EXISTENCE OF INFINITELY MANY ANTI-PERIODIC SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS

SHAPOUR HEIDARKHANI, GHASEM A. AFROUZI, ARMIN HADJIAN, JOHNNY HENDERSON

Abstract. In this article, we establish the existence of infinitely many anti-periodic solutions for a second-order impulsive differential inclusion with a perturbed nonlinearity and two parameters. The technical approach is mainly based on a critical point theorem for non-smooth functionals.

1. Introduction

The aim of this article is to show the existence of infinitely many solutions for the following two parameter second-order impulsive differential inclusion subject to anti-periodic boundary conditions

\[-(\phi_p(u'(x)))' + M\phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x, u(x)) \text{ in } [0, T] \setminus Q,
- \Delta \phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1, 2, \ldots, m,
\]
\[u(0) = -u(T), \quad u'(0) = -u'(T),\]

where \(Q = \{x_1, x_2, \ldots, x_m\}, \quad p > 1, \quad T > 0, \quad M \geq 0, \quad \phi_p(x) := |x|^{p-2}x, \quad 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = T, \quad \Delta \phi_p(u'(x_k)) := \phi_p(u'(x_k^+)) - \phi_p(u'(x_k^-)), \]

and \(u'(x_k^+)\) and \(u'(x_k^-)\) denoting the right and left limits, respectively, of \(u'(x)\) at \(x = x_k\), \(I_k \in C(\mathbb{R}, \mathbb{R})\), \(k = 1, 2, \ldots, m\), \(\lambda\) is a positive parameter, \(\mu\) is a nonnegative parameter, and \(F\) is a multifunction defined on \(\mathbb{R}\), satisfying

(F1) \(F : \mathbb{R} \to 2^\mathbb{R}\) is upper semicontinuous with compact convex values;
(F2) \(\min F, \max F : \mathbb{R} \to \mathbb{R}\) are Borel measurable;
(F3) \(|\xi| \leq a(1 + |s|^{r-1})\) for all \(s \in \mathbb{R}, \xi \in F(s), r > 1 (a > 0)\).

Also, \(G\) is a multifunction defined on \([0, T] \times \mathbb{R}\), satisfying

(G1) \(G(x, \cdot) : \mathbb{R} \to 2^\mathbb{R}\) is upper semicontinuous with compact convex values for a.e. \(x \in [0, T] \setminus Q;\)
(G2) \(\min G, \max G : ([0, T] \setminus Q) \times \mathbb{R} \to \mathbb{R}\) are Borel measurable;
(G3) \(|\xi| \leq a(1 + |s|^{r-1})\) for a.e. \(x \in [0, T], s \in \mathbb{R}, \xi \in G(x, s), r > 1 (a > 0)\).

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Impulsive differential equations are used to describe various models of real-world processes that are subject to a sudden change. These models are studied in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, and so forth. Associated with this development, a theory of impulsive differential equations has been given extensive attention. Differential inclusions arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few. It is very important to study anti-periodic boundary value problems because they can be applied to interpolation problems, antiperiodic wavelets, the Hill differential operator, and so on. It is natural from both a physical standpoint as well as a theoretical view to give considerable attention to a synthesis involving problems for impulsive differential inclusion with anti-periodic boundary conditions.

Recently, multiplicity of solutions for differential inclusions via non-smooth variational methods and critical point theory has been considered and here we cite the papers. For instance, in the author, employing a non-smooth Ricceri-type variational principle, developed by Marano and Motreanu, has established the existence of infinitely many, radially symmetric solutions for a differential inclusion problem in \( \mathbb{R}^N \). Also, in the authors extended a recent result of Ricceri concerning the existence of three critical points of certain non-smooth functionals. Two applications have been given, both in the theory of differential inclusions; the first one concerns a non-homogeneous Neumann boundary value problem, the second one treats a quasilinear elliptic inclusion problem in the whole \( \mathbb{R}^N \). In the author, under convenient assumptions, has investigated the existence of at least three positive solutions for a differential inclusion involving the \( p \)-Laplacian operator on a bounded domain, with homogeneous Dirichlet boundary conditions and a perturbed nonlinearity depending on two positive parameters; his result also ensured an estimate on the norms of the solutions independent of both the perturbation and the parameters. Very recently, Tian and Henderson in the paper, based on a non-smooth version of critical point theory of Ricceri due to Iannizzotto, have established the existence of at least three solutions for the problem whenever \( \lambda \) is large enough and \( \mu \) is small enough.

In the present paper, motivated by employing an abstract critical point result (see Theorem 2.6 below), we are interested in ensuring the existence of infinitely many anti-periodic solutions for the problem (1.1); see Theorem 3.1 below. We refer to in which related variational methods are used for non-homogeneous problems.

To the best of our knowledge, no investigation has been devoted to establishing the existence of infinitely many solutions to such a problem as (1.1). For a couple of references on impulsive differential inclusions, we refer to and .

A special case of our main result is the following theorem.

**Theorem 1.1.** Assume that (F1)–(F3) hold, and \( I_i(0) = 0, I_i(s) < 0, s \in \mathbb{R} \), \( i = 1, 2, \ldots, m \). Furthermore, suppose that

\[
\liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min_{0 \leq j \leq T} f_j(s)ds}{\xi^p} = 0,
\]

\[
\limsup_{\xi \to +\infty} \frac{\int_0^T \min_{0 \leq j \leq T} f_j(\frac{s}{\xi} - x) F(s) dx}{\xi^p} \frac{1}{p+1} (\frac{M}{p+1})^{p+1} \sum_{i=1}^m \int_0^s f_j(\frac{4}{\xi} - x_i) I_i(s)ds = +\infty.
\]
Then, the problem \( (1.1) \), for \( \lambda = 1 \) and \( \mu = 0 \), admits a sequence of pairwise distinct solutions.

2. Basic definitions and preliminary results

Let \((X, \| \cdot \|_X)\) be a real Banach space. We denote by \(X^*\) the dual space of \(X\), while \((\cdot, \cdot)\) stands for the duality pairing between \(X^*\) and \(X\). A function \(\varphi : X \to \mathbb{R}\) is called locally Lipschitz if, for all \(u \in X\), there exist a neighborhood \(U\) of \(u\) and a real number \(L > 0\) such that
\[
|\varphi(v) - \varphi(w)| \leq L\|v - w\|_X \quad \text{for all } v, w \in U.
\]
If \(\varphi\) is locally Lipschitz and \(u \in X\), the generalized directional derivative of \(\varphi\) at \(u\) along the direction \(v \in X\) is
\[
\varphi^\circ(u; v) := \limsup_{w \to u, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.
\]
The generalized gradient of \(\varphi\) at \(u\) is the set
\[
\partial\varphi(u) := \{ u^* \in X^* : \langle u^*, v \rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X \}.
\]
So \(\partial\varphi : X \to 2^{X^*}\) is a multifunction. We say that \(\varphi\) has compact gradient if \(\partial\varphi\) maps bounded subsets of \(X\) into relatively compact subsets of \(X^*\).

**Lemma 2.1** ([14, Proposition 1.1]). Let \(\varphi \in C^1(X)\) be a functional. Then \(\varphi\) is locally Lipschitz and
\[
\varphi^\circ(u; v) = \langle \varphi'(u) , v \rangle \quad \text{for all } u, v \in X;
\]
\[
\partial\varphi(u) = \{ \varphi'(u) \} \quad \text{for all } u \in X.
\]

**Lemma 2.2** ([14, Proposition 1.3]). Let \(\varphi : X \to \mathbb{R}\) be a locally Lipschitz functional. Then \(\varphi^\circ(u; \cdot)\) is subadditive and positively homogeneous for all \(u \in X\), and
\[
\varphi^\circ(u; v) \leq L\|v\| \quad \text{for all } u, v \in X,
\]
with \(L > 0\) being a Lipschitz constant for \(\varphi\) around \(u\).

**Lemma 2.3** ([4]). Let \(\varphi : X \to \mathbb{R}\) be a locally Lipschitz functional. Then \(\varphi^\circ : X \times X \to \mathbb{R}\) is upper semicontinuous and for all \(\lambda \geq 0, u, v \in X\),
\[
(\lambda \varphi)^\circ(u; v) = \lambda \varphi^\circ(u; v).
\]
Moreover, if \(\varphi, \psi : X \to \mathbb{R}\) are locally Lipschitz functionals, then
\[
(\varphi + \psi)^\circ(u; v) \leq \varphi^\circ(u; v) + \psi^\circ(u; v) \quad \text{for all } u, v \in X.
\]

**Lemma 2.4** ([14, Proposition 1.6]). Let \(\varphi, \psi : X \to \mathbb{R}\) be locally Lipschitz functionals. Then
\[
\partial(\lambda \varphi)(u) = \lambda \partial\varphi(u) \quad \text{for all } u \in X, \lambda \in \mathbb{R},
\]
\[
\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \quad \text{for all } u \in X.
\]

**Lemma 2.5** ([4, Proposition 1.6]). Let \(\varphi : X \to \mathbb{R}\) be a locally Lipschitz functional with a compact gradient. Then \(\varphi\) is sequentially weakly continuous.
We say that \( u \in X \) is a (generalized) critical point of a locally Lipschitz functional \( \phi \) if \( 0 \in \partial \phi(u) \); i.e.,

\[
\varphi^0(u; v) \geq 0 \quad \text{for all } v \in X.
\]

When a non-smooth functional, \( g : X \to (-\infty, +\infty) \), is expressed as a sum of a locally Lipschitz function, \( \varphi : X \to \mathbb{R} \), and a convex, proper, and lower semicontinuous function, \( j : X \to (-\infty, +\infty) \); that is, \( g := \varphi + j \), a (generalized) critical point of \( g \) is every \( u \in X \) such that

\[
\varphi^0(u; v - u) + j(v) - j(u) \geq 0
\]

for all \( v \in X \) (see [14, Chapter 3]).

Hereafter, we assume that \( X \) is a reflexive real Banach space, \( \mathcal{N} : X \to \mathbb{R} \) is a sequentially weakly lower semicontinuous functional, \( \Upsilon : X \to \mathbb{R} \) is a sequentially weakly upper semicontinuous functional, \( \lambda \) is a positive parameter, \( j : X \to (-\infty, +\infty) \) is a convex, proper, and lower semicontinuous functional, and \( D(j) \) is the effective domain of \( j \). Write

\[
\mathcal{M} := \Upsilon - j, \quad I_\lambda := \mathcal{N} - \lambda \mathcal{M} = (\mathcal{N} - \lambda \Upsilon) + \lambda j.
\]

We also assume that \( \mathcal{N} \) is coercive and

\[
D(j) \cap \mathcal{N}^{-1}((\infty, r)) \neq \emptyset
\]

for all \( r > \inf_X \mathcal{N} \). Moreover, owing to (2.1) and provided \( r > \inf_X \mathcal{N} \), we can define

\[
\varphi(r) := \inf_{u \in \mathcal{N}^{-1}((\infty, r))} \frac{\left( \sup_{v \in \mathcal{N}^{-1}((\infty, r))} \mathcal{M}(v) \right) - \mathcal{M}(u)}{r - \mathcal{N}(u)},
\]

\[
\gamma := \lim_{r \to +\infty} \varphi(r), \quad \delta := \lim_{r \to (\inf_X \mathcal{N})^+} \varphi(r).
\]

If \( \mathcal{N} \) and \( \Upsilon \) are locally Lipschitz functionals, in [1, Theorem 2.1] the following result is proved; it is a more precise version of [13, Theorem 1.1] (see also [14]).

**Theorem 2.6.** Under the above assumption on \( X, \mathcal{N} \) and \( \mathcal{M} \), one has

(a) For every \( r > \inf_X \mathcal{N} \) and every \( \lambda \in (0, 1/\varphi(r)) \), the restriction of the functional \( I_\lambda = \mathcal{N} - \lambda \mathcal{M} \) to \( \mathcal{N}^{-1}((\infty, r)) \) admits a global minimum, which is a critical point (local minimum) of \( I_\lambda \) in \( X \).

(b) If \( \gamma < +\infty \), then for each \( \lambda \in (0, 1/\gamma) \), the following alternative holds: either

(b1) \( I_\lambda \) possesses a global minimum, or

(b2) there is a sequence \( \{u_n\} \) of critical points (local minima) of \( I_\lambda \) such that \( \lim_{n \to +\infty} \mathcal{N}(u_n) = +\infty \).

(c) If \( \delta < +\infty \), then for each \( \lambda \in (0, 1/\delta) \), the following alternative holds: either

(c1) there is a local minimum of \( \mathcal{N} \) which is a local minimum of \( I_\lambda \), or

(c2) there is a sequence \( \{u_n\} \) of pairwise distinct critical points (local minima) of \( I_\lambda \), with \( \lim_{n \to +\infty} \mathcal{N}(u_n) = \inf_X \mathcal{N} \), which converges weakly to a global minimum of \( \mathcal{N} \).

Now we recall some basic definitions and notation. On the reflexive Banach space \( X := \{ u \in W^{1,p}([0,T]) : u(0) = -u(T) \} \) we consider the norm

\[
\|u\|_X := \left( \int_0^T (|u'(x)|^p + M|u(x)|^p) \, dx \right)^{1/p}
\]
for all $u \in X$, which is equivalent to the usual norm (note that $M \geq 0$). We recall that $X$ is compactly embedded into the space $C^0([0, T])$ endowed with the maximum norm $\| \cdot \|_{C^0}$.

**Lemma 2.7** ([16] Lemma 3.3). Let $u \in X$. Then

$$\|u\|_{C^0} \leq \frac{1}{2} T^{1/q} \|u\|_X,$$

where $1/p + 1/q = 1$.

Obviously, $X$ is compactly embedded into $L^\gamma([0, T])$ endowed with the usual norm $\| \cdot \|_{L^\gamma}$, for all $\gamma \geq 1$.

**Definition 2.8.** A function $u \in X$ is a weak solution of the problem (1.1) if there exists $u^* \in L^\gamma([0, T])$ (for some $\gamma > 1$) such that

$$\int_0^T \left[ \phi_p(u'(x))v'(x) + M\phi_p(u(x))v(x) - u^*(x)v(x) \right] dx - \sum_{i=1}^m I_i(u(x_i))v(x_i) = 0$$

for all $v \in X$ and $u^* \in \lambda F(u(x)) + \mu G(x, u(x))$ for a.e. $x \in [0, T]$.

**Definition 2.9.** By a solution of the impulsive differential inclusion (1.1) we will understand a function $u : [0, T] \setminus Q \to \mathbb{R}$ is of class $C^1$ with $\phi_p(u')$ absolutely continuous, satisfying

$$-(\phi_p(u'(x)))' + M\phi_p(u(x)) = u^* \quad \text{in} \ [0, T] \setminus Q,$$

$$-\Delta \phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1, 2, \ldots, m,$$

$$u(0) = -u(T), \quad u'(0) = -u'(T),$$

where $u^* \in \lambda F(u(x)) + \mu G(x, u(x))$ and $u^* \in L^\gamma([0, T])$ (for some $\gamma > 1$).

**Lemma 2.10** ([16] Lemma 3.5). If a function $u \in X$ is a weak solution of (1.1), then $u$ is a classical solution of (1.1).

We introduce for a.e. $x \in [0, T]$ and all $s \in \mathbb{R}$, the Aumann-type set-valued integral

$$\int_0^s F(t)dt = \left\{ \int_0^s f(t)dt : f : \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } F \right\}$$

and set $\mathcal{F}(u) = \int_0^T \min_{0 \leq s \leq T} \int_0^s F(t)ds \ dx$ for all $u \in L^p([0, T])$; the Aumann-type set-valued integral

$$\int_0^s G(x, t)dt = \left\{ \int_0^s g(x, t)dt : g : [0, T] \times \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } G \right\}$$

and set $\mathcal{G}(u) = \int_0^T \min_{0 \leq s \leq T} \int_0^s G(x, t, s)ds \ dx$ for all $u \in L^p([0, T])$.

**Lemma 2.11** ([10] Lemma 3.1). The functionals $\mathcal{F}, \mathcal{G} : L^p([0, T]) \to \mathbb{R}$ are well defined and Lipschitz on any bounded subset of $L^p([0, T])$. Moreover, for all $u \in L^p([0, T])$ and all $u^* \in \partial(\mathcal{F}(u) + \mathcal{G}(u))$,

$$u^* (x) \in F(u(x)) + G(x, u(x)) \quad \text{for a.e. } x \in [0, T].$$

We define an energy functional for the problem (1.1) by setting

$$I_\lambda(u) = \frac{1}{p} \|u\|_X^p - \lambda \mathcal{F}(u) - \mu \mathcal{G}(u) - \sum_{i=1}^m \int_0^u I_i(s)ds$$

for all $u \in X$. 


Lemma 2.12 ([10] Lemma 4.4). The functional \( I_\lambda : X \to \mathbb{R} \) is locally Lipschitz. Moreover, for each critical point \( u \in X \) of \( I_\lambda \), \( u \) is a weak solution of (1.1).

3. Main results

We formulate our main result using the following assumptions:

\[ (F4) \]
\[
\liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min \int_0^t F(s) ds}{\xi^p} \\
< \frac{1}{p} \int T \frac{2}{p} \limsup_{\xi \to +\infty} \frac{\int_0^T \min \int_0^t F(s) ds dx}{\xi^p} \\
< \frac{1}{p} \int T \frac{2}{p} \limsup_{\xi \to +\infty} \frac{\int_0^T \min \int_0^t F(s) ds dx}{\xi^p} \\
< \frac{1}{p} \int T \frac{2}{p} \limsup_{\xi \to +\infty} \frac{\int_0^T \min \int_0^t F(s) ds dx}{\xi^p} \\
< \frac{1}{p} \int T \frac{2}{p} \limsup_{\xi \to +\infty} \frac{\int_0^T \min \int_0^t F(s) ds dx}{\xi^p}.
\]

\[ (I1) \]
\[ I_\lambda (0) = 0, \quad I_\lambda (s) < 0, \quad s \in \mathbb{R}, \quad i = 1, 2, \ldots, m. \]

Theorem 3.1. Assume that (F1)–(F4), (I1) hold. Let

\[ \lambda_1 := 1 / \limsup_{\xi \to +\infty} \frac{\int_0^T \min \int_0^t F(s) ds dx}{\xi^p} \]
\[ \lambda_2 := 1 / \liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min \int_0^t F(s) ds}{\xi^p}. \]

Then, for every \( \lambda \in (\lambda_1, \lambda_2) \), and every multifunction \( G \) satisfying

\[ (G4) \]
\[ \int_0^T \min \int_0^t G(x, s) ds dx \geq 0 \quad \text{for all} \quad t \in \mathbb{R}, \quad \text{and} \]

\[ (G5) \]
\[ G_\infty := \lim_{\xi \to +\infty} \sup_{|t| \leq \xi} \min \int_0^t G(x, s) ds < +\infty, \]

if we put

\[ \mu_{G, \lambda} := \frac{2}{p G_\infty} \left( 1 - \lambda \frac{2}{p T} \liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min \int_0^t F(s) ds}{\xi^p} \right), \]

where \( \mu_{G, \lambda} = +\infty \) when \( G_\infty = 0 \), problem (1.1) admits an unbounded sequence of solutions for every \( \mu \in (0, \mu_{G, \lambda}) \) in \( X \).

Proof. Our aim is to apply Theorem 2.6(b) to (1.1). To this end, we fix \( \bar{\lambda} \in (\lambda_1, \lambda_2) \) and let \( G \) be a multifunction satisfying (G1)–(G5). Since \( \bar{\lambda} < \lambda_2 \), we have

\[ \mu_{G, \bar{\lambda}} := \frac{1}{p G_\infty} \left( 1 - \lambda \frac{2}{p T} \liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min \int_0^t F(s) ds}{\xi^p} \right) > 0. \]

Now fix \( \bar{\mu} \in (0, \mu_{G, \bar{\lambda}}) \), put \( \nu_1 := \lambda_1 \), and

\[ \nu_2 := \frac{\lambda_2}{1 + \frac{2}{p T} \frac{\bar{\mu} G_\infty}{\bar{\lambda}_2}}. \]

If \( G_\infty = 0 \), then \( \nu_1 = \lambda_1 \), \( \nu_2 = \lambda_2 \) and \( \bar{\lambda} \in (\nu_1, \nu_2) \). If \( G_\infty \neq 0 \), since \( \bar{\mu} < \mu_{G, \bar{\lambda}} \), we have

\[ \frac{\bar{\lambda}}{\lambda_2} + \frac{2}{p T} \frac{\bar{\mu} G_\infty}{\bar{\lambda}_2} < 1, \]

and so

\[ \frac{\lambda_2}{1 + \frac{2}{p T} \frac{\bar{\mu} G_\infty}{\bar{\lambda}_2}} > \bar{\lambda}, \]
namely, $\lambda < \nu_2$. Hence, taking into account that $\lambda > \lambda_1 = \nu_1$, one has $\lambda \in (\nu_1, \nu_2)$.

Now, set

$$J(x, s) := F(s) + \frac{1}{\lambda} G(x, s)$$

for all $(x, s) \in [0, T] \times \mathbb{R}$. Assume $j$ identically zero in $X$ and for each $u \in X$ put

$$\mathcal{N}(u) := \frac{1}{p} \|u\|_X^p - \frac{m}{i=1} \int_0^{u(x_i)} I_i(s) ds, \quad \mathcal{Y}(u) := \int_0^T \min \int_0^u J(x, s) ds dx,$$

$$\mathcal{M}(u) := \mathcal{Y}(u) - j(u) = \mathcal{Y}(u),$$

$$I^*_L(u) := \mathcal{N}(u) - \lambda \mathcal{M}(u) = \mathcal{N}(u) - \lambda \mathcal{Y}(u).$$

It is a simple matter to verify that $\mathcal{N}$ is sequentially weak lower semicontinuous on $X$. Clearly, $\mathcal{N} \in C^1(X)$. By Lemma 2.1, $\mathcal{N}$ is locally Lipschitz on $X$. By Lemma 2.11, $\mathcal{F}$ and $\mathcal{G}$ are locally Lipschitz on $L^p([0, T])$. So, $\mathcal{Y}$ is locally Lipschitz on $L^p([0, T])$. Moreover, $X$ is compactly embedded into $L^p([0, T])$. So $\mathcal{Y}$ is locally Lipschitz on $X$. Furthermore, $\mathcal{Y}$ is sequentially weak upper semicontinuous. For all $u \in X$, by (I_1),

$$\int_0^{u(x_i)} I_i(s) ds < 0, \quad i = 1, 2, \ldots, m.$$ 

So, we have

$$\mathcal{N}(u) = \frac{1}{p} \|u\|_X^p - \frac{m}{i=1} \int_0^{u(x_i)} I_i(s) ds > \frac{1}{p} \|u\|_X^p$$

for all $u \in X$. Hence, $\mathcal{N}$ is coercive and $\inf_X \mathcal{N} = \mathcal{N}(0) = 0$. We want to prove that, under our hypotheses, there exists a sequence $\{\pi_n\} \subset X$ of critical points for the functional $I^*_L$, that is, every element $\pi_n$ satisfies

$$I^*_L(\pi_n, v - \pi_n) \geq 0, \quad \text{for every } v \in X.$$

Now, we claim that $\gamma < +\infty$. To see this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n \to +\infty} \xi_n = +\infty$ and

$$\lim_{n \to +\infty} \sup_{|t| \leq \xi_n} \min_{t_0} \int_0^t J(x, s) ds = \lim_{\xi \to +\infty} \sup_{|t| \leq \xi} \min_{t_0} \int_0^t J(x, s) ds \frac{x^p}{\xi^p}. \quad (3.1)$$

Put

$$r_n := \frac{2 \xi_n}{T^{1/q}} \in \mathbb{N}, \quad \text{for all } n \in \mathbb{N}.$$

Then, for all $v \in X$ with $\mathcal{N}(v) < r_n$, taking into account that $\|v\|_X < r_n$ and $\|v\|_{C^0} \leq \frac{T}{2} T^{1/q} \|v\|_X$, one has $|v(x)| \leq \xi_n$ for every $x \in [0, T]$. Therefore, for all $n \in \mathbb{N}$,

$$\varphi(r_n) = \inf_{u \in \mathcal{N}^{-1}(\{(-\infty, r)\})} \frac{\left(\sup_{v \in \mathcal{N}^{-1}((-\infty, r))} \mathcal{M}(v) - \mathcal{M}(u)\right)}{r - \mathcal{N}(u)} \leq \sup_{\|v\|_X < r_n} \left(\mathcal{F}(v) + \frac{1}{\lambda} \mathcal{G}(v)\right) \leq \sup_{|t| \leq \xi_n} \left(\int_0^T \min_{t_0} \int_0^t F(s) ds dx + \frac{1}{\lambda} \int_0^T \min_{t_0} \int_0^t G(x, s) ds dx\right) \leq \frac{T}{2} \left[p \sup_{|t| \leq \xi_n} \int_0^t F(s) ds + \frac{1}{\lambda} \int_0^T \sup_{|t| \leq \xi_n} \int_0^t G(x, s) ds\right] r_n \leq \frac{T}{2} \left[p \frac{\xi_n^p}{\xi_n^p} + \frac{r_n}{\lambda} \frac{\xi_n^p}{\xi_n^p}\right].$$
Moreover, from Assumptions (F4) and (G5), we have
\[
\lim_{n \to +\infty} \sup_{|t| \leq \xi_n} \min_{\xi_n} \int_0^t F(s) \, ds + \frac{p}{\lambda} \sup_{|t| \leq \xi_n} \min_{\xi_n} \int_0^t G(x, s) \, ds < +\infty,
\]
which follows
\[
\lim_{n \to +\infty} \sup_{|t| \leq \xi_n} \min_{\xi_n} \int_0^t J(x, s) \, ds < +\infty.
\]
Therefore,
\[
\gamma \leq \liminf_{n \to +\infty} \varphi(r_n) \leq p \left( \frac{T}{2} \right)^p \liminf_{n \to +\infty} \frac{\sup_{|t| \leq \xi_n} \min_{\xi_n} \int_0^t J(x, s) \, ds}{\xi_p} < +\infty. \tag{3.2}
\]
Since
\[
\sup_{|t| \leq \xi} \min_{\xi} \int_0^t F(s) \, ds \leq \sup_{|t| \leq \xi} \min_{\xi} \int_0^t F(s) \, ds + \frac{p}{\lambda} \sup_{|t| \leq \xi} \min_{\xi} \int_0^t G(x, s) \, ds,
\]
and taking (G5) into account, we get
\[
\liminf_{\xi \to +\infty} \sup_{|t| \leq \xi} \min_{\xi} \int_0^t F(s) \, ds \leq \liminf_{\xi \to +\infty} \sup_{|t| \leq \xi} \min_{\xi} \int_0^t F(s) \, ds + \frac{p}{\lambda} G_{\infty}. \tag{3.3}
\]
Moreover, from Assumption (G4) we obtain
\[
\limsup_{\xi \to +\infty} \frac{\int_0^T \min_{\xi} f_0^{\xi} (T - x) J(x, s) \, ds \, dx}{\frac{1}{p} \xi^p \left( T + \frac{2M}{p+1} (\frac{T}{2})^{p+1} \right) - \sum_{i=1}^{m} f_0^{\xi} (\frac{T}{2} - x_i) I_i(s) d s} \geq \limsup_{\xi \to +\infty} \frac{\int_0^T \min_{\xi} f_0^{\xi} (T - x) F(s) \, ds \, dx}{\frac{1}{p} \xi^p \left( T + \frac{2M}{p+1} (\frac{T}{2})^{p+1} \right) - \sum_{i=1}^{m} f_0^{\xi} (\frac{T}{2} - x_i) I_i(s) d s} \tag{3.4}
\]
Therefore, from (3.3) and (3.4), we observe that
\[
\lambda \in (\nu_1, \nu_2) \subseteq \frac{1}{\limsup_{\xi \to +\infty} \sup_{|t| \leq \xi} \min_{\xi} \int_0^t J(x, s) \, ds \frac{1}{\xi^p}},
\]
\[
\subseteq (0, 1/\gamma).
\]
For the fixed $\lambda$, the inequality (3.2) ensures that the condition (b) of Theorem 2.6 can be applied and either $I_{\lambda}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n \to +\infty} \|u_n\| = +\infty$. The other step is to show that for the fixed $\lambda$ the functional $I_{\lambda}$ has no global minimum. Let us verify that the functional $I_{\lambda}$ is unbounded from below. Since
\[
\frac{1}{\lambda} < \limsup_{\xi \to +\infty} \frac{\int_0^T \min_{\xi} f_0^{\xi} (T - x) F(s) \, ds \, dx}{\frac{1}{p} \xi^p \left( T + \frac{2M}{p+1} (\frac{T}{2})^{p+1} \right) - \sum_{i=1}^{m} f_0^{\xi} (\frac{T}{2} - x_i) I_i(s) d s} \leq \limsup_{\xi \to +\infty} \frac{\int_0^T \min_{\xi} f_0^{\xi} (T - x) J(x, s) \, ds \, dx}{\frac{1}{p} \xi^p \left( T + \frac{2M}{p+1} (\frac{T}{2})^{p+1} \right) - \sum_{i=1}^{m} f_0^{\xi} (\frac{T}{2} - x_i) I_i(s) d s},
\]
there exists a sequence \( \{\eta_n\} \) of positive numbers and a constant \( \tau \) such that 
\[
\lim_{n \to +\infty} \eta_n = +\infty \quad \text{and} \quad \frac{1}{\lambda} < \tau < \frac{\int_0^T \min_0^\eta_n(x) J(x,s) \, ds \, dx}{\frac{1}{p} \eta^p_n(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}) - \sum_{i=1}^m \int_0^\eta_n(\frac{T}{2}-x_i) I_i(s) \, ds},
\]
for each \( n \in \mathbb{N} \) large enough. For all \( n \in \mathbb{N} \), set
\[
w_n(x) := \eta_n(T - x).
\]
For any fixed \( n \in \mathbb{N} \), it is easy to see that \( w_n \in X \) and, in particular, one has
\[
\|w_n\|_X^p = \eta^p_n(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}) - \sum_{i=1}^m \int_0^{\eta_n(\frac{T}{2}-x_i)} I_i(s) \, ds.
\]
By (3.5) and (3.6), we see that
\[
I_\lambda(w_n) = \frac{1}{\lambda} \eta^n_p(T + \frac{2M}{p+1}(\frac{T}{2})^{p+1}) - \sum_{i=1}^m \int_0^{\eta_n(\frac{T}{2}-x_i)} I_i(s) \, ds
\]
for every \( n \in \mathbb{N} \) large enough. Since \( \lambda \tau > 1 \) and \( \lim_{n \to +\infty} \eta_n = +\infty \), we have
\[
\lim_{n \to +\infty} I_\lambda(w_n) = -\infty.
\]
Then, the functional \( I_\lambda \) is unbounded from below, and it follows that \( I_\lambda \) has no global minimum. Therefore, from part (b) of Theorem 2.6, the functional \( I_\lambda \) admits a sequence of critical points \( \{\pi_n\} \subset X \) such that \( \lim_{n \to +\infty} N(\pi_n) = +\infty \). Since \( N \) is bounded on bounded sets, and taking into account that \( \lim_{n \to +\infty} N(\pi_n) = +\infty \), then \( \{\pi_n\} \) has to be unbounded, i.e.,
\[
\lim_{n \to +\infty} \|\pi_n\|_X = +\infty.
\]
Moreover, if \( \pi_n \in X \) is a critical point of \( I_\lambda \), clearly, by definition, one has
\[
I_\lambda^X(\pi_n, v - \pi_n) \geq 0, \quad \text{for every} \ v \in X,
\]
finally, by Lemma 2.12, the critical points of \( I_\lambda \) are weak solutions for the problem (1.1), and by Lemma 2.10 every weak solution of (1.1) is a solution of (1.1). Hence, the assertion follows.

**Remark 3.2.** Under the conditions
\[
\liminf_{\xi \to +\infty} \sup_{|t| \leq \xi} \frac{\int_0^t F(s) \, ds}{\xi^p} = 0,
\]

the problem

Assume that Theorem 3.3.

problem (1.1) admits a sequence of solutions which is unbounded in \( X \).

from Theorem 3.1, we see that for every \( \lambda > 0 \) and for each \( \mu \in [0, \frac{2p}{p+1}] \), problem (1.1) admits a sequence of solutions which is unbounded in \( X \). Moreover, if \( G_\infty = 0 \), the result holds for every \( \lambda > 0 \) and \( \mu \geq 0 \).

The following result is a special case of Theorem 3.1 with \( \mu = 0 \).

**Theorem 3.3.** Assume that (F1)–(F4), (I1) hold. Then, for each

\[
\lambda \in \left( \limsup_{\xi \to +\infty} \frac{1}{\int_0^T \min_{t \in \mathbb{R}} \frac{p}{p} (T + \frac{2M}{p+1} (T)^p + 1) - \sum_{i=1}^m \int_0^{\xi (\frac{x}{T} - x_i)} I_i(s) ds} \right),
\]

the problem

\[-(\phi_p(u'(x)))' + M \phi_p(u(x)) \in \lambda F(u(x)) \quad \text{in} \ [0, T] \setminus Q,
\]

\[-\Delta \phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1, 2, \ldots, m,
\]

\[u(0) = -u(T), \quad u'(0) = -u'(T)\]

has an unbounded sequence of solutions in \( X \).

Now, we present the following example to illustrate our results.

**Example 3.4.** Consider the problem

\[-(\phi_3(u'(x)))' + \phi_3(u(x)) \in \lambda F(u(x)) \quad \text{in} \ [0, 2] \setminus \{1\},
\]

\[-\Delta \phi_3(u'(x_1)) = I_1(u(x_1)), \quad x_1 = 1,
\]

\[u(0) = -u(2), \quad u'(0) = -u'(2), \quad u'(2) = 0
\]

(3.7)

where, for \( s \in \mathbb{R} \),

\[F(s) = \begin{cases} 
\{0\}, & \text{if } |s| < 2^{-1/3}, \\
[0, 1], & \text{if } |s| = 2^{-1/3}, \\
\{s - 2^{-1/3} + 1\}, & \text{if } s > 2^{-1/3}, \\
\{s + 2^{-1/3} + 1\}, & \text{if } s < -2^{-1/3}.
\end{cases}
\]

Simple calculations show that

\[\sup_{|t| \leq 2^{-1/3}} \min_{|t| \leq 2^{-1/3}} \int_0^t F(s) ds = 0
\]

and

\[
\frac{2}{5} \int_0^2 \min_{t \in \mathbb{R}} \int_0^{\xi (1-x)} F(s) ds dx \\
\frac{5}{6} \xi^3 - \int_0^{\xi (1-x)} I_1(s) ds \\
= \frac{6}{5} \xi^3 \int_{-1}^1 \min_{t \in \mathbb{R}} \int_0^{\xi x} F(s) ds dx
\]
compact convex values, such that
\[ a > 0 \]
for some \( \xi \in \mathbb{R} \). So,
\[
\liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min_{t \in [0,T]} \int_0^t F(s) ds}{\frac{1}{\xi^3}} = 0,
\]
\[
\limsup_{\xi \to +\infty} \frac{\int_0^2 \min_{t \in [0,T]} \int_0^{\xi(1-x)} F(s) ds dx}{\frac{1}{\xi^3} - \int_0^{\xi(1-x)} I_1(s) ds} > 0.
\]
Hence, using Theorem 3.3, problem (3.1), for \( \lambda \) lying in a convenient interval, has an unbounded sequence of solutions in \( X := \{ u \in W^{1,3}([0,2]) : u(0) = -u(2) \} \).

Here we point out the following consequences of Theorem 3.3, using the assumptions
(F5) \( \liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min_{t \in [0,T]} \int_0^t F(s) ds}{\xi^p} < \frac{1}{p} \left( \frac{2}{T} \right)^p \);
(F6) \( \limsup_{\xi \to +\infty} \frac{\int_0^2 \min_{t \in [0,T]} \int_0^{\xi(1-x)} F(s) ds dx}{\frac{1}{\xi^3} - \int_0^{\xi(1-x)} I_1(s) ds} > 1. \)

**Corollary 3.5.** Assume that (F1)–(F3), (F5)–(F6), (11) hold. Then, the problem
\[-(\phi_p(u'(x)))' + M \phi_p(u(x)) \in F(u(x)) \quad \text{in } [0,T] \setminus Q,\]
\[-\Delta \phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1, 2, \ldots, m,\]
\[ u(0) = -u(T), \quad u'(0) = -u'(T) \]
has an unbounded sequence of solutions in \( X \).

**Remark 3.6.** Theorem 1.1 in the Introduction is an immediate consequence of Corollary 3.5.

Now, we give the following consequence of the main result.

**Corollary 3.7.** Let \( F_1 : \mathbb{R} \to 2^\mathbb{R} \) be an upper semicontinuous multifunction with compact convex values, such that \( \min F_1, \max F_1 : \mathbb{R} \to \mathbb{R} \) are Borel measurable and \( |\xi| \leq a(1 + |s|^{r_1-1}) \) for all \( s \in \mathbb{R}, \xi \in F_1(s), r_1 > 1 (a > 0) \). Furthermore, suppose that
\[
\liminf_{\xi \to +\infty} \frac{\sup_{|t| \leq \xi} \min_{t \in [0,T]} \int_0^t F_1(s) ds}{\xi^p} < +\infty;
\]
\[
\limsup_{\xi \to +\infty} \frac{\int_0^2 \min_{t \in [0,T]} \int_0^{\xi(1-x)} F_1(s) ds dx}{\frac{1}{\xi^3} - \int_0^{\xi(1-x)} I_1(s) ds} = +\infty.
\]
Then, for every multifunction \( F_2 : \mathbb{R} \to 2^\mathbb{R} \) which is upper semicontinuous with compact convex values, \( \min F_2, \max F_2 : \mathbb{R} \to \mathbb{R} \) are Borel measurable and \( |\xi| \leq b(1 + |s|^{r_2-1}) \) for all \( s \in \mathbb{R}, \xi \in F_2(s), r_2 > 1 (b > 0) \), and satisfies the conditions
\[
\sup_{t \in \mathbb{R}} \min_{I \in \mathbb{R}} \int_0^t F_2(s) ds \leq 0
\]
and
\[
\liminf_{\xi \to +\infty} \frac{\int_0^T \min_{t \in [0,T]} \int_0^{\xi(1-x)} F_2(s) ds dx}{\frac{1}{\xi^p} - \frac{2M}{p+1} \left( \frac{T}{2} \right)^p} - \sum_{i=1}^m \int_0^{\xi(1-x_i)} I_i(s) ds \quad > -\infty,
\]
and
solutions, which strongly converges to 0 in $X \rightarrow Q$. Remark 3.8. We observe that in Theorem 3.1 we can replace □ by applying Theorem 3.3, we have the desired conclusion.

Moreover, Assumption (C1) and the condition

\[-\Delta \phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1, 2, \ldots, m,\]

\[u(0) = -u(T), \quad u'(0) = -u'(T)\]

has an unbounded sequence of solutions in $X$.

Proof. Set $F(t) = F_1(t) + F_2(t)$ for all $t \in \mathbb{R}$. Assumption (C2) along with the condition

\[\liminf_{\xi \rightarrow +\infty} \frac{1}{1/p - 1/p} \int_0^T \min_{\xi \in \mathbb{R}} \left( T + \frac{2M}{p+1} \right) - \sum_{i=1}^m \int_0^T I_i(s)ds > -\infty\]

yield

\[\limsup_{\xi \rightarrow +\infty} \frac{1}{1/p - 1/p} \int_0^T \min_{\xi \in \mathbb{R}} \left( T + \frac{2M}{p+1} \right) - \sum_{i=1}^m \int_0^T I_i(s)ds = +\infty.\]

Moreover, Assumption (C1) and the condition

\[\sup_{t \in \mathbb{R}} \int_0^t F_2(s)ds \leq 0\]

ensure that

\[\liminf_{\xi \rightarrow +\infty} \frac{\sup_{|t| \leq \xi} \int_0^t F(s)ds}{\xi^p} \leq \liminf_{\xi \rightarrow +\infty} \frac{\sup_{|t| \leq \xi} \int_0^t F_1(s)ds}{\xi^p} < +\infty.\]

Since

\[\liminf_{\xi \rightarrow +\infty} \frac{1}{1/p - 1/p} \int_0^T F(s)ds \geq \liminf_{\xi \rightarrow +\infty} \frac{1}{1/p - 1/p} \int_0^T F_1(s)ds,\]

by applying Theorem 3.3 we have the desired conclusion.

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References


Shapour Heidarkhani
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran
E-mail address: s.heidarkhani@razi.ac.ir

Ghasem A. Afrouzi
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: afrouzi@umz.ac.ir

Armin Hadjian
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
E-mail address: a.hadjian@umz.ac.ir

Johnny Henderson
Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
E-mail address: Johnny.Henderson@baylor.edu