Ω, $j(x) \geq 0$, and $\int_\Omega j(x)dx = C > 0$, then Jensen's inequality states that

$$F\Big[\frac{1}{C}\int_\Omega f(x)j(x)dx\Big] \leq \frac{1}{C}\int_\Omega F[f(x)]j(x)dx.$$

6. Since $\lim_{t\to\infty} k(t) = 0$, then $\lim_{t\to\infty}(-k'(t))$ cannot be equal to a positive number, and so it is natural to assume that $\lim_{t\to+\infty}(-k'(t)) = 0$, and so to also assume that $\lim_{t\to\infty} k''(t) = 0$. Hence, there is $t_1 > 0$ large enough such that $k'(t_1) < 0$ and

$$\max\{k(t), -k'(t), k''(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall t \geq t_1. \tag{2.6}$$

As k' is nondecreasing, $k'(0) < 0$ and $k'(t_1) < 0$, then $k'(t) < 0$ for any $t \in [0, t_1]$ and

$$0 < -k'(t_1) \leq -k'(t) \leq -k'(0), \quad \forall t \in [0, t_1].$$

Therefore, since H is a positive continuous function,

$$a \leq H(-k'(t)) \leq b, \quad \forall t \in [0, t_1]$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$k''(t) \geq H(-k'(t)) \geq a = \frac{a}{k'(0)}k'(0) \geq \frac{a}{k'(0)}k'(t)$$

which gives, for some positive constant d,

$$k''(t) \geq -dk'(t), \quad \forall t \in [0, t_1]. \tag{2.7}$$

3. Proof of the main result

In this section we prove Theorem 2.1. For this purpose, we establish several lemmas.

Lemma 3.1. *Under the assumptions* (A1) *and* (A2)*, the energy functional satisfies, along the solution of* (1.1)*, the estimate*

$$E'(t) = -h\int_\Omega |\nabla\theta|^2 dx - \alpha\int_{\Gamma_1} |u_t|^2\, d\Gamma + \frac{\alpha}{2}k'(t)\int_{\Gamma_1} |u|^2\, d\Gamma - \frac{\alpha}{2}\int_{\Gamma_1} (k'' \circ u)(t)\, d\Gamma \leq 0. \tag{3.1}$$

Proof. Multiplying the first two equations of (1.1) by u_t and θ respectively, integrating by parts over Ω, and using (2.2) give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(|u_t|^2+\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div} u)^2+b\theta^2\big)\,dx\\
&=-h\int_\Omega|\nabla\theta|^2dx+\int_{\Gamma_1}u_t\cdot[\mu\frac{\partial u}{\partial v}+(\mu+\lambda)(\operatorname{div} u)v]\,d\Gamma\\
&=-h\int_\Omega|\nabla\theta|^2dx-\alpha\int_{\Gamma_1}|u_t|^2\,d\Gamma-\alpha k(0)\int_{\Gamma_1}u_tu\,d\Gamma-\alpha\int_{\Gamma_1}u_t\cdot(k'*u)\,d\Gamma
\end{aligned}$$

Then, we make use of the identity

$$(f*w)w'=-\frac{1}{2}f(t)|w(t)|^2+\frac{1}{2}f'\circ w-\frac{1}{2}\frac{d}{dt}\Big[f\circ w-(\int_0^t f(s)ds)|w(t)|^2\Big]. \tag{3.2}$$

to obtain (3.1). □

Lemma 3.2. *Under the assumptions* (A1) *and* (A2)*, the functional*

$$K(t):=\int_\Omega u_t\cdot[M+(n-1)u]dx,$$

where $M=\langle M_1,M_2,\dots,M_n\rangle$ such that $M_i=2m\nabla u^i$ and $m=(x-x_0)$, satisfies, along the solution of (1.1)*, the estimate*

$$\begin{aligned}
K'(t)\le&-\int_\Omega|u_t|^2dx-\frac{\mu}{2}\int_\Omega|\nabla u|^2dx-\frac{\mu+\lambda}{2}\int_\Omega(\operatorname{div} u)^2dx\\
&+c\int_{\Gamma_1}|u_t|^2\,d\Gamma-c\int_{\Gamma_1}(k'\circ u)(t)\,d\Gamma+c\int_\Omega|\nabla\theta|^2dx,\quad\forall t\ge t_1.
\end{aligned}\tag{3.3}$$

Proof. Direct computations, using (1.1), yield

$$\begin{aligned}
K'(t)&=\sum_{i=1}^n\int_\Omega u_t^i(2m\cdot\nabla u_t^i)dx+(n-1)\int_\Omega|u_t|^2dx+\int_\Omega u_{tt}\cdot[M+(n-1)u]dx\\
&=\sum_{i=1}^n\int_\Omega m\cdot\nabla|u_t^i|^2dx+(n-1)\int_\Omega|u_t|^2dx\\
&\quad+\int_\Omega[\mu\Delta u+(\mu+\lambda)\nabla(\operatorname{div} u)-\beta\nabla\theta]\cdot[M+(n-1)u]dx\\
&=-\int_\Omega|u_t|^2dx+\int_{\Gamma_1}(m\cdot v)|u_t|^2\,d\Gamma+\mu\int_\Omega\Delta u\cdot[M+(n-1)u]dx\\
&\quad+(\mu+\lambda)\int_\Omega\nabla(\operatorname{div} u)[M+(n-1)u]dx-\beta\int_\Omega\nabla\theta[M+(n-1)u]dx.
\end{aligned}\tag{3.4}$$

Now, we estimate the last three terms in (3.4) as follows. First, we use the identity

$$2\nabla u^i\cdot\nabla(m\cdot\nabla u^i)=2|\nabla u^i|^2+m\cdot\nabla(|\nabla u^i|^2) \tag{3.5}$$

to obtain

$$\begin{aligned}
&\int_\Omega\Delta u\cdot M\,dx\\
&=-\sum_{i=1}^n\int_\Omega\nabla u^i\cdot\nabla(2m.\nabla u^i)dx+\sum_{i=1}^n\int_{\partial\Omega}(2m\cdot\nabla u^i)\frac{\partial u^i}{\partial v}\,d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n}\int_\Omega [2|\nabla u^i|^2 + m\cdot\nabla(|\nabla u^i|^2)]dx + \sum_{i=1}^{n}\int_{\partial\Omega}(2m\cdot\nabla u^i)\frac{\partial u^i}{\partial v}\,d\Gamma \\
&= (n-2)\int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega}(m\cdot v)|\nabla u|^2\,d\Gamma + \sum_{i=1}^{n}\int_{\partial\Omega}(2m\cdot\nabla u^i)\frac{\partial u^i}{\partial v}\,d\Gamma
\end{aligned}
$$

By the fact that

$$
\nabla u^i = (\frac{\partial u^i}{\partial v})v \quad \text{on } \Gamma_0, \tag{3.6}
$$

we obtain

$$
\begin{aligned}
\int_\Omega \Delta u\cdot M\,dx = (n-2)\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1}(m\cdot v)|\nabla u|^2\,d\Gamma + \int_{\Gamma_0}(m\cdot v)|\nabla u|^2\,d\Gamma \\
+ \sum_{i=1}^{n}\int_{\Gamma_1}(2m\cdot\nabla u^i)\frac{\partial u^i}{\partial v}\,d\Gamma.
\end{aligned}
$$

Since

$$
\int_\Omega \Delta u\cdot u dx = -\int_\Omega |\nabla u|^2 dx + \int_{\Gamma_1} u\cdot\frac{\partial u}{\partial v}\,d\Gamma
$$

and

$$
\begin{gathered}
m\cdot v \le 0 \quad \text{on } \Gamma_0 \\
m\cdot v \ge \delta_0 > 0 \quad \text{on } \Gamma_1,
\end{gathered}
$$

it follows that

$$
\begin{aligned}
&\int_\Omega \Delta u\cdot[M+(n-1)u]dx \\
&= -\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1}(m\cdot v)|\nabla u|^2\,d\Gamma \\
&\quad + \int_{\Gamma_0}(m\cdot v)|\nabla u|^2\,d\Gamma + \sum_{i=1}^{n}\int_{\Gamma_1}[2m\cdot\nabla u^i + (n-1)u^i]\frac{\partial u^i}{\partial v}\,d\Gamma \\
&\le -\int_\Omega |\nabla u|^2 dx - \delta_0\int_{\Gamma_1}|\nabla u|^2\,d\Gamma + \sum_{i=1}^{n}\int_{\Gamma_1}(2m\cdot\nabla u^i)\frac{\partial u^i}{\partial v}\,d\Gamma \\
&\quad + (n-1)\int_{\Gamma_1} u\cdot\frac{\partial u}{\partial v}\,d\Gamma.
\end{aligned} \tag{3.7}
$$

Next, we consider

$$
\begin{aligned}
&\int_\Omega \nabla(\operatorname{div} u)\cdot[M+(n-1)u]dx \\
&= -\int_\Omega(\operatorname{div} u)(\operatorname{div} M)dx + \int_{\partial\Omega}(\operatorname{div} u)(M\cdot v)\,d\Gamma \\
&\quad - (n-1)\int_\Omega(\operatorname{div} u)^2 dx + (n-1)\int_{\Gamma_1}(\operatorname{div} u)(u\cdot v)\,d\Gamma.
\end{aligned} \tag{3.8}
$$

But, one can show that

$$
\operatorname{div} M = 2(\operatorname{div} u) + 2m\cdot\nabla(\operatorname{div} u). \tag{3.9}
$$

Therefore,

$$
-\int_\Omega(\operatorname{div} u)(\operatorname{div} M)dx = -2\int_\Omega(\operatorname{div} u)^2 dx - 2\int_\Omega(\operatorname{div} u)(m\cdot\nabla(\operatorname{div} u))dx
$$

$$
\begin{aligned}
&= -2\int_\Omega (\operatorname{div} u)^2 dx - \int_\Omega m\cdot\nabla(\operatorname{div} u)^2 dx\\
&= (n-2)\int_\Omega (\operatorname{div} u)^2 dx - \int_{\partial\Omega} (\operatorname{div} u)^2 (m\cdot v)\, d\Gamma.
\end{aligned}
$$

Also, using (3.6),

$$
M\cdot v = 2(m\cdot v)(\operatorname{div} u) \quad \text{on } \Gamma_0
$$

which gives

$$
\int_{\partial\Omega} (\operatorname{div} u)(M\cdot v)\, d\Gamma = 2\int_{\Gamma_0} (\operatorname{div} u)^2 (m\cdot v)\, d\Gamma + \sum_{i=1}^n \int_{\Gamma_1} (\operatorname{div} u)(2m\cdot\nabla u^i) v_i\, d\Gamma.
$$

Consequently, (3.8) becomes

$$
\begin{aligned}
&\int_\Omega \nabla(\operatorname{div} u)\cdot[M+(n-1)u]dx\\
&= -\int_\Omega (\operatorname{div} u)^2 dx + \int_{\Gamma_0} (\operatorname{div} u)^2 (m\cdot v)\, d\Gamma\\
&\quad - \int_{\Gamma_1} (\operatorname{div} u)^2 (m\cdot v)\, d\Gamma + \sum_{i=1}^n \int_{\Gamma_1} (\operatorname{div} u)(2m\cdot\nabla u^i) v_i\, d\Gamma\\
&\quad + (n-1)\int_{\Gamma_1} (\operatorname{div} u)(u\cdot v)\, d\Gamma\\
&\le -\int_\Omega (\operatorname{div} u)^2 dx - \delta_0 \int_{\Gamma_1} (\operatorname{div} u)^2\, d\Gamma + \sum_{i=1}^n \int_{\Gamma_1} (\operatorname{div} u)(2m\cdot\nabla u^i) v_i\, d\Gamma\\
&\quad + (n-1)\int_{\Gamma_1} (\operatorname{div} u)(u\cdot v)\, d\Gamma\\
&\le -\int_\Omega (\operatorname{div} u)^2 dx + \sum_{i=1}^n \int_{\Gamma_1} (\operatorname{div} u)(2m\cdot\nabla u^i) v_i\, d\Gamma\\
&\quad + (n-1)\int_{\Gamma_1} (\operatorname{div} u)(u\cdot v)\, d\Gamma.
\end{aligned}
\tag{3.10}
$$

For the last term of (3.4), we find, using (3.9), that

$$
\begin{aligned}
&-\int_\Omega \nabla\theta\cdot[M+(n-1)u]dx\\
&= \int_\Omega (\operatorname{div} M)\theta dx + (n-1)\int_\Omega (\operatorname{div} u)\theta dx\\
&= (n+1)\int_\Omega (\operatorname{div} u)\theta dx + 2\int_\Omega (m\cdot\nabla(\operatorname{div} u))\theta dx\\
&= (n+1)\int_\Omega (\operatorname{div} u)\theta dx - 2\int_\Omega (\operatorname{div} u)(div(m\theta))dx\\
&= -(n-1)\int_\Omega (\operatorname{div} u)\theta dx - 2\int_\Omega (\operatorname{div} u)(m\cdot\nabla\theta)dx.
\end{aligned}
\tag{3.11}
$$

A combination of (3.4), (3.7), (3.10), and (3.11) leads to

$$\begin{aligned} K'(t) \le &-\int_\Omega |u_t|^2dx + \int_{\Gamma_1} (m\cdot v)|u_t|^2\,d\Gamma - \mu\int_\Omega |\nabla u|^2dx - \mu\delta_0\int_{\Gamma_1}|\nabla u|^2\,d\Gamma \\ &+ \sum_{i=1}^n \int_{\Gamma_1} (2m\cdot\nabla u^i)\Big[\mu\frac{\partial u^i}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v_i\Big]\,d\Gamma \\ &+ (n-1)\int_{\Gamma_1} u\cdot\Big[\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v\Big]\,d\Gamma - (\mu+\lambda)\int_\Omega (\operatorname{div} u)^2dx \\ &- (n-1)\int_\Omega (\operatorname{div} u)\theta dx - 2\int_\Omega (\operatorname{div} u)(m\cdot\nabla\theta)dx. \end{aligned} \tag{3.12}$$

By using the boundary condition (2.2), Young's inequality and $|m(x)| \le R$, and noting that

$$k' * u = \int_0^t k'(t-s)[u(s)-u(t)]ds + u(t)[k(t)-k(0)]$$

and

$$\begin{aligned} \Big|\int_0^t k'(t-s)[u(s)-u(t)]dsBig|^2 &\le \Big(\int_0^t -k'(s)ds\Big)(-k'\circ u)(t) \\ &= [k(0)-k(t)](-k'\circ u)(t) \\ &\le -c(k'\circ u)(t), \end{aligned}$$

we obtain

$$\begin{aligned} &\sum_{i=1}^n \int_{\Gamma_1} (2m\cdot\nabla u^i)[\mu\frac{\partial u^i}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v_i]\,d\Gamma \\ &\quad + (n-1)\int_{\Gamma_1} u\cdot[\mu\frac{\partial u}{\partial v} + (\mu+\lambda)(\operatorname{div} u)v]\,d\Gamma \\ &= -\alpha\sum_{i=1}^n \int_{\Gamma_1} (2m\cdot\nabla u^i)[u_t^i + k(0)u^i + k' * u^i]\,d\Gamma \\ &\quad - \alpha(n-1)\int_{\Gamma_1} u\cdot[u_t + k(0)u + k' * u]\,d\Gamma \\ &= -\alpha\sum_{i=1}^n \int_{\Gamma_1} (2m\cdot\nabla u^i)\Big[u_t^i + k(t)u^i + \int_0^t k'(t-s)[u^i(s)-u^i(t)]ds\Big]\,d\Gamma \\ &\quad - \alpha(n-1)\int_{\Gamma_1} u\cdot\Big[u_t + k(t)u + \int_0^t k'(t-s)[u(s)-u(t)]ds\Big]\,d\Gamma \\ &\le \mu\delta_0\int_{\Gamma_1}|\nabla u|^2\,d\Gamma + C_\varepsilon\int_{\Gamma_1}|u_t|^2\,d\Gamma - C_\varepsilon\int_{\Gamma_1}(k'\circ u)\,d\Gamma + (\varepsilon + ck^2(t))\int_{\Gamma_1}|u|^2\,d\Gamma. \end{aligned}$$

Then, using

$$\int_{\Gamma_1}|u|^2\,d\Gamma \le c_0\int_\Omega |\nabla u|^2dx \tag{3.13}$$

and that $\lim_{t\to\infty} k(t) = 0$ and choosing ε small enough, we deduce that for all $t \geq t_1$,

$$\begin{aligned}
&\sum_{i=1}^{n}\int_{\Gamma_1}(2m\cdot\nabla u^i)[\mu\frac{\partial u^i}{\partial v}+(\mu+\lambda)(\operatorname{div} u)v_i]\,d\Gamma\\
&+(n-1)\int_{\Gamma_1}u\cdot[\mu\frac{\partial u}{\partial v}+(\mu+\lambda)(\operatorname{div} u)v]\,d\Gamma\\
&\leq \mu\delta_0\int_{\Gamma_1}|\nabla u|^2\,d\Gamma+c\int_{\Gamma_1}|u_t|^2\,d\Gamma-c\int_{\Gamma_1}(k'\circ u)\,d\Gamma+\frac{\mu}{2}\int_\Omega|\nabla u|^2dx,
\end{aligned} \tag{3.14}$$

where t_1, introduced in (2.6), is large enough. Also, using Young's and Poincaré's inequalities yields

$$-(n-1)\int_\Omega(\operatorname{div} u)\theta dx-2\int_\Omega(\operatorname{div} u)(m\cdot\nabla\theta)dx\leq\frac{(\mu+\lambda)}{2}\int_\Omega(\operatorname{div} u)^2dx+c\int_\Omega|\nabla\theta|^2dx \tag{3.15}$$

By inserting (3.14) and (3.15) in (3.12), the estimate (3.3) is established. □

Proof of Theorem 2.1. For $N > 0$, we define

$$\mathcal{L}(t) := NE(t) + K(t).$$

Combining (3.1) and (3.3), for all $t \geq t_1$, we obtain

$$\begin{aligned}
\mathcal{L}'(t)\leq &-\int_\Omega|u_t|^2dx-\frac{\mu}{2}\int_\Omega|\nabla u|^2dx-\frac{\mu+\lambda}{2}\int_\Omega(\operatorname{div} u)^2dx-(hN-c)\int_\Omega|\nabla\theta|^2dx\\
&-(\alpha N-c)\int_{\Gamma_1}|u_t|^2\,d\Gamma-c\int_{\Gamma_1}(k'\circ u)(t)\,d\Gamma.
\end{aligned}$$

At this point, we choose N large enough so that

$$\gamma := (hN-c) > 0 \quad\text{and}\quad \alpha N - c > 0.$$

So, we arrive at

$$\mathcal{L}'(t)\leq-\int_\Omega\big[|u_t|^2+\frac{\mu}{2}|\nabla u|^2dx+\frac{\mu+\lambda}{2}(\operatorname{div} u)^2+\gamma|\nabla\theta|^2\big]dx-c\int_{\Gamma_1}(k'\circ u)(t)\,d\Gamma$$

which, using Poincaré's inequality and (3.13), yields

$$\mathcal{L}'(t)\leq -mE(t)-c\int_{\Gamma_1}(k'\circ u)(t)\,d\Gamma,\quad \forall t\geq t_1. \tag{3.16}$$

On the other hand, we can choose N even larger (if needed) so that

$$\mathcal{L}(t)\sim E(t). \tag{3.17}$$

Now, we use (2.7) and (3.1) to conclude that, for any $t \geq t_1$,

$$\begin{aligned}
-\int_0^{t_1}k'(s)\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds&\leq\frac{1}{d}\int_0^{t_1}k''(s)\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\\
&\leq -cE'(t).
\end{aligned} \tag{3.18}$$

Next, we take $F(t) = \mathcal{L}(t) + cE(t)$, which is clearly equivalent to $E(t)$, and use (3.16) and (3.18), to obtain: for all $t \geq t_1$,

$$F'(t)\leq -mE(t)-c\int_{t_1}^{t}k'(s)\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds. \tag{3.19}$$

(I) $H(t)=ct^p$ and $1\le p<\frac{3}{2}$:

Case 1. $p=1$**:** Estimate (3.19) yields

$$F'(t)\le -mE(t)+c\int_{\Gamma_1}(k''\circ u)(t)\,d\Gamma\le -mE(t)-cE'(t),\quad \forall t\ge t_1.$$

which gives

$$(F+cE)'(t)\le -mE(t),\quad \forall t\ge t_1.$$

Hence, using the fact that $F+cE\sim E$, we obtain easily that

$$E(t)\le c'e^{-ct}=c'G^{-1}(t).$$

Case 2. $1<p<\frac{3}{2}$**:** One can easily show that $\int_0^{+\infty}[-k'(s)]^{1-\delta_0}ds<+\infty$ for any $\delta_0<2-p$. Using this fact, (3.1), and (3.13) and choosing t_1 even larger if needed, we deduce that, for all $t\ge t_1$,

$$\begin{aligned}\eta(t)&:=\int_{t_1}^{t}[-k'(s)]^{1-\delta_0}\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\\ &\le 2\int_{t_1}^{t}[-k'(s)]^{1-\delta_0}\int_{\Gamma_1}(|u(t)|^2+|u(t-s)|^2)\,d\Gamma ds\\ &\le cE(0)\int_{t_1}^{t}[-k'(s)]^{1-\delta_0}ds<1.\end{aligned}\tag{3.20}$$

Then, Jensen's inequality, (3.1), hypothesis (A2), and (3.20) lead to

$$\begin{aligned}&-\int_{t_1}^{t}k'(s)\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\\ &=\int_{t_1}^{t}[-k'(s)]^{\delta_0}[-k'(s)]^{1-\delta_0}\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\\ &=\int_{t_1}^{t}[-k'(s)]^{(p-1+\delta_0)(\frac{\delta_0}{p-1+\delta_0})}[-k'(s)]^{1-\delta_0}\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\\ &\le \eta(t)\Big[\frac{1}{\eta(t)}\int_{t_1}^{t}[-k'(s)]^{(p-1+\delta_0)}[-k'(s)]^{1-\delta_0}\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\Big]^{\frac{\delta_0}{p-1+\delta_0}}\\ &\le \Big[\int_{t_1}^{t}[-k'(s)]^{p}\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\Big]^{\frac{\delta_0}{p-1+\delta_0}}\\ &\le c\Big[\int_{t_1}^{t}k''(s)\int_{\Gamma_1}|u(t)-u(t-s)|^2\,d\Gamma ds\Big]^{\frac{\delta_0}{p-1+\delta_0}}\\ &\le c[-E'(t)]^{\frac{\delta_0}{p-1+\delta_0}}.\end{aligned}$$

Then, in particular for $\delta_0=1/2$, we find that (3.19) becomes

$$F'(t)\le -mE(t)+c[-E'(t)]^{\frac{1}{2p-1}}.$$

Now, we multiply by $E^\alpha(t)$, with $\alpha=2p-2$, to obtain, using (3.1),

$$(FE^\alpha)'(t)\le F'(t)E^\alpha(t)\le -mE^{1+\alpha}(t)+cE^\alpha(t)[-E'(t)]^{\frac{1}{1+\alpha}}.$$

Then, Young's inequality, with $q=1+\alpha$ and $q'=\frac{1+\alpha}{\alpha}$, gives

$$(FE^\alpha)'(t)\le -mE^{1+\alpha}(t)+\varepsilon E^{1+\alpha}(t)+C_\varepsilon(-E'(t)).$$

Consequently, picking $\varepsilon < m$, we obtain

$$F_0'(t) \le -m' E^{1+\alpha}(t)$$

where $F_0 = FE^\alpha + C_\varepsilon E \sim E$. Hence we have, for some $a_0 > 0$,

$$F_0'(t) \le -a_0 F_0^{1+\alpha}(t)$$

from which we easily deduce that

$$E(t) \le \frac{a}{(a't + a'')^{1/(2p-2)}} \tag{3.21}$$

By recalling that $p < 3/2$ and using (3.21), we find that $\int_0^{+\infty} E(s)ds < +\infty$. Hence, by noting that

$$\int_0^t \int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma ds \le c \int_0^t E(s)ds,$$

estimate (3.19) gives

$$\begin{aligned} F'(t) &\le -mE(t) + c\int_{\Gamma_1} ([-k']^{p\cdot\frac{1}{p}} \circ u)(t)\, d\Gamma \le -mE(t) + c\Big[\int_{\Gamma_1} ([-k']^p \circ u)(t)\, d\Gamma\Big]^{1/p} \\ &\le -mE(t) + c\Big[\int_{\Gamma_1} (k'' \circ u)(t)\, d\Gamma\Big]^{1/p} \le -mE(t) + c[-E'(t)]^{1/p}. \end{aligned}$$

Therefore, repeating the above steps, with $\alpha = p - 1$, we arrive at

$$E(t) \le \frac{a}{(a't + a'')^{1/(p-1)}} = cG^{-1}(c't + c'').$$

(II) The general case: We define

$$I(t) := \int_{t_1}^t \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma \, ds$$

where H_0 is such that (2.4) is satisfied. As in (3.20), we find that $I(t)$ satisfies, for all $t \ge t_1$,

$$I(t) < 1. \tag{3.22}$$

We also assume, without loss of generality that $I(t) \ge b_0 > 0$, for all $t \ge t_1$; otherwise (3.19) yields an exponential decay. In addition, we define $\xi(t)$ by

$$\xi(t) := \int_{t_1}^t k''(s) \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma \, ds$$

and infer from (A2) and the properties of H_0 and D that

$$\frac{-k'(s)}{H_0^{-1}(k''(s))} \le \frac{-k'(s)}{H_0^{-1}(H(-k'(s)))} = \frac{-k'(s)}{D^{-1}(-k'(s))} \le k_0$$

for some positive constant k_0. Then, using (3.1) and choosing t_1 even larger (if needed), one can easily see that $\xi(t)$ satisfies, for all $t \ge t_1$,

$$\begin{aligned} \xi(t) &\le k_0 \int_{t_1}^t k''(s) \int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma ds \\ &\le cE(0) \int_{t_1}^t k''(s) \le -ck'(t_1)E(0) \\ &< \min\{r, H(r), H_0(r)\}. \end{aligned} \tag{3.23}$$

Since H_0 is strictly convex on $(0,r]$ and $H_0(0)=0$, it follows that

$$H_0(\theta x) \le \theta H_0(x),$$

provided $0 \le \theta \le 1$ and $x \in (0,r]$. Using this fact, hypothesis (A2), (2.6), (3.22), (3.23), and Jensen's inequality leads to

$$\begin{aligned}
\xi(t) &= \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(k''(s))] \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma ds \\
&\ge \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(k''(s))] \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma ds \\
&\ge H_0\Big(\frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(k''(s)) \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma ds\Big) \\
&= H_0\Big(-\int_{t_1}^{t} k'(s) \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma ds\Big)
\end{aligned}$$

This implies that

$$-\int_{t_1}^{t} k'(s) \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma ds \le H_0^{-1}(\xi(t))$$

and (3.19) becomes

$$F'(t) \le -mE(t) + cH_0^{-1}(\xi(t)), \quad \forall t \ge t_1. \tag{3.24}$$

Now, for $\varepsilon_0 < r$ and $c_0 > 0$, using (3.24), and the fact that $E' \le 0$, $H_0' > 0, H_0'' > 0$ on $(0,r]$, we find that the functional F_1, defined by

$$F_1(t) := H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) F(t) + c_0 E(t)$$

satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 F_1(t) \le E(t) \le \alpha_2 F_1(t) \tag{3.25}$$

and

$$\begin{aligned}
F_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H_0''(\varepsilon_0 \frac{E(t)}{E(0)}) F(t) + H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) F'(t) + c_0 E'(t) \\
&\le -mE(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) + cH_0'(\varepsilon_0 \frac{E(t)}{E(0)}) H_0^{-1}(\xi(t)) + c_0 E'(t).
\end{aligned} \tag{3.26}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [1, p. 61-64]), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad \text{if } s \in (0, H_0'(r)] \tag{3.27}$$

and H_0^* satisfies the Young's inequality

$$AB \le H_0^*(A) + H_0(B), \quad \text{if } A \in (0, H_0'(r)], B \in (0,r] \tag{3.28}$$

With $A = H_0'\big(\varepsilon_0 \frac{E(t)}{E(0)}\big)$ and $B = H_0^{-1}(\xi(t))$, using (3.1), (3.23) and (3.26)-(3.28), we arrive at

$$\begin{aligned}
F_1'(t) &\le -mE(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) + cH_1^*\Big(H_0'(\varepsilon_0 \frac{E(t)}{E(0)})\Big) + c\xi(t) + c_0 E'(t) \\
&\le -mE(t) H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) + c\varepsilon_0 \frac{E(t)}{E(0)} H_0'(\varepsilon_0 \frac{E(t)}{E(0)}) - cE'(t) + c_0 E'(t).
\end{aligned}$$

Consequently, with a suitable choice of ε_0 and c_0, we obtain, for all $t \geq t_1$,

$$F_1'(t) \leq -\tau\Big(\frac{E(t)}{E(0)}\Big)H_0'\Big(\varepsilon_0\frac{E(t)}{E(0)}\Big) = -\tau H_2(\frac{E(t)}{E(0)}), \tag{3.29}$$

where $H_2(t) = tH_0'(\varepsilon_0 t)$.

Since $H_2'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon_0 t)$, using the strict convexity of H_0 on $(0, r]$, we find that $H_2'(t)$, $H_2(t) > 0$ on $(0, 1]$. Thus, with

$$R(t) = \epsilon\frac{\alpha_1 F_1(t)}{E(0)}, \quad 0 < \epsilon < 1,$$

taking in account (3.25) and (3.29), we have

$$R(t) \sim E(t) \tag{3.30}$$

and, for some $k_0 > 0$,

$$R'(t) \leq -\epsilon k_0 H_2(R(t)), \quad \forall t \geq t_1.$$

Then, a simple integration and a suitable choice of ϵ yield, for some $k_1, k_2 > 0$,

$$R(t) \leq H_1^{-1}(k_1 t + k_2), \quad \forall t \geq t_1, \tag{3.31}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)}ds$.

Here, we have used, based on the properties of H_2, the fact that H_1 is strictly decreasing function on $(0, 1]$ and $\lim_{t\to 0} H_1(t) = +\infty$. A combination of (3.30) and (3.31), estimate (2.3) is established.

Moreover, if $\int_0^1 H_1(t)dt < +\infty$, then

$$\int_0^t \int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma ds \leq c\int_0^t E(s)ds < +\infty.$$

Therefore, we can repeat the same process with

$$I(t) := \int_{t_1}^t \int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma ds,$$

and

$$\xi(t) := \int_{t_1}^t k''(s)\int_{\Gamma_1} |u(t) - u(t-s)|^2 \, d\Gamma ds,$$

to obtain (2.5). □

4. Appendix

Let $0 < q < 1$ and consider

$$k'(t) = -\exp(-t^q).$$

Here, we show how to apply Theorem 2.1 to this specific type of resolvent kernels. First, one can show that $k''(t) = H((-k'(t)))$ where

$$H(t) = \frac{qt}{[\ln(1/t)]^{\frac{1}{q}-1}}.$$

Since

$$H'(t) = \frac{(1-q) + q\ln(1/t)}{[\ln(1/t)]^{1/q}} \quad \text{and} \quad H''(t) = \frac{(1-q)[\ln(1/t) + \frac{1}{q}]}{[\ln(1/t)]^{\frac{1}{q}+1}},$$

then the function H satisfies hypothesis (A2) on the interval $(0, r]$ for any $0 < r < 1$. Also, by taking $D(t) = t^{\alpha}$, (2.4) is satisfied for any $\alpha > 1$. Therefore, an explicit rate of decay can be obtained by Theorem 2.1. The function $H_0(t) = H(t^{\alpha})$ has derivative

$$H_0'(t) = \frac{q\alpha t^{\alpha-1}[\frac{1}{q} - 1 + \ln(1/t^{\alpha})]}{[\ln(1/t^{\alpha})]^{1/q}}$$

Therefore,

$$\begin{aligned}
H_1(t) &= \int_t^1 \frac{[\ln(1/(\varepsilon_0 s)^{\alpha})]^{1/q} q\alpha\varepsilon_0^{\alpha-1} s^{\alpha}[\frac{1}{q} - 1 + \ln(1/(\varepsilon_0 s)^{\alpha})]}{d} s \\
&= \frac{1}{q\alpha^2} \int_{\ln[(\varepsilon_0 t)^{-\alpha}]}^{\ln[\varepsilon_0{}^{-\alpha}]} \frac{u^{1/q} e^{(1-\frac{1}{\alpha})u}}{\frac{1}{q} - 1 + u} du,
\end{aligned}$$

where $u = \ln(1/(\varepsilon_0 s)^{\alpha})$. Using the fact that $(\frac{1}{q} - 1 + u) > (\frac{1}{q} - 1)$ and the function $f(u) = u^{1/q}$ is increasing on $(0, +\infty)$ and taking $\varepsilon_0 < 1$, then

$$\begin{aligned}
H_1(t) &\le \frac{[-\alpha \ln \varepsilon_0 t]^{1/q}}{\alpha^2(1-q)} \int_{-\alpha \ln\, \varepsilon_0 t}^{-\alpha \ln \varepsilon_0} e^{(1-\frac{1}{\alpha})u} du \\
&= \frac{[-\alpha \ln \varepsilon_0 t]^{1/q}[t^{1-\alpha} - 1]}{\alpha(1-q)(\alpha-1)\varepsilon_0{}^{\alpha-1}} = b[-\ln \varepsilon_0 t]^{1/q}[t^{1-\alpha} - 1]
\end{aligned}$$

where $b = \frac{\alpha^{\frac{1}{q}-1}}{(1-q)(\alpha-1)\varepsilon_0^{\alpha}{}^{-1}}$. Next, we find that

$$\begin{aligned}
\int_0^1 H_1(t)dt &\le \int_0^1 b[-\ln \varepsilon_0 t]^{1/q}[t^{1-\alpha} - 1]dt \quad (\text{taking } v = -\ln \varepsilon_0 t) \\
&= \frac{b}{\varepsilon_0} \int_{-\ln \varepsilon_0}^{+\infty} v^{\frac{1}{q}}[\varepsilon_0^{\alpha-1} e^{(\alpha-2)v} - e^{-v}]dv.
\end{aligned}$$

Then, it is easily seen that $\int_0^1 H_1(t)dt < +\infty$ if $(\alpha - 2) < 0$, and so we choose $1 < \alpha < 2$. Therefore, we can use (2.5) to deduce

$$E(t) \le k_3 G^{-1}(k_1 t + k_2)$$

where

$$\begin{aligned}
G(t) &= \int_t^1 \frac{1}{sH'(\varepsilon_0 s)} ds = \int_t^1 \frac{[\ln \frac{1}{\varepsilon_0 s}]^{1/q}}{s[1 - q + q \ln \frac{1}{\varepsilon_0 s}]} ds \\
&= \int_{\ln \frac{1}{\varepsilon_0}}^{\ln \frac{1}{\varepsilon_0 t}} \frac{u^{1/q}}{1 - q + qu} du = \frac{1}{q} \int_{\ln \frac{1}{\varepsilon_0}}^{\ln \frac{1}{\varepsilon_0 t}} u^{\frac{1}{q}-1}[\frac{u}{\frac{1-q}{q} + u}]du \\
&\le \frac{1}{q} \int_{\ln \frac{1}{\varepsilon_0}}^{\ln \frac{1}{\varepsilon_0 t}} u^{\frac{1}{q}-1} du = [\ln \frac{1}{\varepsilon_0 t}]^{1/q} - [\ln \frac{1}{\varepsilon_0}]^{\frac{1}{q}} \\
&\le [\ln \frac{1}{\varepsilon_0 t}]^{1/q}.
\end{aligned}$$

Hence, $G^{-1}(t) \le \frac{1}{\varepsilon_0} \exp(-t^q)$ and the enegy decays at the same rate of g, that is

$$E(t) \le c \exp(-\omega t^q).$$

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