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GENERALIZED RIEMANN DERIVATIVE

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Abstract. Initiated by Marshall Ash in 1966, the study of generalized Riemann derivative draw significant attention of the mathematical community and numerous studies where carried out since then. One of the major areas that benefits from these developments is the numerical analysis, as the use of generalized Riemann derivatives leads to solving a wider class of problems that are not solvable with the classical tools. This article studies the generalized Riemann derivative and its properties and establishes relationships between Riemann generalized derivative and the classical one. The existence of classical derivative implies the existence of the Riemann generalized derivative, and we study conditions necessary for the generalized Riemann derivative to imply the existence of the classical derivative. Furthermore, we provide conditions on the generalized Riemann derivative that are sufficient for the existence of the classical derivative.

1. Introduction

Marshall Ash initiated the study of generalized Riemann derivative in his thesis [2] in In 1966. Urged by Zygmund and starting from his papers [18, 19, 27], Ash begun his studies with symmetric derivative and Schwarz derivative of second order: Given an interval I of real numbers, $x \in \mathrm{Int}(I)$ and $f : I \to \mathbb{R}$ a function, then for all $h \in \mathbb{R} \setminus \{0\}$ such that $x - h \in I$ and $x + h \in I$ we define the following ratios:

$$R_1 f(x,h) = \frac{f\big(x + \frac{h}{2}\big) - f\big(x - \frac{h}{2}\big)}{h}, \quad R_2 f(x,h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

If the limit $\lim_{h\to 0} R_1 f(x,h) = R_1 f(x)$ exists and is finite, then $R_1 f(x)$ is known as the symmetric derivative of f at x, cf. [7, 10]. If the limit $\lim_{h\to 0} R_2 f(x,h) = R_2 f(x)$ exists and is finite, then $R_2 f(x)$ is known as the Schwarz derivative of f at x, cf. [20, 22]. Both derivatives have important applications in trigonometrical series theory and in numerical analysis, see [17, 23, 24, 23, 26]. A natural generalization of these two derivatives is the generalized Riemann derivative of order r of a function f at a point x. We consider a_i, b_i, $i = 1, 2, \dots, n$ real numbers and suppose that the following conditions of consistency are satisfied:

$$\sum_{i=1}^{n} a_i b_i^k = \begin{cases} 0, & k = 0, 1, \dots, r-1 \\ 1, & k = r. \end{cases} \tag{1.1}$$

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If we have $k = 0$ and $b_i = 0$ in relation (1.1), then we denote by $b_i^k = 1$. Further we consider the ratio

$$R_r^{a,b} f(x,h) = \frac{\sum_{i=1}^{n} a_i f(x + b_i h)}{h^r}.$$

In the case when

$$\lim_{h\to 0} R_r^{a,b} f(x,h) = R_r^{a,b} f(x)$$

exists and is finite, we say that the function f is generalized Riemann differentiable of order r at x.

We denote by $D_r^{a,b} f(x) = r! \cdot R_r^{a,b} f(x)$ and we say that $D_r^{a,b} f(x)$ is the generalized Riemann derivative of order r of function f at x. One can easily observe that if f is a function differentiable of order r in classical sense then f is generalized Riemann differentiable of order r and the two derivatives are equal. The converse does not hold. More generally if f is Peano differentiable of order r, then f is generalized Riemann differentiable of order r. The converse does not hold.

Among the most important contributions to the generalized Riemann derivative are those of Humke, Laczkovich and Mukhopadhyay in [14, 15, 21]. In [2]–[6] and [8], Ash gives a number of problems linked to generalized Riemann derivative. These papers deals with the applications of generalized Riemann derivative to some uniqueness theorems in trigonometric series theory. It can be noted that by replacing the classical derivative with the generalized Riemann derivative in the process of solving ordinary differential equations, the resulting solutions are no longer differentiable in the classical sense. Such solutions are known in ordinary differential equations theory as weak solutions. Therefore, it is necessary to study the system of parameters a_i, b_i, $i = 1, \dots, n$ that satisfy conditions of consistency and for which the generalized Riemann derivative coincides with the classical derivative.

It can also be noticed that the speed of convergence of numerical scheme associated to differential equation depends essentially on the type of generalized Riemann derivative and consequently on the parameters a_i, b_i, $i = 1, \dots, n$. In the following section we study the links between generalized Riemann derivative and classical derivative. We give conditions in which the existence of generalized Riemann derivative implies the existence of classical derivative. We give sufficient conditions on the system of vectors (a, b) that define generalized Riemann derivative, such that any function which is generalized Riemann differentiable is also classical differentiable.

2. (σ, τ)-Riemann differentiable functions

We study further a new class of generalized differentiable functions - the functions (σ, τ)-Riemann differentiable. The motivation for this definition is that we mark out the system (σ, τ) in $\mathbb{K}^n \times \mathbb{K}^n$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), on which we give conditions such that a function f is generalized Riemann differentiable and such that a series of theorems hold.

Denote by $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $G \subset \mathbb{K}$ an open subset and the function $f : G \to \mathbb{K}$. For $p, n \in \mathbb{N}^*, p \leq n$, define the set:

$$L(p, n, \mathbb{K}) := \Big\{ (\sigma, \tau) \in \mathbb{K}^{*n} \times \mathbb{K}^n : \sum_{k=1}^{n} \sigma_k \tau_k^j = 0, \text{ for } j \in \{0, 1, \dots, p-1\},$$

$$\sum_{k=1}^{n} \sigma_k \tau_k^p = p! \text{ and } \tau \text{ has all the components distinct}\Big\}.$$

Definition 2.1. The function f is (σ, τ)-differentiable of order p at x (x in G) if $(\sigma, \tau) \in L(p, n, \mathbb{K})$ and if the following limit exists and belongs to $\mathbb{K}$:

$$\lim_{h\to 0} \frac{1}{h^p} \sum_{k=1}^{n} \sigma_k f(x + \tau_k h).$$

If the limit above exists, we denote it by $D_p(\sigma, \tau)f(x)$.

Remark 2.2. If $G \subset \mathbb{K}$ is an open set, $(\sigma, \tau) \in L(p, n, \mathbb{K})$, $f : G \to \mathbb{K}$ is (σ, τ)-differentiable at x, $(x \in G)$, then for all $\lambda \in K$ the function λf is (σ, τ)-differentiable at x and the following relation holds:

$$D_p(\sigma, \tau)(\lambda f)(x) = \lambda D_p(\sigma, \tau)f(x).$$

Remark 2.3. If f, $g : G \to \mathbb{K}$ are two functions (σ, τ) differentiable at x, then the function $f + g$ is (σ, τ) differentiable at x and the following relation holds:

$$D_p(\sigma, \tau)(f + g)(x) = D_p(\sigma, \tau)f(x) + D_p(\sigma, \tau)g(x).$$

The proof is straightforward and we let the reader to complete it, if needed.

Remark 2.4. Let $G \subset \mathbb{R}$ with $\operatorname{Int} G \neq \emptyset$ and $x \in \operatorname{Int} G$, $f : G \to \mathbb{R}$ and $(\sigma, \tau) \in L(p, n, \mathbb{R})$. If f is differentiable of order p at x in the classic sense, then $D_p(\sigma, \tau)f(x)$ exists and these two derivatives are equal:

$$D_p(\sigma, \tau)f(x) = f^{(p)}(x).$$

Theorem 2.5. *Let $G \subset \mathbb{K}$ such that $\operatorname{Int} G \neq \emptyset$, $x \in \operatorname{Int} G$, $f : G \to \mathbb{K}$ and $(\sigma', \tau') \in L(p, m, \mathbb{K})$, $(\sigma'', \tau'') \in L(p, n, \mathbb{K})$. If $D_p(\sigma', \tau')f(x)$ and $D_p(\sigma'', \tau'')f(x)$ exist and belong to $\mathbb{K}$, then they are equal.*

Proof. Using Definition 2.1, we obtain

$$\begin{aligned}
D_p(\sigma', \tau')f(x) &= \frac{1}{p!} p! D_p(\sigma', \tau')f(x) \\
&= \frac{1}{p!} \sum_{j=1}^{n} \sigma''_j \tau''^p_j D_p(\sigma', \tau')f(x) \\
&= \frac{1}{p!} \sum_{j=1}^{n} \sigma''_j \tau''^p_j \lim_{j\to 0} \frac{1}{h^p} \sum_{k=1}^{m} \sigma'_k f(x + \tau'_k h) \\
&= \frac{1}{p!} \sum_{j=1}^{n} \sigma''_j \tau''^p_j \lim_{h\to 0} \frac{1}{(\tau''_j h)^p} \sum_{k=1}^{m} \sigma'_k f(x + \tau'_k \tau''_k h) \\
&= \frac{1}{p!} \lim_{h\to 0} \frac{1}{h^p} \sum_{j=1}^{n} \sum_{k=1}^{m} \sigma''_j \tau''^p_j \sigma'_k \frac{1}{\tau''^p_j} f(x + \tau'_k \tau''_j h) \\
&= \frac{1}{p!} \lim_{h\to 0} \frac{1}{h^p} \sum_{j=1}^{n} \sum_{k=1}^{m} \sigma''_j \sigma'_k f(x + \tau'_k \tau''_j h) \\
&= \frac{1}{p!} \lim_{h\to 0} \sum_{k=1}^{m} \sigma'_k \sum_{j=1}^{n} \sigma''_j f(x + \tau'_k \tau''_j h)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p!}\sum_{k=1}^{m}\sigma'_k\tau_k'^p \lim_{h\to 0}\frac{1}{(h\tau'_k)^p}\sum_{j=1}^{n}\sigma''_j f(x+\tau''_j(\tau'_k h)) \\
&= \frac{1}{p!}\sum_{k=1}^{m}\sigma'_k\tau_k'^p D_p(\sigma'',\tau'')f(x) \\
&= \frac{1}{p!}p! D_p(\sigma'',\tau'')f(x) = D_p(\sigma'',\tau'')f(x).
\end{aligned}
$$

□

This theorem shows that if a function is differentiable in the generalized sense of order p in relation with two systems (σ',τ') and (σ'',τ''), then these two generalized derivatives of order p are equal. We can formulate with the aid of notion of divided difference, the (σ,τ) derivative.

Remark 2.6. *Let $f: G\to\mathbb{K}$, $G\subset\mathbb{K}$ open set.*

1. Then f is (σ,τ) differentiable if and only if the following limit exists:

$$
\lim_{h\to 0}\sum_{j=1}^{n}\sigma_j\tau_j[x+\tau_j h, x; f] = D_1(\sigma,\tau)f(x).
$$

with $(\sigma,\tau)\in L(1,n,\mathbb{K})$.

2. If $p=n-1$ and $(\sigma,\tau)\in L(p,n,\mathbb{K})$ and $G\subset K$ open set and $f: G\to\mathbb{K}$, then

$$
D_p(\sigma,\tau)f(x) = (n-1)!\lim_{h\to 0}[x+\tau_1 h, x+\tau_2 h,\dots,x+\tau_n h; f].
$$

The following theorem gives conditions on a (σ,τ)-differentiable function at a point x and on the system of vectors (σ,τ) for such function to become differentiable in the classical sense and such that the two derivatives to be equal.

Theorem 2.7. *Let $G\subset\mathbb{R}$, $(\sigma,\tau)\in L(1,n,\mathbb{R})$, $x\in \operatorname{Int} G\neq\emptyset$ and $f: G\to\mathbb{R}$. Suppose that the following conditions hold:*

(i) *f is (σ,τ)-differentiable at x;*
(ii) *there exist left and right derivatives $f'_l(x)$ and $f'_r(x)$ and are finite;*
(iii) $\sum_{\tau_k<0}\sigma_k\tau_k \neq \sum_{\tau_k>0}\sigma_k\tau_k$.

Under these conditions f is differentiable at x and $D_1(\sigma,\tau)f(x) = f'(x)$.

Proof. On the one hand, by the definition of (σ,τ)-derivative of f we obtain:

$$
\begin{aligned}
&D_1(\sigma,\tau)f(x) \\
&= \lim_{h\to 0,h>0}\frac{1}{h}\sum_{k=1}^{n}\sigma_k f(x+\tau_k h) \\
&= \lim_{h\to 0,h>0}\frac{1}{h}\sum_{k=1}^{n}\sigma_k[f(x+\tau_k h)-f(x)] \\
&= \lim_{h\to 0,h>0,\tau_k\neq 0}\sum_{k=1}^{n}\sigma_k\tau_k\frac{f(x+\tau_k h)-f(x)}{\tau_k h} \qquad (2.1)\\
&= \lim_{h\to 0,h>0}\Big\{\sum_{\tau_k>0}^{n}\sigma_k\tau_k\frac{f(x+\tau_k h)-f(x)}{\tau_k h} + \sum_{\tau_k<0}\sigma_k\tau_k\frac{f(x+\tau_k h)-f(x)}{\tau_k h}\Big\} \\
&= \Big(\sum_{\tau_k>0}\sigma_k\tau_k\Big)f'_r(x) + \Big(\sum_{\tau_k<0}\sigma_k\tau_k\Big)f'_l(x).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&D_1(\sigma,\tau)f(x)\\
&= \lim_{h\to 0, h<0} \frac{1}{h}\sum_{k=1}^{n} \sigma_k f(x+\tau_k h)\\
&= \lim_{h'\to 0, h'>0} \frac{1}{-h'}\sum_{k=1}^{n} \sigma_k f(x-\tau_k h')\\
&= \lim_{h'\to 0, h'>0} \sum_{k=1}^{n} \sigma_k\tau_k \frac{f(x-\tau_k h')-f(x)}{-\tau_k h'}\\
&= \lim_{h'\to 0, h'>0} \Big\{ \sum_{\tau_k>0} \sigma_k\tau_k \frac{f(x-\tau_k h')-f(x)}{-\tau_k h'} + \sum_{\tau_k<0} \sigma_k\tau_k \frac{f(x-\tau_k h')-f(x)}{-\tau_k h'} \Big\}\\
&= f_l'(x)\sum_{\tau_k>0} \sigma_k\tau_k + f_r'(x)\sum_{\tau_k<0}\sigma_k\tau_k.
\end{aligned}
\tag{2.2}
$$

From (2.1) and (2.2) we obtain

$$\Big(\sum_{\tau_k>0} \sigma_k\tau_k - \sum_{\tau_k<0}\sigma_k\tau_k\Big)(f_r'(x)-f_l'(x)) = 0.$$

Taking into account (iii) it follows that: $f_l'(x) = f_r'(x) = f'(x)$. This lead us to conclude that f is differentiable at x. In addition we have

$$D_1(\sigma,\tau)f(x) = f'(x)\Big[\sum_{\tau_k>0} \sigma_k\tau_k + \sum_{\tau_k<0}\sigma_k\tau_k\Big] = f'(x)\sum_{k=1}^{n}\sigma_k\tau_k = f'(x).$$

□

Remarks. (1) If G is an interval, $f : G \to \mathbb{R}$ and conditions (i) and (ii) from Theorem 2.7 hold, and

(iii') $\sum_{\tau_k<0} \sigma_k\tau_k = \sum_{\tau_k>0}\sigma_k\tau_k$,

then

$$D_1(\sigma,\tau)f(x) = \frac{1}{2}[f_l'(x) + f_r'(x)].$$

(2) Condition (iii') is very important because if the system (σ,τ) satisfies it, then a large class of non-differentiable functions at a point or on a finite set becomes (σ,τ)-differentiable. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \alpha x, & x<0\\ \beta x, & x\ge 0\end{cases}$$

for $\alpha,\beta\in\mathbb{R}^*$, $\alpha\neq\beta$. This function is not differentiable at $x=0$. However we have

$$
\begin{aligned}
(D_1(\sigma,\tau)f)_r(x) &= \lim_{h\to 0, h>0}\frac{1}{h}\sum_{k=1}^{n}\sigma_k f(\tau_k h)\\
&= \lim_{h\to 0, h>0}\frac{1}{h}\Big[\sum_{\tau_k>0}\sigma_k\beta\tau_k h + \sum_{\tau_k<0}\sigma_k\alpha\tau_k h\Big]\\
&= \beta\sum_{\tau_k>0}\sigma_k\tau_k + \alpha\sum_{\tau_k<0}\sigma_k\tau_k.
\end{aligned}
$$

For
$$\sum_{\tau_k>0}\sigma_k\tau_k=\sum_{\tau_k<0}\sigma_k\tau_k=\frac{1}{2},$$
we obtain
$$(D_1(\sigma,\tau)f)_r(x)=\frac{\alpha+\beta}{2}.$$
Similarly we can prove that
$$(D_1(\sigma,\tau)f)_l(x)=\frac{\alpha+\beta}{2}.$$
We conclude that function f is (σ,τ)-differentiable at $x=0$ and moreover, is (σ,τ)-differentiable on $\mathbb{R}$.

(3) As we can easily observe, condition (iii) from Theorem 2.7 holds for the classical derivative.

The (σ,τ)-derivative has a lot of interesting properties that shall be further studied. We shall give a theorem for the Riemann generalized derivative of the product of two functions.

Theorem 2.8. *Let $G\subset\mathbb{K}$ open set, f, $g:G\to\mathbb{K}$ and $(\sigma,\tau)\in L(1,n,\mathbb{K})$. If the following conditions hold:*

1. *f is Lipschitz and (σ,τ) differentiable;*
2. *g is continuous and (σ,τ) differentiable,*

then $f\cdot g$ is (σ,τ) differentiable and we have the following relation
$$D_1(\sigma,\tau)(f\cdot g)(x)=f(x)\cdot D_1(\sigma,\tau)g(x)+g(x)\cdot D_1(\sigma,\tau)f(x),\quad x\in G \tag{2.3}$$

Proof. It is easy to show that
$$\begin{aligned}
&\frac{1}{h}\sum_{j=1}^{n}\sigma_j[f(x+\tau_jh)-f(x)][g(x+\tau_jh)-g(x)]\\
&=\frac{1}{h}\sum_{j=1}^{n}\sigma_jf(x+\tau_jh)\cdot g(x+\tau_jh)\\
&\quad-f(x)\cdot\frac{1}{h}\sum_{j=1}^{n}\sigma_jg(x+\tau_jh)-g(x)\cdot\frac{1}{h}\sum_{j=1}^{n}\sigma_jf(x+\tau_jh)\\
&\quad+f(x)\cdot g(x)\cdot\frac{1}{h}\sum_{j=1}^{n}\sigma_j.
\end{aligned}\tag{2.4}$$
for $x\in G$, $h\neq 0$, $|h|$ small enough. From (1) there exists $L\geq 0$ such that $|f(x)-f(y)|\leq L|x-y|$, for all x, $y\in G$. By applying the modulus, the left side of (2.4) becomes
$$\begin{aligned}
&\Big|\sum_{j=1}^{n}\sigma_j\frac{f(x+\tau_jh)-f(x)}{h}(g(x+\tau_jh)-g(x))\Big|\\
&\leq\sum_{j=1}^{n}|\sigma_j|\cdot\Big|\frac{f(x+\tau_jh)-f(x)}{h}\Big|\cdot|g(x+\tau_jh)-g(x)|\\
&\leq L\sum_{j=1}^{n}|\sigma_j|\cdot|\tau_j|\cdot|g(x+\tau_jh)-g(x)|
\end{aligned}$$

for all $x \in G$, $|h|$ small enough. Using that g is continuous, the left side of (2.4) tends to 0 when $h \to 0$. So we obtain

$$0 = D_1(\sigma,\tau)(f \cdot g)(x) - f(x) \cdot D_1(\sigma,\tau)g(x) - g(x) \cdot D_1(\sigma,\tau)f(x) \quad \text{for } x \in G$$

which proves completes the proof. □

Corollary 2.9. *Let $G \subset \mathbb{K}$ open set, f, $g : G \to \mathbb{K}$ and $(\sigma,\tau) \in L(1,n,\mathbb{K})$. If the following conditions hold:*

1. *f is locally Lipschitz,*
2. *g is continuous and (σ,τ) differentiable,*

then $f \cdot g$ is (σ,τ) differentiable almost everywhere and relation (2.3) *holds almost everywhere.*

Proof. We will use that any Lipschitz function is almost everywhere differentiable. Many properties of the classical derivative correspond to (σ,τ) derivative and when these properties does not hold on general case, we shall find conditions on f or on the system (σ,τ) such as these properties remain available. Such a situation shall be reveal in a theorem bellow. □

Notation. Let $G \subset \mathbb{K}$ such that $\operatorname{Int} G \neq \emptyset$, $f : G \to \mathbb{K}$, $x \in \operatorname{Int} G$ and $\lambda \in \mathbb{K}$, $\lambda \neq -1$. We define:

$$R(\lambda)f(x) := \lim_{h\to 0} \frac{f(x+h) - f(x-\lambda h)}{(1+\lambda)h}$$

in the hypotheses in which this limit belongs to $\mathbb{K}$, and we say that f is $R(\lambda)$ differentiable at x. We define:

$$\sigma = \Big(\frac{1}{1+\lambda}, -\frac{1}{1+\lambda}\Big), \quad \tau = \Big(1, -\lambda\Big).$$

We notice that $(\sigma,\tau) \in L(1,2,\mathbb{K})$ and $R(\lambda)f(x) = D_1(\sigma,\tau)f(x)$.

Theorem 2.10. *Let $G \subset K$ open set, $x \in G$ and $f, g : G \to \mathbb{K}$, $\lambda \in \mathbb{K}$, $\lambda \neq -1$, such that:*

1. *f, g are continuous at x;*
2. *f, g are $R(\lambda)$-differentiable at x.*

Under these conditions the function $h = fg$ is $R(\lambda)$ differentiable at x and we have the formula

$$R(\lambda)(fg)(x) = g(x)R(\lambda)f(x) + f(x)R(\lambda)g(x), \quad x \in G.$$

Proof. The proof is based on

$$\begin{aligned}
&R(\lambda)(fg)(x)\\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x-\lambda h)g(x-\lambda h)}{(1+\lambda)h}\\
&= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x-\lambda h)] + g(x-\lambda h)[f(x+h) - f(x-\lambda h)]}{(1+\lambda)h}\\
&= f(x)R(\lambda)g(x) + g(x)R(\lambda)f(x).
\end{aligned}$$

□

Remarks. (1) For $p = 1$, $n = 2$, $\sigma_1 = \frac{1}{2}$, $\sigma_2 = -\frac{1}{2}$, $\tau_1 = 1$, $\tau_2 = -1$ we obtain the symmetric Riemann derivative.

(2). For $p = 2$, $n = 3$, $\sigma_1 = 1$, $\sigma_2 = -2$, $\sigma_3 = 1$, $\tau_1 = 1$, $\tau_2 = 0$, $\tau_3 = -1$ we find the Schwarz derivative.

(3) It is also easy to notice that $R(\lambda)$ extend the symmetric Riemann derivative, which is $R(1)$ and the classical derivative, which is $R(0)$.

In conclusion, the set of (σ, τ)-differentiable functions is larger than the set of classical differentiable functions.

If $(\sigma, \tau) \in L(p, n, \mathbb{K})$ and G an open set, then we denote

$$\mathfrak{T}_p(\sigma, \tau)(G, \mathbb{K}) = \{f : G \to \mathbb{K} | f \text{ is } (\sigma, \tau)\text{-differentiable everywhere on } G\}.$$

Theorem 2.11. *If $(\sigma, \tau) \in L(p, n, \mathbb{K})$ and if any function from $\mathfrak{T}_p(\sigma, \tau)(G, \mathbb{K})$ is p times differentiable in classical sense, then there exists $j \in \{1, 2, \dots, n\}$ such that $\tau_j = 0$.*

Proof. Indeed, if $\tau_j \neq 0$, for all $j \in \{1, 2, \dots, n\}$ we prove that there exist a function in $\mathfrak{T}_p(\sigma, \tau)(G, \mathbb{K})$ which is (σ, τ) differentiable and does not have the property in the theorem. This function is $f : G \to \mathbb{K}$,

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \in G \setminus \{a\}. \end{cases}$$

This function is discontinuous at a and consequently is not differentiable in the classical sense at a. However $D(\sigma, \tau)f(x) = 0$ and this leads us to a contradiction. □

More properties of (σ, τ)-Riemann differentiable functions were studied in [1, 13, 25].

3. Main Results

If we replace the classical derivative of the function with the Riemann generalized derivative in the process of solving numerical ordinary differential equations, we shall obtain new solutions that are not differentiable in the classical sense (see citeref9,ref12). This kind of solutions are well known in ordinary differential equations theory as weak solutions. Therefore is needed to study the system of parameters (σ, τ) that satisfy the consistency conditions and for which, the Riemann generalized derivative which is defined by this system of parameters, is equal to classical derivative.

The speed of convergence of the numerical scheme associated to the ordinary differential equation, depends on the type of derivative, consequently on the system of parameters (σ, τ). We shall further give some conditions in which a (σ, τ)-differentiable function is classical differentiable.

Lemma 3.1. *Let a, $b \in \mathbb{K}$ such that $|a| \leq |b| < 1$ and the function $\varphi : \mathbb{K} \to \mathbb{K}$. Suppose that the conditions below hold:*

(i) $\lim_{x \to 0} x\varphi(x) = 0$;
(ii) $\lim_{x \to 0} [\varphi(x) - a\varphi(bx)] = 0$, $x \in \mathbb{K}$.

Then $\lim_{x \to 0} \varphi(x) = 0$.

Proof. If $b=0$ results that $a=0$ and the statement is available in this case. For b in $\mathbb{K}^*$, we denote

$$g(x) := \varphi(x) - a\varphi(bx), \quad x \in \mathbb{K}. \tag{3.1}$$

This condition is equivalent to $\varphi(x) = g(x) + a\varphi(bx)$, $x \in \mathbb{K}$. Repeating the transformation: $x \to bx$, we obtain

$$\begin{gathered}
\varphi(bx) = g(bx) + a\varphi(b^2x), \quad x \in \mathbb{K}\\
\varphi(b^2x) = g(b^2x) + a\varphi(b^3x), \quad x \in \mathbb{K}\\
\dots\\
\varphi(b^{n-1}x) = g(b^{n-1}x) + a\varphi(b^nx), \quad x \in \mathbb{K},\ n \in \mathbb{N}^*.
\end{gathered}$$

It follows that

$$\varphi(x) = g(x) + ag(bx) + a^2g(b^2x) + \dots + a^{n-1}g(b^{n-1}x) + a^n\varphi(b^nx),$$

for $x \in \mathbb{K}$, $n \in \mathbb{N}^*$. From (1) and (ii), we have $\lim_{x\to 0} g(x) = 0$. For $0 < |x| < r$, $n \in \mathbb{N}^*$ we obtain

$$\begin{aligned}
|\varphi(x)| \le \sup_{0<|y|\le r} |g(y)| + |a| \sup_{0<|y|\le r} |g(by)| + \dots\\
+ |a|^{n-1} \sup_{0<|y|\le r} |g(b^{n-1}y)| + |\frac{a}{b}|^n \cdot \frac{1}{|x|} \cdot |(b^nx)\varphi(b^nx)|.
\end{aligned}$$

For $\epsilon > 0$ we consider V_ϵ - the disc of radius ϵ centered in 0 such that for any $x \in V_r \setminus \{0\}$ we have

$$|\varphi(x)| \le \frac{1}{1-|a|} \cdot \sup_{0<|y|\le r} |g(y)| + \frac{1}{|x|} \cdot \overline{\lim_{n\to\infty}} |(b^nx)\varphi(b^nx)|.$$

As a consequence we obtain

$$|\varphi(x)| \le \frac{1}{1-|a|} \cdot \sup_{0<|y|\le r} |g(y)|, \quad \text{for } x \text{ in } V_r \setminus \{0\}$$

and because $\lim_{r\to 0}\Big(\sup_{0<|y|\le r} |g(y)|\Big) = 0$ we conclude that $\lim_{x\to 0}\varphi(x) = 0$. □

We remark that Lemma 3.1 remains true if we have the condition below instead of condition (ii):

(ii') $\lim_{x\to 0}[\varphi(x) + a\varphi(bx)] = 0$, $x \in \mathbb{K}$.

In this chase the auxiliary function is $g(x) := \varphi(x) + a\varphi(bx)$, $x \in \mathbb{K}$, and

$$\varphi(x) = g(x) - ag(bx) + a^2g(b^2x) - a^3g(b^3x) + \dots + (-a)^{n-1}g(b^{n-1}x) + (-1)^na^n\varphi(b^nx),$$

for $x \in \mathbb{K}$ and $n \in \mathbb{N}^*$.

Lemma 3.2. *Let a, $b \in \mathbb{K}$ such that $|a| \ge |b| > 1$ and $\varphi : \mathbb{K} \to \mathbb{K}$ for which the following conditions hold:*

(i) $\lim_{x\to 0} x\varphi(x) = 0$,

(ii) $\lim_{x\to 0}[\varphi(x) + a\varphi(bx)] = 0$.

Then $\lim_{x\to 0}\varphi(x) = 0$.

Proof. To prove this we shall use Lemma 3.1 with transformations: $x = y/b = b'y$, $b \in \mathbb{K}^*$, $b' = 1/b$, $a' = 1/a$, $a' \in \mathbb{K}^*$ with $1 > |b'| \ge |a'|$. For the case $b = 0$ results that $a = 0$ and condition (ii) validate the statement. □

Lemma 3.3. *Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{K}^n$, $b = (b_1, b_2, \dots, b_n) \in \mathbb{K}^n$ with $|a_j| \le |b_j| < 1$, for all $j \in \{1, 2, \dots, n\}$ and the function $\varphi : \mathbb{K} \to \mathbb{K}$ with the properties:*

(i) $\lim_{x\to 0} x\varphi(x) = 0$;

(ii)

$$\lim_{x\to 0} \Big\{\varphi(x) + \sum_{j=1}^{n} a_j\varphi(b_j x) + \sum_{j<k} a_j a_k \varphi(b_j b_k x) + \dots + a_1 a_2 \dots a_n \varphi(b_1 \dots b_n x)\Big\} = 0.$$

Then $\lim_{x\to 0} \varphi(x) = 0$.

Proof. We shall repeatedly use Lemma 3.1 with condition (ii'). For $1 \le j \le n$ we define

$$(L_j\varphi)(x) := \varphi(x) + a_j\varphi(b_j x), \quad x \in \mathbb{K}$$

and hence $\lim_{x\to 0}(L_j\varphi)(x) = 0$ which implies $\lim_{x\to 0} \varphi(x) = 0$, for $j \in \{1, \dots, n\}$. Using the relation

$$(L_j L_k\varphi)(x) = \varphi(x) + [a_k\varphi(b_k x) + a_j\varphi(b_j x)] + a_j a_k \varphi(b_j b_k x),$$

for $1 \le j$, $k \le n$, $x \in \mathbb{K}$. By induction we obtain

$$(L_1 L_2 \dots L_n\varphi)(x) = \varphi(x) + \sum_{j=1}^{n} a_j\varphi(b_j x) + \sum_{j<k} a_j a_k \varphi(b_j b_k x) + \dots + a_1 a_2 \dots a_n \varphi(b_1 \dots b_n x),$$

for $x \in \mathbb{K}$. Therefore, the proof has the following logical scheme:

$$\lim_{x\to 0}(L_j L_k\varphi)(x) = \lim_{x\to 0} L_j(L_k\varphi(x)) = 0$$

implies

$$\lim_{x\to 0}(L_k\varphi)(x) = 0 \Rightarrow \lim_{x\to 0} \varphi(x) = 0, \quad j, k \in \{1, \dots, n\}.$$

This completes the proof. □

Lemma 3.4. *Let the polynomial function*

$$P(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n \in \mathbb{K}[x]$$

with roots $(x_j)_{1\le j\le n}$, that have the property $|x_j| < 1$, for all $j \in \{1, \dots, n\}$. Let $\varphi : \mathbb{K} \to \mathbb{K}$ satisfy the following conditions:

(1) $\lim_{x\to 0} x\varphi(x) = 0$;

(2) $\lim_{x\to 0} \sum_{k=0}^{n} (-1)^k \alpha_k \varphi(b^k x) = 0$ *where* $\max_{1\le j\le n} |x_j| \le |b| < 1$.

Then $\lim_{x\to 0} \varphi(x) = 0$.

For the proof of the above lemma, it is sufficient to consider $b_1 = b_2 = \dots = b_n = b$, $a_j = x_j$ in Lemma 3.3 and to take into account Viète relations.

Lemma 3.5. *Let a_k, $b_k \in \mathbb{K}^*$, $k \in \{1, 2, \dots, n\}$ and the function $\varphi : \mathbb{K} \to \mathbb{K}$ with the following properties:*

(1) *φ is bounded on a neighborhood of the origin;*

(2) $\sum_{j=1}^{n} |a_j| < 1$;

(3) $b = \max_{1\le j\le n} |b_j| \le 1$;

(4) $\lim_{x\to 0}[\varphi(x) - \sum_{j=1}^{n} a_j\varphi(b_j x)] = 0$.

Then $\lim_{x\to 0} \varphi(x) = 0$.

Proof. We define the function

$$g(x) := \varphi(x) - \sum_{j=1}^{n} a_j \varphi(b_j x), \quad \text{for } x \in \mathbb{K}.$$

Note that $\lim_{x\to 0} g(x) = 0$. We define the operator $L : \mathbb{K}^{\mathbb{K}} \to \mathbb{K}^{\mathbb{K}}$,

$$L\psi(x) := \sum_{j=1}^{n} a_j \psi(b_j x), \quad \psi \in \mathbb{K}^{\mathbb{K}}, x \in \mathbb{K}.$$

Note that the operator L is linear. We can write: $\varphi - L\varphi = g$ as $\varphi = L\varphi + g$ and iterate to obtain $L\varphi = L^2\varphi + Lg$, but $L\varphi = \varphi - g$, which implies

$$\varphi = L^2\varphi + Lg + g.$$

By induction we obtain

$$\varphi = L^k\varphi + L^{k-1}g + \dots + Lg + g, \quad k \geq 1.$$

As $L\varphi(x) = \sum_{j=1}^{n} a_j\varphi(b_j x)$, $x \in \mathbb{K}$, we deduce that

$$|L\varphi(x)| \leq \sum_{j=1}^{n} |a_j| \cdot |\varphi(b_j x)|, \quad x \in \mathbb{K}.$$

Denote

$$\widetilde{\varphi}(r) := \sup_{0<|x|\leq r} |\varphi(x)|, \quad r \in (0, \infty).$$

Then we have

$$\widetilde{L\varphi}(r) \leq \Big(\sum_{j=1}^{n} |a_j|\Big)\widetilde{\varphi}(br), \quad r \in (0, \infty).$$

In the same manner we obtain

$$\big(\widetilde{L^2\varphi}\big)(r) \leq \Big(\sum_{j=1}^{n} |a_j|\Big)\widetilde{L\varphi}(br) \leq \Big(\sum_{j=1}^{n} |a_j|\Big)^2 \widetilde{\varphi}(b^2 r), \quad r \in (0, \infty)$$

and similarly,

$$\Big(\widetilde{L^k\varphi}\Big)(r) \leq \Big(\sum_{j=1}^{n} |a_j|^k\Big)\widetilde{\varphi}(b^k r), \quad r \in (0, \infty),\ k \in \mathbb{N}^*.$$

If we denote $a := \sum_{j=1}^{n} |a_j|$, then

$$\widetilde{L^k\varphi}(r) \leq a^k \widetilde{\varphi}(b^k r), \quad r \in (0, \infty),\ k \in \mathbb{N}^*.$$

If follows that

$$\begin{aligned}
\widetilde{\varphi}(r) &\leq \widetilde{L^k\varphi}(r) + \widetilde{L^{k-1}g}(r) + \dots + \widetilde{Lg}(r) + \widetilde{g}(r) \\
&\leq a^k\widetilde{\varphi}(r) + a^{k-1}\widetilde{g}(r) + \dots + a\widetilde{g}(r) + \widetilde{g}(r) \\
&\leq a^k\widetilde{\varphi}(r) + \frac{1}{1-a}\widetilde{g}(r).
\end{aligned}$$

The above relation holds for all $k \in \mathbb{N}^*$ and under hypotheses (1) and (2), that is φ is bounded on a neighborhood of the origin and $a \in (0, 1)$. This implies that

$$\widetilde{\varphi}(r) \leq \frac{1}{1-a}\widetilde{g}(r), \quad r \in (0, \infty).$$

As $\lim_{r\to 0}\widetilde{g}(r)=0$, it follows that $\lim_{r\to 0}\widetilde{\varphi}(r)=0$ and in conclusion we obtain $\lim_{x\to 0}\varphi(x)=0$. $\square$

Lemma 3.6. *Let $a_k,\ b_k\in\mathbb{K}^*$, $k\in\{1,2,\dots,n\}$ and the functions $\varphi:\mathbb{K}\to\mathbb{K}$ with the following properties:*

1. $\lim_{x\to 0} x\varphi(x)=0$;
2. $\sum_{j=1}^n \big|\frac{a_j}{b_j}\big|\le 1$;
3. $b=\max_j |b_j|<1$;
4. $\lim_{x\to 0}\big[\varphi(x)-\sum_{j=1}^n a_j\varphi(b_jx)\big]=0$.

Then $\lim_{x\to 0}\varphi(x)=0$.

Proof. We define the operator $L:\mathbb{K}^{\mathbb{K}}\to\mathbb{K}^{\mathbb{K}}$:

$$L\psi(x):=\sum_{j=1}^n a_j\psi(b_jx),\quad \psi\in\mathbb{K}^{\mathbb{K}},\ x\in\mathbb{K}$$

and the function $g(x):=\varphi(x)-L\varphi(x)$, $x\in\mathbb{K}$. Iterating, we have

$$\varphi=L^k\varphi+L^{k-1}g+\dots+Lg+g.$$

This leads us to define for a function $\psi:\mathbb{K}\to\mathbb{K}$ bounded on a neighborhood of origin

$$\widetilde{\psi}(r):=\sup_{0<|x|\le r}|\psi(x)|,\quad r\in(0,\infty).$$

We notice that $\widetilde{\psi}$ is increasing. Denote $u(x):=x\varphi(x),\quad x\in\mathbb{K}$. Then we can write

$$|L\varphi(x)|\le\sum_{j=1}^n|a_j|\cdot|\varphi(b_jx)|,\quad x\in\mathbb{K}$$

and further we have

$$|xL\varphi(x)|\le\sum_{j=1}^n\Big|\frac{a_j}{b_j}\Big|\cdot|b_jx\varphi(b_jx)|=\sum_{j=1}^n\Big|\frac{a_j}{b_j}\Big|\cdot|u(b_jx)|,\quad x\in\mathbb{K}.$$

Similarly, we obtain

$$|xL^2\varphi(x)|\le\sum_{j=1}^n\Big|\frac{a_j}{b_j}\Big|\cdot|b_jxL\varphi(b_jx)|\le\sum_{j,k=1}^n\Big|\frac{a_j}{b_j}\Big|\cdot\Big|\frac{a_k}{b_k}\Big|\cdot|b_jb_kx\varphi(b_jb_k)|,\quad x\in\mathbb{K}.$$

As

$$|L^2\varphi(x)|\le\frac{1}{|x|}\sum_{j,k=1}^n\Big|\frac{a_j}{b_j}\Big|\cdot\Big|\frac{a_k}{b_k}\Big|\cdot|u(b_jb_kx)|,\quad x\in\mathbb{K}^*,$$

we obtain

$$|L^k\varphi(x)|\le\frac{1}{|x|}\sum_{j_1,\dots,j_k}\Big|\frac{a_{j_1}}{b_{j_1}}\Big|\cdot\dots\cdot\Big|\frac{a_{j_k}}{b_{j_k}}\Big|\cdot|u(b_{j_1}\dots b_{j_k}x)|,\quad x\in\mathbb{K}^*,k\in\mathbb{N}^*$$

So we have

$$|L^k\varphi(x)|\le\frac{1}{|x|}\Big(\sum_{j=1}^n\Big|\frac{a_j}{b_j}\Big|\Big)^k|\widetilde{u}(b^k|x|)|\le\frac{1}{|x|}\widetilde{u}(b^k|x|),\quad x\in\mathbb{K}^*$$

where $a := \sum_{j=1}^{n} |a_j|$, $a = \sum_{j=1}^{n} \left|\frac{a_j}{b_j}\right| \cdot |b_j| \le \sum_{j=1}^{n} |b_j| = b < 1$. Then we have

$$|Lg(x)| \le \Big(\sum_{j=1}^{n} |a_j|\Big)\widetilde{g}(b|x|), \quad x \in \mathbb{K}^*,$$

$$|\widetilde{L^k g}(r)| \le \Big(\sum_{j=1}^{n} |a_j|\Big)^k \widetilde{g}(b^k r), \quad r \in (0,\infty),\ k \in \mathbb{N}^*.$$

As

$$|\varphi(x)| \le \frac{1}{|x|} \cdot \widetilde{g}(b^k|x|) + \frac{1}{1-a}\widetilde{g}(|x|), \quad x \in \mathbb{K}^*, k \in \mathbb{N}^*,$$

letting $k \to \infty$ we obtain

$$|\varphi(x)| \le \frac{1}{1-a}\widetilde{g}(|x|), \quad x \in \mathbb{K}^*.$$

We can now conclude that $\lim_{x\to 0} \varphi(x) = 0$. □

From Lemma 3.6 can be easy obtained the following lemma.

Lemma 3.7. *Let $n \ge 2$, $\varphi : \mathbb{K} \to \mathbb{K}$, α_j, $\beta_j \in \mathbb{K}^*$, $j \in \{1,\dots,n\}$, such that conditions below hold:*

1. *$\lim_{x\to 0} x\varphi(x) = 0$;*
2. *$\max_{2\le j\le n} |\beta_j| < |\beta_1|$;*
3. *$\sum_{j=2}^{n} \left|\frac{\alpha_j}{\beta_j}\right| \le \left|\frac{\alpha_1}{\beta_1}\right|$;*
4. *$\lim_{x\to 0} \sum_{j=1}^{n} \alpha_j\varphi(\beta_j x) = 0$.*

Then $\lim_{x\to 0} \varphi(x) = 0$.

Proof. For this proof, we denote

$$\frac{\alpha_j}{\alpha_1} = -a_{j-1}, \quad \frac{\beta_j}{\beta_1} = b_{j-1}, \quad \beta_1 x = t, \quad j \in \{2,\dots,n\}.$$

Then

$$\begin{aligned}\lim_{x\to 0}\sum_{j=1}^{n} \alpha_j\varphi(\beta_j x) &= \alpha_1 \lim_{x\to 0}\sum_{j=1}^{n} \frac{\alpha_j}{\alpha_1}\varphi\Big(\frac{\beta_j}{\beta_1}\beta_1 x\Big)\\ &= \alpha_1 \lim_{t\to 0}\Big[\varphi(t) - \sum_{j=2}^{n} a_{j-1}\varphi(b_{j-1}t)\Big] = 0.\end{aligned}$$

Therefore, condition (4) in Lemma 3.6 is verified. Further we have

$$\sum_{j=2}^{n} \Big|\frac{\alpha_j}{\beta_j}\Big| = \sum_{j=2}^{n} \Big|\frac{a_{j-1}\alpha_1}{b_{j-1}\beta_1}\Big| = \Big|\frac{\alpha_1}{\beta_1}\Big|\sum_{k=1}^{n-1}\Big|\frac{a_k}{b_k}\Big| < \Big|\frac{\alpha_1}{\beta_1}\Big|$$

if and only if $\sum_{k=1}^{n-1} \left|\frac{a_k}{b_k}\right| < 1$. Also

$$\max_{2\le j\le n} |\beta_j| = \max_{2\le j\le n} |\beta_1 b_{j-1}| < |\beta_1|.$$

Applying Lemma 3.6, it results that $\lim_{x\to 0} \varphi(x) = 0$. □

Lemma 3.8. *Let $n \ge 2$, $\varphi : \mathbb{K} \to \mathbb{K}$, α_j, $\beta_j \in \mathbb{K}^*$, $j \in \{1,2,\dots,n\}$, such that the following conditions hold:*

1. *φ is bounded on a neighborhood of the origin;*
2. *$\sum_{j=2}^{n} |\alpha_j| < |\alpha_1|$;*

(3) $\beta = \max_{1\le j\le n} |\beta_j| < |\beta_1|$;
(4) $\lim_{x\to 0} \sum_{j=1}^{n} \alpha_j \varphi(\beta_j x) = 0$.

Then $\lim_{x\to 0} \varphi(x) = 0$.

Proof. We proceed analogously as in Lemma 3.7 and we make the transformations:

$$\alpha_j = -\alpha_1 a_{j-1}, \quad \beta_j = \beta_1 b_{j-1}, \quad \beta_1 x = t, \quad j \in 1, \dots, n.$$

Further, we notice that the conditions from Lemma 3.5 hold. Indeed,

$$\sum_{j=2}^{n} |\alpha_j| = \sum_{j=2}^{n} |a_{j-1}\alpha_1| = |\alpha_1| \sum_{k=1}^{n-1} |a_k| < |\alpha_1|$$

if and only if $\sum_{k=1}^{n-1} |a_k| < 1$. As

$$\begin{aligned} 0 &= \lim_{x\to 0} \sum_{j=1}^{n} \alpha_j \varphi(\beta_j x) \\ &= \alpha_1 \lim_{x\to 0} \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_1} \varphi\Big(\frac{\beta_j}{\beta_1}\beta_1 x\Big) \\ &= \alpha_1 \lim_{t\to 0} \Big[\varphi(t) - \sum_{j=2}^{n} a_{j-1}\varphi(b_{j-1}t)\Big] \\ &= \alpha_1 \lim_{t\to 0} \Big[\varphi(t) - \sum_{k=1}^{n-1} a_k \varphi(b_k t)\Big], \end{aligned}$$

according to Lemma 3.5, it results that $\lim_{x\to 0} \varphi(x) = 0$. □

The following theorem establishes conditions in which a (σ, τ)-differentiable function at a point is classical differentiable at that point.

Theorem 3.9. *We consider the function* $f : \mathbb{K} \to \mathbb{K}$, $x \in \mathbb{K}$ *and the numbers* a_j, $b_j \in \mathbb{K}^*$, $j \in \{1, 2, \dots, n\}$ *such that the following conditions hold:*
(1) $\sum_{j=1}^{n} a_j = 1$;
(2) $\max_{2\le j\le n} |b_j| < |b_1|$;
(3) $\sum_{j=2}^{n} \big|\frac{a_j}{b_j}\big| \le \big|\frac{a_1}{b_1}\big|$;
(4) *f is continuous at x;*
(5) *f is (σ, τ) differentiable at x, where* $\sigma = \Big(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}, -\sum_{j=1}^{n} \frac{a_j}{b_j}\Big)$, $\tau = (b_1, \dots, b_n, 0)$.

Then f is classical differentiable at the point x and $D_1(\sigma, \tau) f(x) = f'(x)$.

Proof. We shall use the Lemma 3.7 with the following notation

$$\varphi(h) := \frac{f(x+h) - f(x) - \ell h}{h}, \quad h \in \mathbb{K}^*,$$

where $\ell := D_1(\sigma, \tau) f(x)$. We notice that f is continuous at x, which is equivalent to:

$$0 = \lim_{h\to 0} [f(x+h) - f(x)] = \lim_{h\to 0} \Big[h \cdot \frac{f(x+h) - f(x)}{h} - \ell h\Big] = \lim_{h\to 0} h\varphi(h);$$

that is, condition (1) from Lemma 3.7 holds. On the one hand we have

$$\ell = \lim_{h\to 0} \frac{1}{h} \sum_{k=1}^{n+1} \sigma_k f(x+\tau_k h) = \lim_{h\to 0} \frac{1}{h} \Big[\sum_{k=1}^{n} \frac{a_k}{b_k} (f(x+b_k h) - f(x)) \Big].$$

On the other hand side we have

$$\begin{aligned} \lim_{h\to 0} \sum_{k=1}^{n} a_k \varphi(b_k h) &= \lim_{h\to 0} \sum_{k=1}^{n} a_k \frac{f(x+b_k h) - f(x) - \ell b_k h}{b_k h} \\ &= \lim_{h\to 0} \frac{1}{h} \sum_{k=1}^{n} \frac{a_k}{b_k} (f(x+b_k h) - f(x)) - \ell \\ &= \ell - \ell = 0. \end{aligned}$$

The conditions for Lemma 3.7 being satisfied, it results that $\lim_{x\to 0} \varphi(x) = 0$. This is equivalent with f differentiable at x and $f'(x) = \ell$. ☐

Definition 3.10. Let $V \subset \mathbb{K}$, a neighborhood of 0, $a = (a_1, \dots, a_n) \in \mathbb{K}^n$, $b = (b_1, \dots, b_n) \in \mathbb{K}^n$ and the function $\varphi : V \to \mathbb{K}$. We say that the system (a, b) satisfies *condition* (C_1) if the following conditions are satisfied:

(i) $\lim_{x\to 0} x\varphi(x) = 0$,
(ii) $\lim_{x\to 0} \sum_{k=1}^{n} a_k \varphi(b_k x) = 0$

imply $\lim_{x\to 0} \varphi(x) = 0$.

Proposition 3.11. *Let a, $b \in \mathbb{K}^n$, c, $d \in \mathbb{K}^m$. If the systems (a, b) and (c, d) satisfy condition (C_1), then the system $((a_i, c_j), (b_k, d_l))$, i, $k \in \{1, \dots, n\}$, j, $l \in \{1, \dots, m\}$ also satisfy condition (C_1).*

Proof. Indeed, it is sufficient to consider the function

$$g(x) = \sum_{j=1}^{m} c_j \varphi(d_j x), \quad x \in \mathbb{K}.$$

Then

$$\sum_{k=1}^{n} a_k g(b_k x) = \sum_{k=1}^{n} a_k \sum_{j=1}^{m} c_j \varphi(b_k d_j x) = \sum_{k=1}^{n} \sum_{j=1}^{m} a_k c_j \varphi(b_k d_j x),$$

for $x \in \mathbb{K}$. ☐

Definition 3.12. The system (a, b) with $a \in \mathbb{K}^n$, $b \in \mathbb{K}^n$ satisfies *condition* (C_2) if for all functions $\varphi : \mathbb{K} \to \mathbb{K}$ with the properties:

(i) φ is bounded on a neighborhood of origin;
(ii) $\lim_{x\to 0} \sum_{k=1}^{n} a_k \varphi(b_k x) = 0$; results that $\lim_{x\to 0} \varphi(x) = 0$.

Remarks. (1) It is easy to observe that for Definition 3.12, we can state a result similar to Proposition 3.11.

(2) Condition (C_1) and condition (C_2) are related as follows: $(C_2) \Rightarrow (C_1)$. Indeed, let φ bounded on an arbitrary neighborhood of origin which satisfies the condition

$$\lim_{x\to 0} \sum_{k=1}^{n} a_k \varphi(b_k x) = 0.$$

As φ is bounded on a neighborhood of origin results that $\lim_{x\to 0} x\varphi(x) = 0$.

Theorem 3.13. *Let $f : \mathbb{K} \to \mathbb{K}$, $x_0 \in \mathbb{K}$ fixed, $a_j \in \mathbb{K}$, $b_j \in \mathbb{K}^*$ $j \in 1, \dots, n$ such that the following condition hold:*

1. $\sum_{j=1}^{n} a_j = 1$;
2. *f is continuous at x_0;*
3. *f is (σ,τ)- differentiable at x_0, where*
$$\sigma = \Big(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}, -\sum_{k=1}^{n} \frac{a_k}{b_k}\Big), \quad \tau = (b_1, \dots, b_n, 0);$$
4. *the system (a,b) satisfies condition (C_1).*

Then f is classical differentiable at x_0 and $D_1(\sigma,\tau)f(x_0) = f'(x_0)$.

Proof. Let $\varphi : \mathbb{K}^* \to \mathbb{K}$,
$$\varphi(h) := \frac{f(x_0 + h) - f(x_0) - \ell h}{h}$$
where $\ell := D_1(\sigma,\tau)f(x_0)$.
On the right hand side, because f is (σ,τ)-differentiable at x_0, $(\sigma,\tau) \in L(1, n+1, \mathbb{K})$ then
$$\begin{aligned}
D_1(\sigma,\tau)f(x_0) &= \lim_{h\to 0} \frac{1}{h} \sum_{k=1}^{n+1} \sigma_k f(x_0 + \tau_k h) \\
&= \lim_{h\to 0} \frac{1}{h} \Big[\sum_{k=1}^{n} \frac{a_k}{b_k} f(x_0 + b_k h) - \sum_{k=1}^{n} \frac{a_k}{b_k} f(x_0) \Big] \\
&= \lim_{h\to 0} \sum_{k=1}^{n} a_k \frac{f(x_0 + b_k h) - f(x_0)}{b_k h} = \ell.
\end{aligned}$$
On the left hand side, because f is continuous at x_0 we have
$$\lim_{h\to 0} h\varphi(h) = \lim_{h\to 0} [f(x_0 + h) - f(x_0) - \ell h] = 0$$
and as f is (σ,τ)-differentiable at x_0 we obtain
$$\begin{aligned}
\lim_{h\to 0} \sum_{k=1}^{n} a_k \varphi(b_k h) &= \lim_{h\to 0} \sum_{k=1}^{n} a_k \frac{f(x_0 + b_k h) - f(x_0) - \ell b_k h}{b_k h} \\
&= \lim_{h\to 0} \sum_{k=1}^{n} a_k \Big(\frac{f(x_0 + b_k h) - f(x_0)}{b_k h} - \ell \Big) \\
&= \lim_{h\to 0} \sum_{k=1}^{n} a_k \frac{f(x_0 + b_k h) - f(x_0)}{b_k h} - \ell \sum_{k=1}^{n} a_k = \ell - \ell = 0.
\end{aligned}$$
Therefore, conditions (1) and (2) from Definition 3.10 are satisfied; that is, the system (σ,τ) satisfies condition (C_1). As a consequence we have: $\lim_{h\to 0} \varphi(h) = 0$ which is equivalent to $f'(x_0) = \ell$. □

Theorem 3.14. *Let $A \subset \mathbb{K}$ such that* $\operatorname{Int} A \neq \emptyset$, *$x \in \operatorname{Int} A$, a_j, $b_j \in \mathbb{K}^*$, $1 \le j \le n$, $n \ge 2$ and $f : A \to \mathbb{K}$ with the following properties:*

1. *f is Lipschitz on a neighborhood of x;*
2. $\max_{1\le j\le n} |b_j| \le |b_1|$;
3. $\sum_{j=2}^{n} |a_j| \le |a_1|$;
4. $\sum_{j=1}^{n} a_j = 1$;

(5) *f is (σ,τ) differentiable at x, where*
$$\sigma=\Big(\frac{a_1}{b_1},\dots,\frac{a_n}{b_n},-\sum_{k=1}^n\frac{a_k}{b_k}\Big),\quad \tau=(b_1,\dots,b_n,0).$$

Then f is differentiable at x and $D_1(\sigma,\tau)f(x)=f'(x)$.

Proof. We shall use Lemma 3.8. First we observe the equivalence between the following two statements: "φ bounded on a neighborhood of origin" if and only if "there exists $M>0$ such that $|\varphi(h)|\le M$, for all $h\in V_\epsilon(0)$, where $V_\epsilon(0)$ is a symmetric neighborhood of origin of length 2ϵ, ($\epsilon>0$)" if and only if
$$\Big|\frac{f(x+h)-f(x)-\ell h}{h}\Big|\le M,\quad \forall h\in V_\epsilon(0),$$
where we denoted $\ell:=D_1(\sigma,\tau)f(x)$. This is further equivalent to
$$|f(x+h)-f(x)|\le (M+|\ell|)|h|,\quad \forall h\in V_\epsilon(0)$$
if and only if
$$|f(y)-f(x)|\le (M+|\ell|)|y-x|,\quad \forall y\in V_\epsilon(0)\subset A,$$
where $y=x+h$, which implies that f is Lipschitz at x. Similarly, we have
$$0=\lim_{h\to 0}\sum_{j=1}^n a_j\varphi(b_jh)=\lim_{h\to 0}\sum_{j=1}^n a_j\frac{f(x+b_jh)-f(x)-\ell b_jh}{b_jh}$$
if and only if
$$\lim_{h\to 0}\sum_{j=1}^n a_j\frac{f(x+b_jh)-f(x)}{b_jh}=\ell\sum_{j=1}^n a_j=\ell\,.$$
However,
$$\begin{aligned}\ell=D_1(\sigma,\tau)f(x)&=\lim_{h\to 0}\frac{1}{h}\sum_{k=1}^{n+1}\sigma_kf(x+\tau_kh)\\ &=\lim_{h\to 0}\frac{1}{h}\sum_{k=1}^{n}\frac{a_k}{b_k}[f(x+b_kh)-f(x)].\end{aligned}$$
As the conditions for Lemma 3.8 are satisfied, it results that
$$\lim_{h\to 0}\varphi(h)=0\Leftrightarrow\frac{f(x+h)-f(x)}{h}=\ell\Leftrightarrow f'(x)=D_1(\sigma,\tau)f(x).$$
□

Theorem 3.15. *Let $G\subset\mathbb{K}$ such that $\operatorname{Int}G\neq 0$, $x\in\operatorname{Int}G$, $a_j\in\mathbb{K}$, $b_j\in\mathbb{K}^*$, $j\in\{1,\dots,n\}$ and $f:G\to\mathbb{K}$ with properties:*

(1) *f is Lipschitz on a neighborhood of x;*
(2) *$\sum_{j=1}^n a_j=1$;*
(3) *f is (σ,τ)-differentiable at x, where*
$$\sigma=\Big(\frac{a_1}{b_1},\dots,\frac{a_n}{b_n},-\sum_{k=1}^n\frac{a_k}{b_k}\Big),\tau=(b_1,\dots,b_n,0);$$
(4) *the system (a,b) satisfy condition (C_2).*

Then f is classical differentiable at x and $D_1(\sigma,\tau)f(x)=f'(x)$.

Proof. The condition f-Lipschitz at x is equivalent to φ bounded on a neighborhood of origin, where $\varphi(h) = \frac{1}{h}[f(x+h) - f(x) - \ell h]$ as we could observe in Theorem 3.15. Moreover, condition (ii) from Definition 3.12 of condition (C_2) is also satisfied. From here results that $\lim_{h\to 0} \varphi(h) = 0$ which is equivalent to

$$\ell = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

that is $D_1(\sigma,\tau)f(x) = f'(x)$. $\square$

Conclusion. Theorems 3.13–3.15 are general criteria which state that if we find systems (a, b), $((a, b) \in \mathbb{K}^n \times \mathbb{K}^{*n})$ that satisfy conditions (C_1) and (C_2), then any (σ,τ)-differentiable function at a point, satisfying the conditions from these theorems, is classical differentiable at that point and the two derivatives are equal.

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