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ENTIRE SOLUTIONS OF FERMAT TYPE q-DIFFERENCE DIFFERENTIAL EQUATIONS

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Abstract. In this article, we describe the finite-order transcendental entire solutions of Fermat type q-difference and q-difference differential equations. In addition, we investigate the similarities and other properties among those solutions.

1. Introduction

Fermat's last theorem [20] states that do not exist nonzero rational numbers x, y and an integer $n \geq 3$ such that $x^n + y^n = 1$. The equation $x^2 + y^2 = 1$ does admit nontrivial rational solutions. Replacing x, y in above equation by entire or meromorphic functions f, g, Fermat type functional equations were studied by Gross [4, 5] and many others thereafter, such as [14, 16]. Yang [16] investigated the Fermat type functional equation

$$a(z)f(z)^n + b(z)g(z)^m = 1, \tag{1.1}$$

where $a(z), b(z)$ are small functions with respect to $f(z)$. Recall that $\alpha(z) \not\equiv 0, \infty$ is a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(T(r, f))$, and $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. In fact, Yang [16, Theorem 1] obtained the following result.

Theorem 1.1. *Let m, n be positive integers satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Then there are no nonconstant entire solutions $f(z)$ and $g(z)$ that satisfy* (1.1).

The above theorem implies that there is no nonconstant entire solutions with the assumption of $n > 2$, $m > 2$ in (1.1). However, when $m = n = 2$ and $g(z)$ has a specific relationship with $f(z)$ in (1.1), the problem that can we obtain the accurate expressions of entire solutions is deserve to be considered. This article is devoted to considering Fermat type functional equations in the cases where an entire function $f(z)$ together with one the following: derivative $f'(z)$, shift $f(z + c)$ or q-difference $f(qz)$. This article is organized as follows. In Section 2, we mainly consider Fermat type q-difference equations. Some similarities or other properties among the Fermat

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functional equations of different types can be found in the section. In Section 3, some results on entire solutions of Fermat type q-difference differential equations are given. In this article, we assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [8, 19].

2. Fermat type q-difference equations

Let us recall some results on Fermat type differential equations. Yang and Li [18] considered the entire solutions of

$$f(z)^2 + f'(z)^2 = 1. \tag{2.1}$$

In fact, they considered a generalization of above equation. The solutions of (2.1) can be described as follows.

Theorem 2.1 ([18, Theorem 1])**.** *Transcendental meromorphic solutions of* (2.1) *satisfy* $f(z) = \frac{1}{2}\left(Pe^{-iz} + \frac{1}{P}e^{iz}\right)$*, where* P *is a nonzero constant.*

Let $e^A = P$ in Theorem 2.1. The solutions of (2.1) also can be written as $f(z) = \sin(z + Ai + \frac{\pi}{2})$. In addition, they obtained the following result [18, Theorem 3].

Theorem 2.2. *Let* a_1*,* a_2*,* a_3 *be nonzero meromorphic functions in the complex plane* $\mathbb{C}$*. Then a necessary condition for the differential equation*

$$a_1 f^2 + a_2 (f')^2 = a_3 \tag{2.2}$$

to have a transcendental meromorphic solution satisfying $T(r, a_k) = S(r, f)$*,* $k = 1, 2, 3$*, is* $\frac{a_3}{a_1} \equiv a$*, where* a *is a constant.*

Here, we will show the different properties among results on the existence of Fermat type differential equations, difference equations and q-difference equations. If we replace f' with $f(z+c)$ in (2.2), Theorem 2.2 is not valid for difference equation

$$a_1 f^2 + a_2 f(z+c)^2 = a_3.$$

For example, the equation

$$(z+c)^2 f(z)^2 + z^2 f(z+c)^2 = z^2 (z+c)^2$$

has an entire solution $f(z) = z \sin z$ with the order $\rho(f) = 1$, where $c = \frac{\pi}{2}$. Here $\frac{a_3}{a_1} = z^2$ is not a constant.

Theorem2.2 is not valid for the q-difference equation

$$a_1 f^2 + a_2 f(qz)^2 = a_3. \tag{2.3}$$

The equation $f(z)^2 + f(-z)^2 = z^2$ has a transcendental entire solution

$$f(z) = \frac{z e^{z + \frac{\pi}{4} i} + z e^{-z - \frac{\pi}{4} i}}{2}, \tag{2.4}$$

here $\frac{a_3}{a_1} = z^2$. A natural question is that how to describe the entire solutions of Fermat type difference equations or q-difference equations. The difference analogue of the logarithmic derivative lemma [3, 6] for meromorphic function with finite order has been developed to study difference equations [6, 7], also can be used to consider Fermat type difference equations [10, 11, 12]. One of the results can be stated as follows.

Theorem 2.3 ([11, Theorem 1.1]). *The transcendental entire solutions with finite order of $f(z)^2 + f(z+c)^2 = 1$ satisfy $f(z) = \sin(Az + B)$, where B is a constant and $A = \frac{(4k+1)\pi}{2c}$, k is an integer, c is a nonzero constant.*

Li [9] considered the meromorphic solutions of $f^2 + a(f')^2 = 1$, where a is a nonzero function. Tang and Liao [15] investigated the entire solutions of a generalization of (2.1) as follows

$$f(z)^2 + P(z)^2 f^{(k)}(z)^2 = Q(z), \tag{2.5}$$

where $P(z), Q(z)$ are nonzero polynomials and obtained the following result.

Theorem 2.4 ([15, Theorem 1]). *Let $P(z)$, $Q(z)$ be nonzero polynomials. If (2.5) has a transcendental meromorphic solution $f(z)$, then $P(z) \equiv A$, $Q \equiv B$, $k = 2n+1$ for some nonnegative integer n and $f(z) = b\sin(az + c)$, where a, b, c are constants such that $Aa^k = \pm 1$, $b^2 = B$.*

In a recent article [13], Liu considered an improvement of Theorem 2.3 and obtained the following result which can be seen as the difference analogue of Theorem 2.4.

Theorem 2.5 ([13, Theorem 2.1]). *Let $P(z)$, $Q(z)$ be nonzero polynomials. If the difference equation*

$$f(z)^2 + P(z)^2 f(z+c)^2 = Q(z) \tag{2.6}$$

admits a transcendental entire solution of finite order, then $P(z) \equiv \pm 1$ and $Q(z)$ reduces to a constant q. Thus, $f(z) = \sqrt{q}\sin(Az + B)$, where B is a constant and $A = \frac{(4k+1)\pi}{2c}$, k is an integer, c is a nonzero constant.

In [1, Theorem1.1], a q-difference analogue of the logarithmic derivative lemma was given. Similarly, as the finite-order solutions play a key role in complex difference equations, solutions of order zero are in focus for q-difference equations. A natural idea is how to describe the entire solutions with zero order of Fermat type q-difference equations. However, we will consider the entire solutions with finite order, not limited to zero order in the following, we mainly study the entire solutions of Fermat type q-difference equations

$$f(z)^2 + P(z) f(qz)^2 = Q(z), \tag{2.7}$$

where $P(z), Q(z)$ are nonzero polynomials, and $|q| = 1$. We obtain the following result.

Theorem 2.6. *Let $P(z)$, $Q(z)$ be nonzero polynomials and $|q| = 1$. If the q-difference equation*

$$f(z)^2 + P(z)^2 f(qz)^2 = Q(z) \tag{2.8}$$

admits a transcendental entire solution of finite order, then $P(z)$ must be a constant P. This solution can be written as

$$f(z) = \frac{Q_1(z)e^{p(z)} + Q_2(z)e^{-p(z)}}{2}$$

satisfying one of the following conditions:

- (i) *q satisfies $p(qz) = p(z)$ and $Q_1(z) - iPQ_1(qz) \equiv 0$, $Q_2(z) + iPQ_2(qz) \equiv 0$, $P^4Q(q^2z) = Q(z)$;*
- (ii) *q satisfies $p(qz) + p(z) = 2a_0$, and $iPQ_1(qz)e^{2a_0} \equiv -Q_2(z)$, $iPQ_2(qz) \equiv Q_1(z)e^{2a_0}$, $P^4Q(q^2z) = Q(z)$, $e^{8a_0} = 1$,*

where $Q(z) = Q_1(z)Q_2(z)$ and $p(z)$ is a nonconstant polynomial.

Before giving some examples, we want to explain why we assume the condition $|q| = 1$. From the proof of Theorem 2.6 blew, it is easy to find that the entire solutions with zero order of (2.8) must be polynomials. In addition, Bergweiler, Ishizaki and Yanagihara [2] considered the linear q-difference equation

$$\sum_{j=0}^{n} a_j(z)f(c^j z) = Q(z), \tag{2.9}$$

where $Q(z)$ and $a_j(z)$ $(j = 0, 1, \dots, n)$ are polynomials without common zeros, $a_n(z)a_0(z) \neq 0$ and $0 < |c| < 1$. They proved that any meromorphic solution $f(z)$ of (2.9) satisfies $T(r, f) = O((\log r)^2)$, which implies that the order of $f(z)$ is zero. Thus, assume that $f(z)$ is an entire solution of (2.8). Let $F(z) = f(z)^2$. If $|q| < 1$, then (2.8) changes into $F(z) + P(z)^2F(qz) = 1$, we have $\rho(F) = 0$, implies that $\rho(f) = 0$, thus $f(z)$ should be a polynomial. If $|q| > 1$, thus $|\frac{1}{q}| < 1$, we assume that $G(z) = f(qz)^2$. Thus, (2.8) takes into $G(\frac{1}{q}z) + P(z)^2G(z) = 1$. Using the observation (see [2, p. 2]), we have $T(r, f(z)) \le T(|q|r, f(z)) = T(r, f(qz)) + O(1) = \frac{1}{2}T(r, G(z)) + O(1) = O((\log r)^2) + O(1)$. We also get $\rho(f) = 0$, thus $f(z)$ should be a polynomial. So, when considering the transcendental entire solution of (2.8), we need the condition $|q| = 1$.

Example 2.7. If $P(z) \equiv i$, $q = -1$ and $Q(z) \equiv z^3$, then $f(z)^2 - f(-z)^2 = z^3$ has a transcendental entire solution $f(z) = \frac{ze^{p(z)} + z^2e^{-p(z)}}{2}$, where $p(z)$ is a polynomial satisfies $p(-z) = p(z)$, which implies that we can assume that $p(z) = a_{2n}z^{2n} + a_{2n-2}z^{2(n-1)} + \dots + a_2z^2 + a_0$. Thus, $\rho(f) = 2n$.

Example 2.8. If $P(z) \equiv 1$, $q = -i$ and $Q(z) \equiv z^4$, then $f(z)^2 + f(-iz)^2 = z^4$ has a transcendental entire solution $f(z) = \frac{ze^{p(z)} + z^3e^{-p(z)}}{2}$, where $p(z)$ is a polynomial satisfies $p(-iz) = p(z)$, which implies that we can assume that $p(z) = a_{4n}z^{4n} + a_{4n-4}z^{4(n-1)} + \dots + a_4z^4 + a_0$. Thus, $\rho(f) = 4n$.

Example 2.9. If $P(z) \equiv -1$, $q = -1$ and $Q(z) \equiv z^2$, then $f(z)^2 + f(-z)^2 = z^2$ has a transcendental entire solution $f(z) = \frac{ze^{p(z)} + ze^{-p(z)}}{2}$, where $p(z)$ is a polynomial satisfies $p(-z) + p(z) = 4ki\pi + \frac{i\pi}{2}$, which implies that we can assume that $p(z) = a_{2n+1}z^{2n+1} + a_{2n-1}z^{2n-1} + \dots + a_1z + 2ki\pi + \frac{i\pi}{4}$, where $k \in \mathbb{Z}$. Thus, $\rho(f) = 2n+1$. We also remark that (2.4) is a solution of above equation.

Remark 2.10. From Examples 2.7–2.9, we find that the order of entire solutions of q-difference equation (2.8) can be large enough which is different from the growth of entire solutions on (2.5) and (2.6).

Corollary 2.11. *If $P(z)$, $Q(z)$ are nonconstant polynomials, then there does not exist transcendental entire solutions of finite order of q-difference equation*

$$f(z)^2 + P(z)^2f(qz)^2 = Q(z). \tag{2.10}$$

Theorem 2.12. *Let $P(z)$, $Q(z)$ be nonzero polynomials. Then the q-difference equation*

$$f(z)^2 + zP(z)^2f(qz)^2 = Q(z) \tag{2.11}$$

has no transcendental entire solutions of finite order.

Proof. Assume that $f(z)$ is a transcendental entire solutions of finite order. Let $z = z_1^2$ and $F(z_1) = f(z_1^2)$, $M(z_1) = P(z_1^2)$, $N(z_1) = Q(z_1^2)$. Then $F(z_1)$ is entire function in z_1, and $M(z_1)$, $N(z_1)$ are polynomials in z_1. Thus, from (2.11), we have the following equation

$$F(z_1)^2 + z_1^2 M(z_1)^2 F(\sqrt{q}z_1)^2 = N(z_1) \tag{2.12}$$

From Theorem 2.6, we obtain $z_1^2 M(z_1)^2$ reduce to a constant, which is impossible. □

If we replace $f(qz)$ with $f(qz) - f(z)$, then we obtain the following result.

Theorem 2.13. *If $P(z)$ is a nonzero polynomial with nonzero constant term, then there is no finite order entire solution $f(z)$ satisfying*

$$f(z)^2 + P(z)^2[f(qz) - f(z)]^2 = Q(z). \tag{2.13}$$

The following result plays an important part in the proofs of our theorems.

Lemma 2.14 ([19, Theorem 1.62])**.** *Let $f_j(z)$ be meromorphic functions, $f_k(z)$ be not constant functions ($k = 1, 2, \dots, n-1$), satisfying $\sum_{j=1}^{n} f_j = 1$ and $n \geq 3$. If $f_n(z) \not\equiv 0$ and*

$$\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda < 1$ and $k = 1, 2, \dots, n-1$, then $f_n(z) \equiv 1$.

Lemma 2.15. *Let $p(z)$ be a nonzero polynomial with degree n. If $p(qz) - p(z)$ is a constant, then $q^n = 1$ and $p(qz) \equiv p(z)$. If $p(qz) + p(z)$ is a constant, then $q^n = -1$ and $p(qz) + p(z) \equiv 2a_0$, where a_0 is the constant term of $p(z)$.*

Proof. Assume that $p(z) = a_n z^n + \dots + a_1 z + a_0$. Then

$$p(qz) - p(z) = a_n(qz)^n + \dots + a_1 qz - a_n z^n + \dots - a_1 z,$$

we obtain $a_n(qz)^n = a_n z^n$, which implies that $q^n = 1$.

From $p(qz) + p(z) = a_n(qz)^n + \dots + a_1 qz + a_n z^n + \dots + a_1 z + 2a_0$, we obtain $a_n(qz)^n + a_n z^n = 0$, which implies that $q^n = -1$. We have completed the proof. □

Proof of Theorem 2.6. Assume that $f(z)$ is a transcendental entire solution of finite order of (2.8), then

$$[f(z) + iP(z)f(qz)][f(z) - iP(z)f(qz)] = Q(z). \tag{2.14}$$

Thus, both $f'(z) + iP(z)f(qz)$ and $f'(z) - iP(z)f(qz)$ have finitely many zeros. Combining (2.14) with the Hadamard factorization theorem, we assume that

$$f(z) + iP(z)f(qz) = Q_1(z)e^{p(z)}$$

and

$$f(z) - iP(z)f(qz) = Q_2(z)e^{-p(z)},$$

where $p(z)$ is a nonconstant polynomial, otherwise $f(z)$ is a polynomial, and $Q(z) = Q_1(z)Q_2(z)$, where $Q_1(z)$, $Q_2(z)$ are nonzero polynomials. Thus, we have

$$f(z) = \frac{Q_1(z)e^{p(z)} + Q_2(z)e^{-p(z)}}{2} \tag{2.15}$$

and

$$f(qz) = \frac{Q_1(z)e^{p(z)} - Q_2(z)e^{-p(z)}}{2iP(z)}. \tag{2.16}$$

Combining (2.15) with (2.16), we obtain

$$f(qz) = \frac{Q_1(qz)e^{p(qz)} + Q_2(qz)e^{-p(qz)}}{2} = \frac{Q_1(z)e^{p(z)} - Q_2(z)e^{-p(z)}}{2iP(z)}.$$

Thus,

$$\frac{iP(z)Q_1(qz)e^{p(qz)+p(z)}}{-Q_2(z)} + \frac{iP(z)Q_2(qz)e^{p(z)-p(qz)}}{-Q_2(z)} + \frac{Q_1(z)e^{2p(z)}}{Q_2(z)} = 1. \tag{2.17}$$

Since $p(z)$ is not a constant, then $p(qz) - q(z)$ and $p(qz) + q(z)$ are not constants simultaneously. From Lemma 2.14, we have one of $p(qz) - q(z)$ and $p(qz) + q(z)$ must be a constant. The following, we will discuss two cases.

Case 1. Suppose that $p(qz) - q(z)$ is a constant. From Lemma 2.15, then $p(qz) - p(z) \equiv 0$. It implies that $q^n = 1$. Thus,

$$(Q_1(z) - iP(z)Q_1(qz))e^{2p(z)} = Q_2(z) + iP(z)Q_2(qz)$$

follows from (2.17). Since $p(z)$ is not a constant, then we have

$$Q_1(z) - iP(z)Q_1(qz) \equiv 0,$$

and

$$Q_2(z) + iP(z)Q_2(qz) \equiv 0.$$

Thus, $P(z)$ must be a constant P. From above two equations, we have $P^4Q(q^2z) = Q(z)$, where $Q(z) = Q_1(z)Q_2(z)$.

Case 2. Suppose that $p(qz) + p(z)$ is a constant. Thus, from Lemma 2.15, we obtain

$$iP(z)Q_1(qz)e^{2a_0} \equiv -Q_2(z). \tag{2.18}$$

From (2.17), we obtain

$$iP(z)Q_2(qz) \equiv Q_1(z)e^{2a_0}. \tag{2.19}$$

Combining (2.18) with (2.19), we have

$$P(z)P(qz)Q_2(q^2z) \equiv Q_2(z), \tag{2.20}$$

which implies that $P(z)$ must be a constant P. We also have

$$P(z)P(qz)Q_1(q^2z) \equiv Q_1(z). \tag{2.21}$$

Thus, from (2.20) and (2.21), we have $P^4Q(q^2z) = Q(z)$ and

$$\frac{Q_1(q^2z)}{Q_2(q^2z)} = \frac{Q_1(z)}{Q_2(z)}.$$

Combining (2.18) with (2.19), we also have

$$\frac{Q_1(qz)}{Q_2(qz)}e^{4a_0} = -\frac{Q_2(z)}{Q_1(z)}.$$

Hence, $\frac{Q_1(q^2z)}{Q_2(q^2z)}e^{4a_0} = -\frac{Q_2(qz)}{Q_1(qz)} = \frac{1}{e^{4a_0}}\frac{Q_1(z)}{Q_2(z)}$, thus $e^{8a_0} = 1$. □

For the proof of Theorem 2.13, we need the following two results.

Lemma 2.16. *Let $Q(z)$ be a nonzero polynomial, $|t| \neq 1$. If $Q(qz) \equiv tQ(z)$, then $Q(z)$ should be reduce to a monomial and $Q(z) = a_nz^n$.*

Proof. Obviously, we have $Q(0) = 0$. Assume that

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z,$$

where $a_n \neq 0$. Then,

$$Q(qz) = a_n (qz)^n + a_{n-1}(qz)^{n-1} + a_{n-2}(qz)^{n-2} + \dots + a_1 qz.$$

Since $Q(qz) \equiv tQ(z)$, then $a_n(qz)^n = ta_n z^n$, thus $q^n = t$. From $a_{n-1}(qz)^{n-1} = ta_{n-1}z^{n-1}$, we have either $a_{n-1} = 0$ or $q^{n-1} = t$. If $q^{n-1} = t$, combining with $q^n = t$, we have $q = 1$, which is impossible. Thus, $a_{n-1} = 0$. From $a_{n-2}(qz)^{n-2} = ta_{n-2}z^{n-2}$, then $a_{n-2} = 0$ or $q^{n-2} = t$. If $q^{n-2} = t$, combining with $q^n = t$, we have $q^2 = 1$, hence $t^2 = q^{2n} = 1$, thus $|t| = 1$, a contradiction with the condition. Using this method, we can get $a_j = 0 (j = 1, 2, \dots, n-1)$. □

Lemma 2.17. *Let $P(z)$ be a polynomial with nonzero constant term. If $P(z)$ and $Q(z)$ satisfy $P(z)^2 Q(qz) = (1+P(z)^2)Q(z)$, then $P(z)$ must be a constant and $Q(z)$ must be a monomial.*

Proof. Obviously, $Q(0) = 0$. Assume that $P(z)^2 = b_m z^m + b_{m-1}z^{m-1} + \dots + b_1 z + b_0$, where $b_0 \neq 0$ and $Q(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_1 z$. From $P(z)^2 Q(qz) = (1 + P(z)^2)Q(z)$, if $P(z)^2 \equiv b_0$, we obtain $Q(qz) = \frac{1+b_0}{b_0} Q(z)$, from Lemma 2.16, we obtain $Q(z)$ should be reduce to a monomial $Q(z) = a_n z^n$. If $P(z)^2$ is not a constant, that is $b_m \neq 0$, then $P(z)^2 Q(qz) = (1 + P(z)^2)Q(z)$, thus $b_m q^n z^{n+m} = b_m z^{n+m}$, $q^n = 1$ follows, and $b_0 q^n z^n = (1 + b_0)z^n$, we obtain $b_0 = (1 + b_0)$ which is a contradiction. □

Proof of Theorem 2.13. Assume that $f(z)$ is a transcendental entire solution of finite order of (2.13). Similar idea as the beginning of the proof of Theorem 2.6, we can obtain

$$f(z) = \frac{Q_1(z)e^{h(z)} + Q_2(z)e^{-h(z)}}{2} \tag{2.22}$$

and

$$f(qz) - f(z) = \frac{Q_1(z)e^{h(z)} - Q_2(z)e^{-h(z)}}{2iP(z)}, \tag{2.23}$$

where $h(z)$ is a nonconstant polynomial with $\deg(h(z)) = s$. Thus, from the expression of $f(qz)$, we have

$$\frac{Q_1(qz)e^{h(qz)} + Q_2(qz)e^{-h(qz)}}{2} = \frac{Q_1(z)(1 + P(z)i)e^{h(z)} - Q_2(z)(1 - P(z)i)e^{-h(z)}}{2iP(z)}. \tag{2.24}$$

Hence, we obtain

$$\frac{Q_1(z)(1 + P(z)i)}{iQ_2(qz)P(z)} e^{h(z)+h(qz)} - \frac{Q_2(z)(1 - P(z)i)}{iQ_2(qz)P(z)} e^{h(qz)-h(z)} - \frac{Q_1(qz)}{Q_2(qz)} e^{2h(qz)} = 1. \tag{2.25}$$

If $P(z) \equiv i$, from (2.25),

$$\frac{2Q_2(z)}{Q_2(qz)} e^{h(qz)-h(z)} - \frac{Q_1(qz)}{Q_2(qz)} e^{2h(qz)} = 1 \tag{2.26}$$

follows. It is easy to find that $2h(qz)$ and $h(qz) - q(z)$ are not constants simultaneously. Thus, the equation (2.26) is impossible. If $P(z) \equiv -i$, from (2.25), we have $\frac{2Q_1(z)}{Q_2(qz)} e^{h(qz)+h(z)} - \frac{Q_1(qz)}{Q_2(qz)} e^{2h(qz)} = 1$, and $2h(qz)$ and $h(qz) + q(z)$ are not constants simultaneously, above equation also is impossible.

Thus, we have $P(z) \not\equiv \pm i$. Since that $2h(z)$ is not a constant, then we discuss two cases from Lemma 2.14 in the following:

Case 1. Suppose that $h(qz) - h(z)$ is a constant, which implies that $h(qz) = h(z)$ and $q^s = 1$. Thus,

$$Q_1(z)(1 + P(z)i) \equiv iQ_1(qz)P(z)$$

and

$$Q_2(z)(1 - P(z)i) \equiv -iQ_2(qz)P(z).$$

From above two equations, we have

$$Q(z)(1 + P(z)^2) \equiv Q(qz)P(z)^2. \tag{2.27}$$

Thus, we have $Q(0) = 0$. If $P(z)$ is a constant, then equation (2.27) can be written as $Q(qz) \equiv \frac{1+P^2}{P^2}Q(z)$, where $|\frac{1+P^2}{P^2}| \neq 1$. Using Lemma 2.16, we obtain $Q(z) = a_nz^n$. Thus, $a_n(1+P^2) = a_nq^nP^2$, which implies that $|q| \neq 1$, a contradiction with $q^s = 1$. If $P(z)$ is a polynomial with nonzero constant term, from Lemma 2.17, which is impossible.

Case 2. Suppose that $h(qz) + h(z)$ is a constant $2a_0$. We have

$$Q_1(z)(1 + iP(z))e^{2a_0} \equiv iQ_2(qz)P(z),$$

and

$$Q_2(z)(1 - iP(z)) \equiv -iQ_1(qz)P(z)e^{2a_0}$$

From above two equations, we have

$$Q(z)(1 + P(z)^2) \equiv Q(qz)P(z)^2.$$

Similar discussions as the Case 1, we can get a contradiction. Thus, we have completed the proof of Theorem 2.13. □

3. Fermat type q-difference differential equations

If an equation includes some of $f(z)$, $f(qz)$, $f^{(k)}(z)$, $f(z+c)$, then this equation can be called q-difference differential equation. Yang and Laine [17] considered entire solutions of difference-differential equation. Liu, Cao and Cao have considered Fermat type differential-difference equation in [11, Theorem 1.3], and obtained the following result.

Theorem 3.1. *Transcendental entire solutions with finite order of the differential-difference equation*

$$f'(z)^2 + f(z+c)^2 = 1 \tag{3.1}$$

satisfy $f(z) = \sin(z \pm Bi)$*, where* B *is a constant and* $c = 2k\pi$ *or* $c = 2k\pi + \pi$*,* k *is an integer.*

Next we will consider entire solutions for Fermat type q-difference differential equation, such as

$$f'(z)^2 + f(qz)^2 = 1, \tag{3.2}$$

$$f(z+c)^2 + f(qz)^2 = 1, \tag{3.3}$$

$$f'(z+c)^2 + f(qz)^2 = 1. \tag{3.4}$$

Using a method similar to the one in this paper, we can consider some generalizations of above equations, such as $f'(z)^2 + P(z)^2f(qz)^2 = Q(z)$, but they will not be considered here. We also find that the methods for the proofs of the following two theorems are similar, so we just give the details of proof of Theorem 3.3.

Theorem 3.2. *The transcendental entire solutions with finite order of* (3.2) *satisfy* $f(z) = \sin(z + B)$ *when* $q = 1$*, and* $f(z) = \sin(z + k\pi)$ *or* $f(z) = -\sin(z + k\pi + \frac{\pi}{2})$ *when* $q = -1$*. There are no transcendental entire solutions with finite order when* $q \neq \pm 1$*.*

Theorem 3.3. *The transcendental entire solutions with finite order of* (3.4) *satisfy* $f(z) = \sin(z - Ai - c)$*, where* $2A - ic = 2ki\pi$ *or* $f(z) = \sin(z - Ai + c)$*, where* $2A + ic = 2ki\pi + i\pi$ *when* $q = -1$*, and* $f(z) = \sin(-z - Ai)$*,* $c = 2i\pi$ *or* $f(z) = \sin(-z - Ai + \pi)$*,* $c = 2k\pi + \pi$ *when* $q = 1$*.*

Proof. As in the beginning of the proof of Theorem 2.6, we have

$$f'(z + c) = \frac{e^{p(z)} + e^{-p(z)}}{2} \tag{3.5}$$

and

$$f(qz) = \frac{e^{p(z)} - e^{-p(z)}}{2i}. \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$\frac{p'(z + \frac{c}{q})e^{p(z+\frac{c}{q})+p(qz)}}{iq} + \frac{p'(z + \frac{c}{q})e^{p(qz)-p(z+\frac{c}{q})}}{iq} - e^{2p(qz)} = 1. \tag{3.7}$$

From Lemma 2.14, if $p(z + \frac{c}{q}) + p(qz) = B$, then we have $\frac{p'(z+\frac{c}{q})e^{B}}{iq} = 1$ and $\frac{p'(z+\frac{c}{q})e^{-B}}{iq} = 1$. Thus $e^{2B} = 1$. If $e^B = 1$, then $p(z) = iqz + A - ic$, where A is a constant. Thus, from $p(z + \frac{c}{q}) + p(qz) = 2ki\pi$, we have $q = -1$ and $2A - ic = 2ki\pi$, k is an integer. Hence $p(z) = -iz + A - ic$, and

$$f(z) = \frac{e^{iz+A-ic} - e^{-iz-A+ic}}{2i} = \sin(z - Ai - c),$$

where $2A - ic = 2ki\pi$. If $e^B = -1$, then $p(z) = -iqz + A + ic$, where A is a constant. Thus, from $p(z + \frac{c}{q}) + p(qz) = 2ki\pi + i\pi$, we have $q = -1$ and $2A + ic = 2ki\pi + i\pi$, k is an integer. Hence $p(z) = -iz + A + ic$, and

$$f(z) = \frac{e^{iz+A+ic} - e^{-iz-A-ic}}{2i} = \sin(z - Ai + c),$$

where $2A + ic = 2ki\pi + i\pi$.

If $p(qz) - p(z + \frac{c}{q}) = D$, where D is a constant. Then we have $\frac{p'(z+\frac{c}{q})e^{D}}{iq} = 1$ and $\frac{p'(z+\frac{c}{q})e^{-D}}{iq} = 1$. If $e^D = 1$, thus $p(z) = iqz + A - ic$, where A is a constant. Thus, we have $q = 1$ and $c = 2k\pi$, k is an integer. Hence $p(z) = iz + A - 2ki\pi$, and

$$f(z) = \frac{e^{iz+A-2ki\pi} - e^{-iz-A+2ki\pi}}{2i} = \sin(-z - Ai).$$

If $e^D = -1$, thus $p(z) = -iqz + A + ic$, where A is a constant. Thus, we have $q = 1$ and $c = 2k\pi + \pi$, k is an integer. Hence $p(z) = -iz + A - 2ki\pi - i\pi$, and

$$f(z) = \frac{e^{-iz+A-2ki\pi-i\pi} - e^{iz-A+2ki\pi+i\pi}}{2i} = \sin(-z - Ai + \pi).$$

□

Remark 3.4. Obviously, Theorem 3.2 is an improvement of Theorem 2.1. However, it may be difficult to give all entire solutions of (3.3). Because, we can not get the precise expression of $p(z)$ satisfying $p(z+\frac{c}{q})-p(qz)=B$ or $p(z+\frac{c}{q})+p(qz)=B$. If $c=0$, it is the special case of Theorem 2.6. If $q=1$, it is the case of Theorem 2.3. Here, we can construct entire solutions of (3.3). For example, if $q=-1$, $c=\frac{\pi}{2}$, thus $f(z)=\sin z$ satisfies $f(z+\frac{\pi}{2})^2+f(-z)^2=1$. If $q=\frac{1+i\sqrt{3}}{2}$, $c=\frac{1-i\sqrt{3}}{2}$, and $p(z)=\frac{1}{3}z^3+z^2+z+\frac{3i}{4}\pi+\frac{1}{3}+ki\pi$, thus $p(z+\frac{c}{q})+p(qz)=\frac{3i\pi}{2}+2ki\pi$ and k is an integer. Thus, $f(z)=\frac{e^{p(z-\frac{1-i\sqrt{3}}{2})}-e^{-p(z-\frac{1-i\sqrt{3}}{2})}}{2}$ satisfies $f(z+\frac{1-i\sqrt{3}}{2})^2+f(\frac{1+i\sqrt{3}}{2}z)^2=1$.

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References

[1] D. Barnett, R. G. Halburd, R. J. Korhonen, W. Morgan; *Applications of Nevanlinna theory to q-difference equations,* Proc. Roy. Soc. Edin., Sect. A, Math. **137** (2007), 457–474.
[2] W. Bergweiler, K. Ishizaki, N. Yanagihara; *Meromorphic solutions of some functional equations,* Methods Appl. Anal. **5** (1998), 248–258. Correction: Methods Appl. Anal. **6**(4) (1999).
[3] Y. M. Chiang, S. J. Feng; *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane,* Ramanujan. J. **16** (2008), 105–129.
[4] F. Gross; *On the equation $f^n+g^n=1$*, Bull. Amer. Math. Soc. **72** (1966), 86–88.
[5] F. Gross; *On the equation $f^n+g^n=h^n$,* Amer. Math. Monthy. **73** (1966), 1093–1096.
[6] R. G. Halburd, R. J. Korhonen; *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations,* J. Math. Anal. Appl. **314** (2006), 477–487.
[7] R. G. Halburd, R. J. Korhonen; *Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations*, J. Phys. A. **40** (2007), 1–38.
[8] W. K. Hayman; *Meromorphic Functions*, Oxford at the Clarendon Press, 1964.
[9] B. Q. Li; *On certain non-linear differential equations in complex domains*, Arch. Math. **91** (2008), 344–353.
[10] K. Liu; *Meromorphic functions sharing a set with applications to difference equations*, J. Math. Anal. Appl. **359** (2009), 384–393.
[11] K. Liu, T. B. Cao, H. Z. Cao; *Entire solutions of Fermat type differential-difference equations*, Arch. Math. **99** (2012), 147–155.
[12] K. Liu, L. Z. Yang, X. L. Liu; *Existence of entire solutions of nonlinear difference equations,* Czech. Math. J. **61** (2) (2011), 565–576.
[13] K. Liu; *On entire solutions of some differential-difference equations,* submitted.
[14] P. Montel; *Leçons sur les familles normales de fonctions analytiques et leurs applications,* Gauthier-Villars, Paris, (1927), 135–136. (French)
[15] J. F. Tang, L. W. Liao; *The transcendental meromorphic solutions of a certain type of nonlinear differential equations*, J. Math. Anal. Appl. **334** (2007), 517–527.
[16] C. C. Yang; *A generalization of a theorem of P. Montel on entire functions,* Proc. Amer. Math. Soc. **26** (1970), 332–334.
[17] C. C. Yang, I. Laine; *On analogies between nonlinear difference and differential equations*, Proc. Japan Acad., Ser. A. **86** (2010), 10–14.
[18] C. C. Yang, P. Li; *On the transcendental solutions of a certain type of nonlinear differential equations*, Arch. Math. **82** (2004), 442–448.
[19] C. C. Yang, H. X. Yi; *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers (2003).
[20] A. Wiles; *Modular elliptic curves and Fermats last theorem*, Ann. Math. **141** (1995), 443–551.

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