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RIESZ BASES GENERATED BY THE SPECTRA OF STURM-LIOUVILLE PROBLEMS

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Abstract. Let $\{\lambda_n^2\}_{n=0}^{\infty}$ be the spectra of a Sturm-Liouville problem on $[0,\pi]$. We investigate the question: Do the systems $\{\cos(\lambda_n x)\}_{n=0}^{\infty}$ or $\{\sin(\lambda_n x)\}_{n=0}^{\infty}$ form Riesz bases in $L^2[0,\pi]$? The answer is almost always positive.

1. Introduction and statement of results

Let $\mu_n = \lambda_n^2(q,\alpha,\beta)$, $n = 0,1,2,\dots$ be the eigenvalues of the Sturm-Liouville boundary-value problem $L(q,\alpha,\beta)$

$$-y'' + q(x)y = \mu y, \quad x \in (0,\pi),\ \mu \in \mathbb{C}, \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi], \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0,\pi), \tag{1.3}$$

where $q \in L^1_R[0,\pi]$, that is q is a real, summable on $[0,\pi]$ function. In the simplest case, when $q(x) = 0$ almost everywhere (a.e.) on $[0,\pi]$, eigenfunctions of the problem $L(0,\alpha,\beta)$, which satisfy the initial conditions $y(0) = \sin\alpha$, $y'(0) = -\cos\alpha$, have the form

$$\varphi_n^0(x,\alpha,\beta) = \cos(\lambda_n(0,\alpha,\beta)x)\sin\alpha - \frac{\sin(\lambda_n(0,\alpha,\beta)x)}{\lambda_n(0,\alpha,\beta)}\cos\alpha, \quad n = 0,1,2,\dots \tag{1.4}$$

and form an orthogonal basis in $L^2[0,\pi]$. Here rises a natural question: Do the systems of functions $\{\cos(\lambda_n(0,\alpha,\beta)x)\}_{n=0}^{\infty}$ and $\{\sin(\lambda_n(0,\alpha,\beta)x)\}_{n=0}^{\infty}$ separately form basis in $L^2[0,\pi]$? Examples show, that the answer is not always positive and depends on α and β. When $\alpha = \beta = \frac{\pi}{2}$, then $\lambda_n(0,\frac{\pi}{2},\frac{\pi}{2}) = n$, $n = 0,1,2,\dots$ and the system $\{\cos(\lambda_n(0,\frac{\pi}{2},\frac{\pi}{2})x)\}_{n=0}^{\infty} = \{\cos(nx)\}_{n=0}^{\infty}$ forms an orthogonal basis, but the system $\{\sin(\lambda_n(0,\frac{\pi}{2},\frac{\pi}{2})x)\}_{n=0}^{\infty} = \{0\} \cup \{\sin(nx)\}_{n=1}^{\infty}$ is not a basis because of the "unnecessary" member $\sin(0x) \equiv 0$. However, throwing away this "unnecessary" member, we obtain an orthogonal basis $\{\sin(nx)\}_{n=1}^{\infty}$. In the case of $\alpha = \pi$, $\beta = 0$ (see below section 2), $\lambda_n(0,\pi,0) = n+1$, $n = 0,1,2,\dots$ and the system $\{\sin(\lambda_n(0,\pi,0)x)\}_{n=0}^{\infty} = \{\sin((n+1)x)\}_{n=0}^{\infty}$ forms an orthogonal basis, but the system $\{\cos(\lambda_n(0,\pi,0)x)\}_{n=0}^{\infty} = \{\cos((n+1)x)\}_{n=0}^{\infty}$ is not complete in $L^2[0,\pi]$, there is a lack of constant, but adding it, thus, taking the system

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$\{1\} \cup \{\cos((n+1)x)\}_{n=0}^{\infty} = \{\cos(nx)\}_{n=0}^{\infty}$ we obtain a basis in $L^2[0,\pi]$. The question that we want to answer in this paper is the following: Do the systems $\{\cos(\lambda_n(q,\alpha,\beta)x)\}_{n=0}^{\infty}$ and $\{\sin(\lambda_n(q,\alpha,\beta)x)\}_{n=0}^{\infty}$ form Riesz bases in $L^2[0,\pi]$? The answer we formulate in theorems 1.1 and 1.2 below.

Theorem 1.1. *The system of functions $\{\cos(\lambda_n(q,\alpha,\beta)x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0,\pi]$ for each triple $(q,\alpha,\beta) \in L^1_R[0,\pi] \times (0,\pi] \times [0,\pi)$, except one case: when $\alpha = \pi$, $\beta = 0$, the system $\{\cos(\lambda_n(q,\pi,0)x)\}_{n=0}^{\infty}$ is not a basis, but the system $\{f(x)\} \cup \{\cos(\lambda_n(q,\pi,0)x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0,\pi]$, if $f(x) = \cos(\lambda x)$, where $\lambda^2 \neq \lambda_n^2$ for every $n = 0,1,2,\dots$.*

Theorem 1.2. 1. *Let α, $\beta \in (0,\pi)$. Then the systems*
- (a) *$\{\sin(\lambda_n x)\}_{n=1}^{\infty}$, if there is no zeros among $\lambda_n = \lambda_n(q,\alpha,\beta)$, $n = 0,1,2,\dots$ (i.e. in this case we "throw away" $\sin(\lambda_0 x)$),*
- (b) *$\{\sin(\lambda_n x)\}_{n=0}^{n_0-1} \cup \{\sin(\lambda_n x)\}_{n=n_0+1}^{\infty}$, if $\lambda_{n_0}(q,\alpha,\beta) = 0$ (we "throw away" $\sin(\lambda_{n_0} x) \equiv 0$).*

are Riesz bases in $L^2[0,\pi]$.

2. *Let $\alpha = \pi$, $\beta \in (0,\pi)$ or $\alpha \in (0,\pi)$, $\beta = 0$. Then the systems*
- (a) *$\{\sin(\lambda_n x)\}_{n=0}^{\infty}$, if there is no zeros among $\lambda_n = \lambda_n(q,\alpha,\beta)$, $n = 0,1,2,\dots$,*
- (b) *$\{\sin(\lambda_n x)\}_{n=0}^{n_0-1} \cup \{x\} \cup \{\sin(\lambda_n x)\}_{n=n_0+1}^{\infty}$, if $\lambda_{n_0} = 0$.*

are Riesz bases in $L^2[0,\pi]$.

3. *Let $\alpha = \pi$, $\beta = 0$. The answer is the same as in case 2.*

The Riesz basicity of the systems of functions of sines and cosines in $L^2[0,\pi]$ has been studied in many papers (see, for example, [1, 6, 9, 13, 14, 16]) and is also associated with Riesz basicity in $L^2[-\pi,\pi]$ the systems of the form $\{e^{i\lambda_n x}\}_{n=-\infty}^{\infty}$ (see, e.g. [7, 8, 15]). Completeness and Riesz basicity of systems of sines and cosines are used in many related areas of mathematics, in particular, in solutions of the inverse problems in spectral theory of operators (see, e.g. [1, 2, 3, 10, 11]).

This article is organized as follows. In section 2 we give some necessary information and the results of [6], which are more similar to ours. In section 3 we prove theorems 1.1 and 1.2.

2. Preliminaries

Eigenvalues of the problem $L(q,\alpha,\beta)$. The dependence of the eigenvalues of the Sturm-Liouville problem on parameters α and β from the boundary conditions (1.2) and (1.3) was investigated in [5], where the following theorem was proved.

Theorem 2.1. *The smallest eigenvalue has the property*

$$\lim_{\alpha\to 0} \mu_0(q,\alpha,\beta) = -\infty, \quad \lim_{\beta\to\pi} \mu_0(q,\alpha,\beta) = -\infty. \tag{2.1}$$

For eigenvalues $\mu_n(q,\alpha,\beta)$, $n \geq 2$, the formula

$$\mu_n(q,\alpha,\beta) = [n + \delta_n(\alpha,\beta)]^2 + [q] + r_n(q,\alpha,\beta) \tag{2.2}$$

holds, where $[q] = \frac{1}{\pi}\int_0^\pi q(x)dx$,

$$\begin{aligned}\delta_n(\alpha,\beta) = \frac{1}{\pi}\Big[&\arccos \frac{\cos\alpha}{\sqrt{[n+\delta_n(\alpha,\beta)]^2 \sin^2\alpha + \cos^2\alpha}} \\ &- \arccos \frac{\cos\beta}{\sqrt{[n+\delta_n(\alpha,\beta)]^2 \sin^2\beta + \cos^2\beta}}\Big]\end{aligned} \tag{2.3}$$

and $r_n(q,\alpha,\beta) = o(1)$*, when* $n \to \infty$*, uniformly in* $\alpha, \beta \in [0,\pi]$ *and* q *from the bounded subsets of* $L^1_R[0,\pi]$ *(we will write* $q \in BL^1_R[0,\pi]$*).*

Note that the formula (2.2) is the generalization of the asymptotic formulas known prior to [5] for the eigenvalues of the Sturm-Liouville problem (see [1, 10, 11, 12]. More detailed table of the asymptotic formulas for eigenvalues of the problem $L(q,\alpha,\beta)$ is in [11]). From (2.2) for $\lambda_n(q,\alpha,\beta)$ ($\mu_n = \lambda_n^2$) we obtain the formula

$$\lambda_n(q,\alpha,\beta) = n + \delta_n(\alpha,\beta) + \frac{[q]}{2[n+\delta_n(\alpha,\beta)]} + l_n(q,\alpha,\beta) \tag{2.4}$$

where $l_n = l_n(q,\alpha,\beta) = o(n^{-1})$ when $n \to \infty$ uniformly for all $q \in BL^1_R[0,\pi]$ and $\alpha,\beta \in [0,\pi]$. From (2.3) easily follows that $\delta_n(\alpha,\beta) = O(n^{-1})$ for $\alpha,\beta \in (0,\pi)$; $\delta_n(\alpha,\beta) = \frac{1}{2} + O(n^{-1})$ for $\alpha = \pi, \beta \in (0,\pi)$ and $\alpha \in (0,\pi), \beta = 0$; and $\delta_n(\pi,0) = 1$ for all $n = 2,3,\dots$. Thus, we distinguish 3 cases:

1. $\alpha,\beta \in (0,\pi)$; i.e. the interior points of the square $[0,\pi]\times[0,\pi]$, where $\lambda_n = \lambda_n(q,\alpha,\beta)$ have the asymptotic property $\lambda_n = n + O(n^{-1})$,
2. $\alpha = \pi,\ \beta \in (0,\pi)$ or $\alpha \in (0,\pi),\ \beta = 0$ (i.e. right and bottom edges of the square $[0,\pi]\times[0,\pi]$), where λ_n have the asymptotic property $\lambda_n = n + \frac{1}{2} + O(n^{-1})$,
3. $\alpha = \pi,\ \beta = 0$, where $\lambda_n(q,\pi,0) = n + 1 + O(n^{-1})$.

Riesz bases. The following three definitions and two lemmas are taken from [1]. Equivalent definitions and statements are available in other studies (see, e.g. [4, 6, 8]).

Definition 2.2. A basis $\{f_j\}_{j=1}^\infty$ of a separable Hilbert space H is called a Riesz basis if it is derived from an orthonormal basis $\{e_j\}_{j=1}^\infty$ by linear bounded invertible operator A, i.e., if $f_j = Ae_j$, $j = 1,2,\dots$.

Definition 2.3. Two sequences of elements $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ from H are called quadratically close if $\sum_{j=1}^\infty \|f_j - g_j\|^2 < \infty$.

Definition 2.4. A sequence $\{g_n\}_{n=0}^\infty$ is called ω-linearly independent, if the equality $\sum_{n=0}^\infty c_n g_n = 0$ is possible only when $c_n = 0$ for $n = 0,1,2,\dots$.

Lemma 2.5. *Let* $\{f_n\}_{n=0}^\infty$ *be a Riesz basis in* H*,* $\{f_n\}_{n=0}^\infty$ *and* $\{g_n\}_{n=0}^\infty$ *are quadratically close. If* $\{g_n\}_{n=0}^\infty$ *is* ω*-linearly independent, then* $\{g_n\}_{n=0}^\infty$ *is a Riesz basis in* H*.*

Lemma 2.6. *Let* $\{f_n\}_{n=0}^\infty$ *be a Riesz basis in* H*,* $\{f_n\}_{n=0}^\infty$ *and* $\{g_n\}_{n=0}^\infty$ *are quadratically close. If* $\{g_n\}_{n=0}^\infty$ *is complete in* H*, then* $\{g_n\}_{n=0}^\infty$ *is* ω*-linearly independent (and therefore, is a Riesz basis in* H*).*

The following two theorems are proved in [6].

Theorem 2.7. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers with the property that $\lambda_k \neq \lambda_m$ for $k \neq m$ and of the form $\lambda_n = n + \delta + \delta_n$, with $\delta_n \in [-l, l]$ for sufficiently large n, where the constants $\delta \in [0, \frac{1}{2}]$ and $l \in (0, \frac{1}{4})$ satisfy $[1 + \sin(2\pi\delta)]^{\frac{1}{2}}(1 - \cos(\pi l)) + \sin(\pi l) < 1$. Then $\{\cos(\lambda_n x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$.*

Theorem 2.8. *Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers of the form $\lambda_n = n - \delta + \delta_n$, having the same properties as in Theorem 2.7. Then $\{\sin(\lambda_n x)\}_{n=1}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$.*

It follows from (2.4) that the only circumstance, (essentially) preventing us to apply theorems 2.7 and 2.8 for proving Riesz basicity of systems $\{\cos(\lambda_n(q, \alpha, \beta)x)\}_{n=0}^{\infty}$ and $\{\sin(\lambda_n(q, \alpha, \beta)x)\}_{n=0}^{\infty}$, is that among the eigenvalues $\mu_n = \lambda_n^2(q, \alpha, \beta)$ may be negative (see (2.1)), and accordingly among $\lambda_n(q, \alpha, \beta)$ may be (in a finite number) pure imaginary ones. Can these λ_n interfere the Riesz basicity of the mentioned systems? Our answer is contained in theorems 1.1 and 1.2.

3. Proofs of main Theorems

We will start with a lemma, which is an analogue of [6, Lemma 4].

Lemma 3.1. *Let $\{\nu_n^2\}_{n=0}^{\infty}$ and $\{\lambda_n^2\}_{n=0}^{\infty}$ be two real sequences such that $\nu_k^2 \neq \nu_m^2$ and $\lambda_k^2 \neq \lambda_m^2$, for $k \neq m$, and among which only a finite number of members $(\nu_0^2, \nu_1^2, \dots, \nu_{n_1}^2; \lambda_0^2, \lambda_1^2, \dots, \lambda_{n_2}^2)$ can be negative, and the sequences are enumerated in increasing order $(\nu_0^2 < \nu_1^2 < \dots < \nu_n^2 < \dots; \lambda_0^2 < \lambda_1^2 < \dots < \lambda_n^2 < \dots)$. Let $\{\nu_n\}$ and $\{\lambda_n\}$ have the asymptotic properties*

$$\nu_n = n + \delta + O(n^{-1}), \quad 0 \leq \delta \leq 1, \tag{3.1}$$

$$\lambda_n = n + \delta_n(\alpha, \beta) + O(n^{-1}), \tag{3.2}$$

when $n \to \infty$ and, furthermore,

$$\sum_{n=0}^{\infty} |\lambda_n - \nu_n|^2 < \infty. \tag{3.3}$$

Then $\{\cos(\nu_n x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$ if and only if $\{\cos(\lambda_n x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$.

Proof. Set $f_n(x) = \cos(\nu_n x)$ and $g_n(x) = \cos(\lambda_n x)$, $n = 0, 1, 2 \dots$. Assume $\{f_n\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0, \pi]$. Since for real numbers ν_n and λ_n,

$$\begin{aligned} |\cos(\nu_n x) - \cos(\lambda_n x)| &= |2 \sin \frac{(\lambda_n - \nu_n)x}{2} \sin \frac{(\lambda_n + \nu_n)x}{2}| \\ &\leq 2|\sin \frac{(\lambda_n - \nu_n)x}{2}| \leq |\nu_n - \lambda_n| x \leq \pi |\nu_n - \lambda_n|, \end{aligned}$$

we obtain that

$$\|\cos(\nu_n x) - \cos(\lambda_n x)\|^2 = \int_0^{\pi} |\cos(\nu_n x) - \cos(\lambda_n x)|^2 dx \leq \pi^3 |\nu_n - \lambda_n|^2.$$

Therefore,

$$\sum_{n=0}^{\infty} \|f_n - g_n\|^2 = \sum_{n=0}^{n_0} \|f_n - g_n\|^2 + \sum_{n=n_0+1}^{\infty} \|f_n - g_n\|^2$$

$$\leq M_0 + \pi^3 \sum_{n=n_0+1}^{\infty} |\lambda_n - \nu_n|^2 < \infty;$$

i.e., $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$ are quadratically close ($n_0 = \max\{n_1, n_2\}$). According to Lemma 2.5, to prove the Riesz basicity of the system $\{g_n\}_{n=0}^{\infty}$ it is enough to prove its ω-linearly independence. Assume the contrary, i.e. let there is a sequence $\{c_n\}_{n=0}^{\infty} \in l^2$, not identically zero, such that

$$\sum_{n=0}^{\infty} c_n g_n = 0. \tag{3.4}$$

Let $\lambda \in \mathbb{C}$ be such that $\lambda \neq \pm\lambda_n$, $n = 0, 1, 2, \dots$, and define the function

$$g(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n^2 - \lambda^2} g_n(x). \tag{3.5}$$

It follows from (2.2) that this series is uniformly convergent for $x \in [0, \pi]$. Similarly, the series

$$g'(x) = -\sum_{n=0}^{\infty} \frac{c_n \lambda_n}{\lambda_n^2 - \lambda^2} \sin(\lambda_n x) \tag{3.6}$$

converges uniformly on $[0, \pi]$. Since $g_n'' = -\lambda_n^2 g_n$, we have (note, that here we repeat the proof of [6])

$$\sum_{n=0}^{m} \frac{c_n}{\lambda_n^2 - \lambda^2} g_n''(x) = -\sum_{n=0}^{m} \frac{c_n \lambda_n^2}{\lambda_n^2 - \lambda^2} g_n(x) = -\sum_{n=0}^{m} c_n g_n(x) - \lambda^2 \sum_{n=0}^{m} \frac{c_n}{\lambda_n^2 - \lambda^2} g_n(x).$$

Taking into account (3.4), we conclude that the sequence on the left-side of the last equality converges in $L^2[0, \pi]$ to $-\lambda^2 g(x)$, when $m \to \infty$. This implies that g is twice differentiable and satisfies the differential equation $-g''(x) = \lambda^2 g(x), x \in (0, \pi)$, and initial conditions (see (3.5) and (3.6)):

$$g(0) = h(\lambda) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n^2 - \lambda^2},\ g'(0) = 0; \tag{3.7}$$

i.e., g is the solution of the corresponding Cauchy problem, which is unique and given by the formula

$$g(x) = h(\lambda) \cos(\lambda x). \tag{3.8}$$

The function $h(\lambda)$ defined by (3.7) is meromorphic, and taking into account that $\{c_n\}_{n=0}^{\infty} \neq \{0\}_{n=0}^{\infty}$, is not an identically zero function. Then it has no more than countable number of isolated zeros. If $h(\lambda) \neq 0$, (3.5) and (3.8) show that $\cos(\lambda x)$ belongs to the closed linear span of the system $\{g_n\}_{n=0}^{\infty}$ in $L^2[0, \pi]$. Since $\cos(\lambda x)$ is a continuous function of (λ, x), we obtain that $\cos(\lambda x)$ belongs to closed linear span of the system $\{g_n\}_{n=0}^{\infty}$ for all $\lambda \in \mathbb{C}$. Particularly, the all $\cos(nx)$, $n = 0, 1, 2, \dots$ belong to the closed linear span of the system $\{g_n\}_{n=0}^{\infty}$, so the system $\{g_n\}_{n=0}^{\infty}$ is a complete system in $L^2[0, \pi]$. From Lemma 2.6 follows the ω-linearly independence of the system $\{g_n\}_{n=0}^{\infty}$; i.e. we come to contradiction, and the Riesz basicity of the system $\{g_n\}_{n=0}^{\infty}$ is proved. If we assume, that $\{g_n\}_{n=0}^{\infty}$ is a Riesz basis, then similarly we can prove the Riesz basicity of the system $\{f_n\}_{n=0}^{\infty}$. Lemma 3.1 is proved. $\square$

Let us now turn to the proof of the theorem 1.1. Let us start from the first case: $\alpha, \beta \in (0, \pi)$. Let us take in Lemma 3.1 $\nu_n = n, f_n(x) = \cos(nx)$, and $g_n(x) = \cos(\lambda_n(q,\alpha,\beta)x)$, $n = 0, 1, 2, \dots$. In this case $\lambda_n(q,\alpha,\beta) = n + O(n^{-1})$, and, therefore, (3.3) holds; i.e., $\{f_n\}$ and $\{g_n\}$ are quadratically close. Since $\{f_n\}_{n=0}^\infty = \{\cos(nx)\}_{n=0}^\infty$ is a Riesz basis, then from the Lemma 3.1 follows the Riesz basicity of the system $\{\cos(\lambda_n(q,\alpha,\beta)x)\}_{n=0}^\infty$.

In the second case in Lemma 3.1 we take $\nu_n = n + \frac{1}{2}$, $f_n(x) = \cos((n + \frac{1}{2})x)$, and $g_n(x) = \cos(\lambda_n(q,\alpha,\beta)x)$, $n = 0, 1, 2, \dots$. In the second case $\lambda_n(q,\alpha,\beta) = n + \frac{1}{2} + O(n^{-1})$ and therefore again holds (3.3); i.e. quadratically closeness. As $\{\cos((n + \frac{1}{2})x)\}_{n=0}^\infty$ is the system of eigenfunctions of the Sturm-Liouville problem $L(0, \frac{\pi}{2}, 0)$, it is an orthogonal basis in $L^2[0,\pi]$ (and, particularly, is a Riesz basis). From the Lemma 3.1 follows the Riesz basicity of the system $\{\cos(\lambda_n(q,\alpha,\beta)x)\}_{n=0}^\infty$ in this case.

In the third case in Lemma 3.1 we take $\nu_n = n+1$ ($\delta = 1$), $f_n(x) = \cos((n+1)x)$ and $g_n(x) = \cos(\lambda_n(q,\pi,0)x)$, $n = 0, 1, 2, \dots$. If we assume that $\{g_n\}_{n=0}^\infty$ is a Riesz basis, then from the asymptotic property $\lambda_n(q,\pi,0) = n + 1 + O(n^{-1})$ and Lemma 3.1 follows the Riesz basicity of the system $\{f_n\}_{n=0}^\infty = \{\cos((n+1)x)\}_{n=0}^\infty$, which is incorrect, since it is even not complete. Therefore, $\{\cos\lambda_n(q,\pi,0)x\}_{n=0}^\infty$ does not form a Riesz basis. But adding to this system a function $f(x) = \cos(\lambda x)$, where $\lambda^2 \neq \lambda_n^2$ for every $n = 0, 1, 2, \dots$ and noticing that the system $\{f(x)\} \cup \{\cos(\lambda_n(q,\pi,0)x)\}_{n=0}^\infty$ is ω-linearly independent and quadratically close to the system $\{\cos(nx)\}_{n=0}^\infty$, according to the Lemma 3.1, we get its Riesz basicity. Theorem 1.1 is proved.

Lemma 3.2. *Let $\{\nu_n^2\}_{n=0}^\infty$ and $\{\lambda_n^2\}_{n=0}^\infty$ are the same as in Lemma 3.1. Then $\{\sin(\nu_n x)\}_{n=0}^\infty$ is a Riesz basis in $L^2[0,\pi]$ if and only if $\{\sin(\lambda_n x)\}_{n=0}^\infty$ is a Riesz basis in $L^2[0,\pi]$.*

Proof. Set $f_n(x) = \sin(\nu_n x)$, $g_n(x) = \sin(\lambda_n x)$, $n = 0, 1, 2, \dots$. Quadratic closeness of the systems $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ can be showed in the same way as in Lemma 3.1. Function $g(x)$ (see (3.5)) in this case is the solution of the Cauchy problem $-g'' = \lambda^2 g$, $g(0) = 0$, $g'(0) = h_1(\lambda) = \sum_{n=0}^\infty \frac{c_n\lambda_n}{\lambda_n^2-\lambda^2}$, and, therefore, has the form $g(x) = h_1(\lambda)\sin(\lambda x)/\lambda$. From the continuity of $\sin(\lambda x)/\lambda$ as a function of two variables (λ, x) follows that the equality

$$\frac{\sin(\lambda x)}{\lambda} = \frac{1}{h_1(\lambda)} \sum_{n=0}^\infty \frac{c_n}{\lambda_n^2 - \lambda^2} \sin(\lambda_n x) \tag{3.9}$$

holds not only when $h_1(\lambda) \neq 0$, but also for all $\lambda \in \mathbb{C}$. Hence (3.9) is right for $\lambda = 1, 2, 3, \dots$; i.e., the all elements of the orthogonal basis $\{\sin(nx)\}_{n=1}^\infty$ are in the closed linear span of the system $\{\sin(\lambda_n x)\}_{n=0}^\infty$; i.e., the system $\{\sin(\lambda_n x)\}_{n=0}^\infty$ is complete in $L^2[0,\pi]$. The rest of the proof is as in Lemma 3.1. □

Now the proof of Theorem 1.2 is: In stated in following cases:

(1.a) we take $\nu_n = n + 1$ and accordingly $\{f_n(x)\}_{n=0}^\infty = \{\sin((n+1)x)\}_{n=0}^\infty = \{\sin(nx)\}_{n=1}^\infty$ and $\{g_n(x)\}_{n=1}^\infty = \{\sin(\lambda_n x)\}_{n=1}^\infty$, as stated in Theorem 1.2. Since $\{f_n\}_{n=0}^\infty$ is a Riesz basis (and even an orthogonal basis) and from the asymptotic property $\lambda_n = n + O(n^{-1})$ it follows that $\{f_n\}$ and $\{g_n\}$ are quadratically close, therefore from Lemma 3.2 follows the Riesz basicity of the system $\{\sin(\lambda_n x)\}_{n=1}^\infty$.

(1.b) also $\{f_n(x)\}_{n=1}^{\infty} = \{\sin(nx)\}_{n=1}^{\infty}$ and the system

$$\{\sin(\lambda_n x)\}_{n=0}^{n_0-1} \cup \{\sin(\lambda_n x)\}_{n=n_0+1}^{\infty}$$

is again quadratically close to $\{f_n\}_{n=1}^{\infty}$.

(2.a) we take $\nu_n = n + \frac{1}{2}$, accordingly, $f_n(x) = \sin((n+\frac{1}{2})x)$, and $g_n(x) = \sin(\lambda_n x)$, $n = 0, 1, \dots$. Since $\{\sin((n+\frac{1}{2})x)\}_{n=0}^{\infty}$ is the system of eigenfunctions of the self-adjoint problem $L(0, \pi, \frac{\pi}{2})$, than it is an orthogonal basis in $L^2[0,\pi]$. The asymptotic property $\lambda_n = n + \frac{1}{2} + O(n^{-1})$ ensures the quadratically closeness of the systems $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$, therefore in this case the Riesz basicity of the system $\{g_n\}_{n=0}^{\infty}$ is proved.

(2.b) again $f_n(x) = \sin((n+\frac{1}{2})x)$, $n = 0, 1, \dots$, and $\{g_n\}$ is different from the case (2.a) with only one element g_{n_0}, which has not any effect on quadratically closeness of the systems $\{f_n\}_{n=0}^{\infty}$ and $\{g_n\}_{n=0}^{\infty}$.

(3) we take $\nu_n = n + 1$ and $f_n(x) = \sin((n+1)x)$, $n = 0, 1, 2, \dots$; i.e., $\{f_n(x)\}_{n=0}^{\infty} = \{\sin(nx)\}_{n=1}^{\infty}$. The rest is followed from the asymptotic property $\lambda_n(q, \pi, 0) = n + 1 + O(n^{-1})$, if we take $g_n(x) = \sin(\lambda_n(q, \pi, 0)x)$, $n = 0, 1, \dots$.

Therefore, theorem 1.2 is proved.

Remark 3.3. From lemmas 3.1 and 3.2 it easily follows that $\{\cos(\lambda_n(q, \alpha, \beta)x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0,\pi]$ if and only if $\{\cos(\lambda_n(0, \alpha, \beta)x)\}_{n=0}^{\infty}$ is a Riesz basis in $L^2[0,\pi]$. Similarly for sines. This means that the stability of Riesz basicity is not affected by adding the potential $q(\cdot)$.

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