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STABILITY AND PERIODICITY OF SOLUTIONS FOR DELAY DYNAMIC SYSTEMS ON TIME SCALES

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Abstract. This article concerns the stability and periodicity of solutions to the delay dynamic system

$$x^{\triangle}(t) = A(t)x(t) + F(t, x(t), x(g(t))) + C(t)$$

on a time scale. By the inequality technique for vectors, we obtain some stability criteria for the above system. Then, using the Horn fixed point theorem, we present some conditions under which our system is asymptotically periodic and its periodic solution is unique. In particular, the periodic solution is positive under proper assumptions.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^m$ the m-dimensional Euclidean space and $\mathbb{T}$ a time scale, i.e., an nonempty closed subset of $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

and the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

For the other terminologies on time scales, we refer the reader to [1, 2, 3].

In this paper, we consider the stability and periodicity of solutions for the delay dynamic system

$$x^{\triangle}(t) = A(t)x(t) + F(t, x(t), x(g(t))) + C(t), \quad t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is a τ-periodic time scale, that is, $t + \tau \in \mathbb{T}$ for all $t \in \mathbb{T}$, $\inf \mathbb{T} = -\tau < 0$ and $\sup \mathbb{T} = \infty$.

Since functional differential (or difference) equations with delays appear in a number of ecological models, there have been many research activities concerning the qualitative behavior for the special cases of (1.1); see, for example, the references [4, 5, 6, 9, 10, 11, 12, 13, 14, 15], where stability and periodicity of solutions of

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related equations have been investigated. In particular, for $\mathbb{T} = [-\tau, \infty)$ and the scalar case, Cheng and Zhang [4] in 2001 and Franco et al [5] in 2007 established some existence criteria of periodic solutions to the equation

$$x'(t) = -a(t)x(t) + \lambda b(t)f(x(t - h(t))), \quad t \in [-\tau, \infty), \tag{1.2}$$

where λ is a postive constant, $a, b \in C(\mathbb{R}, [0, \infty))$ are ω-periodic functions with $\int_0^\omega a(s)\mathrm{d}s > 0$ and $\int_0^\omega b(s)\mathrm{d}s > 0$, $h \in C(\mathbb{R}, \mathbb{R})$ is ω-periodic, and $f \in C(\mathbb{R}, [0, \infty))$. Precisely speaking, Franco et al improved the work in [4] and obtained the following result [5, Theorem 2].

Theorem 1.1. *Assume that*

$$\sigma = e^{\int_0^\omega a(s)\mathrm{d}s}, \quad G(t, s) = \frac{\int_t^s a(r)\mathrm{d}r}{\sigma - 1},$$

$$A = \max_{t\in[0,\omega]} \int_0^\omega G(t, s)b(s)\mathrm{d}s, \quad B = \min_{t\in[0,\omega]} \int_0^\omega G(t, s)b(s)\mathrm{d}s.$$

Assume further that $f_0, f_\infty \in (0, \infty)$, and

$$\frac{1}{B \max\{f_0, f_\infty\}} < \lambda < \frac{1}{A \min\{f_0, f_\infty\}},$$

where f_0 and f_∞ are defined respectively by

$$f_0 = \lim_{x\to 0^+} \frac{f(x)}{x}, \quad f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}.$$

Then (1.2) has a positive ω-periodic solution.

In 2003, Zhao [14] studied the asymptotic periodic solutions of (1.1) for the case when $\mathbb{T} = [-\tau, \infty)$ and the function F is independent of the second variable. In 2008, Zhu and Cheng [15] considered the periodic solutions of (1.1) when $\mathbb{T} = \{-\tau, -\tau + 1, -\tau + 2, \dots\}$ with positive integer τ.

The aim of this article is to extend the techniques in [15] and establish some criteria for the stability and periodicity of (1.1). As we will see in the sequel, our results will generalize or improve those in [14, 15] and, to some extent, possess more comprehensive suitability than [5].

2. Preliminaries

Let $[a, b]_\mathbb{T}$ be an interval in $\mathbb{T}$ which is defined by $[a, b]_\mathbb{T} = \{t \in \mathbb{T} : a \le t \le b\}$. The symbol $[a, \infty)_\mathbb{T}$ has a similar meaning. Let $C[a, b]_\mathbb{T}$ with $a, b \in \mathbb{T}$ denote the set of all rd-continuous functions mapping $[a, b]_\mathbb{T}$ into $\mathbb{R}^m$, endowed with the linear structure and the norm $\|\cdot\|$ defined by $\|\varphi\| = \max_{t\in[a,b]_\mathbb{T}} \|\varphi(t)\|$, where $\|\varphi(t)\|$ denotes the norm of vector $\varphi(t)$. Then $C[a, b]_\mathbb{T}$ is a Banach space.

For a given $\varphi \in C[-\tau, 0]_\mathbb{T}$, by a solution $x(\varphi)$ of (1.1) we mean that $x(\varphi)$ is a function defined on $[-\tau, \infty)_\mathbb{T}$ which coincides with φ on $[-\tau, 0]_\mathbb{T}$ and satisfies (1.1) on $[0, \infty)_\mathbb{T}$.

For any $x = (x_1, x_2, \dots, x_m)^T, y = (y_1, y_2, \dots, y_m)^T \in \mathbb{R}^m$, the symbols $x \le y$ and $x < y$ mean, respectively, that $x_i \le y_i$ and $x_i < y_i$ for $i = 1, 2, \dots, m$. The absolute $|x|$ of vector x means that $|x| = (|x_1|, |x_2|, \dots, |x_m|)^T$. For any matrices $A = (a_{ij})_{m\times m}$ and $B = (b_{ij})_{m\times m}$ in $\mathbb{R}^{m\times m}$, the symbols $A \le B$, $A < B$ and $|A|$ can be defined accordingly.

In this paper, we assume throughout that $A : \mathbb{T} \to \mathbb{R}^{m\times m}$, $C : \mathbb{T} \to \mathbb{R}^m$ are all rd-continuous, $g : [0, \infty)_\mathbb{T} \to \mathbb{T}$ is rd-continuous and transforms $[t, t + \tau]_\mathbb{T}$ into

$[t-\tau,t]_{\mathbb{T}}$ for $t\in[0,\infty)_{\mathbb{T}}$, and $F:\mathbb{T}\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ is continuous and satisfies $F(\cdot,0,0)\equiv 0$. Further, we assume that there exist $\alpha>0$, $A_i\in\mathbb{R}^{m\times m}$ with $A_i\ge 0$ for $i\in\{0,1,2\}$, and $C_0\in\mathbb{R}^m$ with $C_0>0$ such that

- (H1) $1-\alpha\mu(t)>0$ for $t\in\mathbb{T}$ and $\lim_{t\to\infty}e_{-\alpha}(t,0)=0$;
- (H2) $|A(t)+\alpha I|\le A_0$ for all $t\in[0,\infty)_{\mathbb{T}}$;
- (H3) $|F(t,x_1,x_2)-F(t,y_1,y_2)|\le A_1|x_1-y_1|+A_2|x_2-y_2|$ and $|C(t)|\le C_0$ for all $t\in[0,\infty)_{\mathbb{T}}$;
- (H4) $\widetilde{\rho}(A_0+A_1+A_2)<\alpha$, where $\widetilde{\rho}$ denotes the spectral radius of square matrices.

We remark that the real number $\alpha>0$ depends on the concrete time scale. For example, let $\mathbb{T}$ be the set of nonnegative integers. Then, $1-\alpha\mu(t)>0$ when $0<\alpha<1$. In this case we have $e_{-\alpha}(t,0)=(1-\alpha)^t$ and hence $\lim_{t\to\infty}e_{-\alpha}(t,0)=0$. While $\mathbb{T}=[0,\infty)$, $e_{-\alpha}(t,0)=e^{-\alpha t}$ and $1-\alpha\mu(t)>0$ for any real number $\alpha>0$. In this case $\lim_{t\to\infty}e_{-\alpha}(t,0)=0$. We remark further that $e_{-\alpha}(t,0)>0$ by the assumption (H1) (see [2, Theorem 2.44]).

Note that (1.1) can be rewritten as

$$x^{\Delta}(t)=-\alpha x(t)+(A(t)+\alpha I)x(t)+F(t,x(t),x(g(t)))+C(t),\quad t\in\mathbb{T},$$

where I is the $m\times m$ identity matrix. Then, similar to [2, Theorem 5.24] (see also [2, Theorems 8.16, 8.24]), the solution $x(\varphi)$ (x for short) of (1.1) is existent and unique on $[-\tau,\infty)_{\mathbb{T}}$, and given by

$$\begin{gathered}
x(t)=\varphi(t),\ t\in[-\tau,0]_{\mathbb{T}},\\
\begin{aligned}
x(t)=e_{-\alpha}(t,0)\varphi(0)+\int_0^t e_{-\alpha}(t,\sigma(s))\Big[&(A(s)+\alpha I)x(s)\\
&+F(s,x(s),x(g(s)))+C(s)\Big]\Delta s,\quad t\in[0,\infty)_{\mathbb{T}}.
\end{aligned}
\end{gathered}\tag{2.1}$$

For a subset $\mathbb{S}$ of $\mathbb{R}^m$ and $\varepsilon\in\mathbb{R}^m$ with $\varepsilon>0$, let $O(\mathbb{S},\varepsilon)$ be defined by

$$O(\mathbb{S},\varepsilon)=\{x\in\mathbb{R}^m:\inf_{s\in\mathbb{S}}|x-s|\le\varepsilon\}.$$

Definition 2.1. Let $x(\varphi_0)$ be a solution of (1.1) and $\varphi\in C[-\tau,0]_{\mathbb{T}}$. If $\varphi\to\varphi_0$ implies $x(\varphi)(t)\to x(\varphi_0)(t)$ for all $t\in[0,\infty)_{\mathbb{T}}$, then the solution $x(\varphi_0)$ is said to be stable.

Definition 2.2. System (1.1) is said to be equi-bounded if, for any $H\in\mathbb{R}^m$ with $H>0$, there exists an $M(H)\in\mathbb{R}^m$ such that $\varphi\in C[-\tau,0]_{\mathbb{T}}$ with $|\varphi(t)|\le H$ on $[-\tau,0]_{\mathbb{T}}$ implies $|x(\varphi)(t)|\le M(H)$ for all $t\in[0,\infty)_{\mathbb{T}}$.

Definition 2.3. A set $\mathbb{S}\subset\mathbb{R}^m$ is said to be a global attractor of (1.1) if, for any $\varepsilon\in\mathbb{R}^m$ with $\varepsilon>0$ and $\varphi\in C[-\tau,0]_{\mathbb{T}}$, there exists a positive number $T(\varepsilon,\varphi)\in\mathbb{T}$ such that the solution $x(\varphi)$ of (1.1) satisfies $x(\varphi)(t)\in O(\mathbb{S},\varepsilon)$ for all $t\in[T(\varepsilon,\varphi),\infty)_{\mathbb{T}}$.

Definition 2.4. System (1.1) is said to be extremely stable if any two solutions $x(\varphi_1)$ and $x(\varphi_2)$ of (1.1) satisfy $x(\varphi_1)(t)-x(\varphi_2)(t)\to 0$ as $t\to\infty$.

The subset B of $C[a,b]_{\mathbb{T}}$ is said to be equi-continuous if, for any real number $\varepsilon>0$, there exists a $\delta>0$ such that $\|x(t_1)-x(t_2)\|<\varepsilon$ for all $x\in B$ provided $|t_1-t_2|\le\delta$ for $t_1,t_2\in[a,b]_{\mathbb{T}}$. In what follows, we will require the time scale version of Arzela-Ascoli theorem. Although a similar result has been given in [16], for the completeness we present the following lemma and its proof.

Lemma 2.5. *If $B \subset C[a,b]_{\mathbb{T}}$ is bounded and equi-continuous, then B is relatively compact.*

Proof. Note that $C[a,b]_{\mathbb{T}}$ is a Banach space, we need only to show that B is completely bounded. Since B is equi-continuous, for any real number $\varepsilon > 0$, there exists a $\delta > 0$ such that when $t_1, t_2 \in [a,b]_{\mathbb{T}}$ with $|t_1 - t_2| \le \delta$, it follows that

$$\|f(t_1) - f(t_2)\| < \varepsilon \quad \text{for all } f \in B. \tag{2.2}$$

For this δ, there exists a partition on $[a,b]_{\mathbb{T}}$ as follows

$$a = t_1 < t_2 < \dots < t_n = b,$$

which satisfies $t_i - t_{i-1} \le \delta$, or $t_i - t_{i-1} > \delta$ but $\rho(t_i) = t_{i-1}$ (see [3, Lemma 5.7]). Let

$$\widetilde{B} = \{(f(t_1), f(t_2), \dots, f(t_n)) : f \in B\}.$$

Since B is bounded, $\widetilde{B} \in \mathbb{R}^{m\times n}$ is also bounded and hence is completely bounded. Therefore, there exists an ε-net $\{f_1, f_2, \dots, f_k\}$ of $\widetilde{B}$. We assert that it is also an ε-net of B. Indeed, for any $f \in B$, $(f(t_1), f(t_2), \dots, f(t_n)) \in \widetilde{B}$ implies that there exists some $v \in \{1, 2, \dots, k\}$ such that

$$\|f(t_i) - f_v(t_i)\| < \varepsilon, \quad i = 1, 2, \dots, n. \tag{2.3}$$

Note that $t \in [a,b]_{\mathbb{T}}$ implies $t \in [t_i, t_{i+1}]_{\mathbb{T}}$ for some $i \in \{1, 2, \dots, n-1\}$. Now there are two cases to consider.

Case 1: $t_{i+1} - t_i \le \delta$. In this case, we have from (2.2) and (2.3) that

$$\|f(t) - f_v(t)\| \le \|f(t) - f(t_i)\| + \|f(t_i) - f_v(t_i)\| + \|f_v(t_i) - f_v(t)\| < 3\varepsilon.$$

Case 2: $t_{i+1} - t_i > \delta$. In this case, $\rho(t_{i+1}) = t_i$ and hence, $t = t_i$ or $t = t_{i+1}$. Thus $\|f(t) - f_v(t)\| < \varepsilon$ by (2.3).

To sum up, $\{f_1, f_2, \dots, f_k\}$ is an ε-net of B. The proof is complete. $\square$

In the sequel, the following standard results will be imposed.

Lemma 2.6 ([7, 14]). *Let $R \in \mathbb{R}^{m\times m}$ and $\widetilde{\rho}(R)$ be the spectral radius of R. If $R \ge 0$ and $\widetilde{\rho}(R) < 1$, then $(I - R)^{-1} \ge 0$.*

Lemma 2.7 ([8]). *Let $\mathbb{X}$ be a Banach space and $u : \mathbb{X} \to \mathbb{X}$ be a completely continuous. If there exists a bounded set $\mathbb{E}$ such that for each $x \in \mathbb{X}$, there exists integer $n = n(x)$ such that $u^n(x) \in \mathbb{E}$, then u has a fixed point in $\mathbb{E}$.*

3. Stability analysis

Note that, by assumption (H4) and Lemma 2.6, the matrix $I - \frac{1}{\alpha}(A_0 + A_1 + A_2)$ is invertible. For the sake of convenience, we let

$$U = \big(I - \frac{A_0 + A_1 + A_2}{\alpha}\big)^{-1}$$

whenever it is defined.

Theorem 3.1. *For any $\varphi_0 \in C[-\tau, 0]_{\mathbb{T}}$, the solution $x(\varphi_0)$ of (1.1) is stable.*

Proof. Let $\varphi \in C[-\tau,0]_{\mathbb{T}}$. From (2.1) we have

$$\begin{aligned}
&|x(\varphi)(t)-x(\varphi_0)(t)| \\
&\le e_{-\alpha}(t,0)|\varphi(0)-\varphi_0(0)| + \int_0^t e_{-\alpha}(t,\sigma(s))(A_0+A_1)|x(\varphi)(s)-x(\varphi_0)(s)|\Delta s \\
&\quad + \int_0^t e_{-\alpha}(t,\sigma(s))A_2|x(\varphi)(g(s))-x(\varphi_0)(g(s))|\Delta s.
\end{aligned} \tag{3.1}$$

For any $\varepsilon \in \mathbb{R}^m$ with $\varepsilon > 0$, we assert that $|x(\varphi)(t)-x(\varphi_0)(t)| \le \frac{1}{\alpha}U\varepsilon$ for all $t \in [0,\infty)_{\mathbb{T}}$ when $|\varphi(t)-\varphi_0(t)| \le \frac{1}{\alpha}U\varepsilon$ on $[-\tau,0]_{\mathbb{T}}$. Otherwise, there exists a $t_1 \in \mathbb{T}\cap(0,\infty)$ and a real number $\beta > 1$ such that

$$|x(\varphi)(t)-x(\varphi_0)(t)| \le \frac{\beta}{\alpha}U\varepsilon, t \in [0,t_1]_{\mathbb{T}}, \tag{3.2}$$

$$|x_v(\varphi)(t_1)-x_v(\varphi_0)(t_1)| = \big(\frac{\beta}{\alpha}U\varepsilon\big)_v, \tag{3.3}$$

where $x_v(\varphi)(t_1)-x_v(\varphi_0)(t_1)$ stands for some component of the vector $x(\varphi)(t_1) - x(\varphi_0)(t_1)$. Then, we have from (3.1) and (3.2) that

$$\begin{aligned}
&|x(\varphi)(t_1)-x(\varphi_0)(t_1)| \\
&\le e_{-\alpha}(t_1,0)|\varphi(0)-\varphi_0(0)| + \int_0^{t_1} e_{-\alpha}(t_1,\sigma(s))\Delta s \cdot \frac{\beta}{\alpha}(A_0+A_1+A_2)U\varepsilon \\
&< \frac{\beta}{\alpha}e_{-\alpha}(t_1,0)U\varepsilon + \int_0^{t_1} e_{-\alpha}(t_1,\sigma(s))\Delta s \cdot \frac{\beta}{\alpha}[(A_0+A_1+A_2)U\varepsilon+\alpha\varepsilon] \\
&= \frac{\beta}{\alpha}e_{-\alpha}(t_1,0)U\varepsilon + \frac{1}{\alpha}[1-e_{-\alpha}(t_1,0)] \cdot \frac{\beta}{\alpha}[(A_0+A_1+A_2)U\varepsilon+\alpha\varepsilon] \\
&= \frac{\beta}{\alpha}U\varepsilon,
\end{aligned}$$

which is in conflict with (3.3), where we have used the formula

$$\int_0^{t_1} e_{-\alpha}(t_1,\sigma(s))\Delta s = -\frac{1}{\alpha}[e_{-\alpha}(t_1,0)-1],$$

see [2, Theorem 2.39] for the details. Since $\varepsilon > 0$ is arbitrary, the solution $x(\varphi_0)$ of (1.1) is stable. The proof is complete. □

Theorem 3.2. *System* (1.1) *is equi-bounded and the set*

$$\mathbb{S} = \big\{s \in \mathbb{R}^m : |s| \le \frac{1}{\alpha}UC_0\big\}$$

is a global attractor of (1.1).

Proof. For the first part, we note that $\frac{1}{\alpha}UC_0 > 0$. Then, for any $H \in \mathbb{R}^m$ with $H > 0$, there exists a real number $\beta_0 \ge 1$ such that $H \le \frac{\beta_0}{\alpha}UC_0$. Now, for any given $\varphi \in C[-\tau,0]_{\mathbb{T}}$ with $|\varphi(t)| \le H$ on $[-\tau,0]_{\mathbb{T}}$, we assert that the solution $x(\varphi)$ of (1.1) satisfies $|x(\varphi)(t)| \le \frac{\beta_0}{\alpha}UC_0$ on $[0,\infty)_{\mathbb{T}}$. Otherwise, similar to the proof of Theorem 3.1, there exist a $t_1 \in \mathbb{T}\cap(0,\infty)$ and a real number $\beta_1 > 1$ such that

$$|x(\varphi)(t)| \le \frac{\beta_1\beta_0}{\alpha}UC_0, t \in [0,t_1]_{\mathbb{T}}, \tag{3.4}$$

$$|x_v(\varphi)(t_1)| = \big(\frac{\beta_1\beta_0}{\alpha}UC_0\big)_v. \tag{3.5}$$

By the same method as above, from (2.1) and (3.4) we have

$$|x(\varphi)(t_1)| < \frac{\beta_1\beta_0}{\alpha}UC_0,$$

which contradicts (3.5).

Next we show the second part. By the equi-boundedness, it follows that the solution $x(\varphi)$ of (1.1) is bounded for any $\varphi \in C[-\tau, 0]_{\mathbb{T}}$ and hence we may assume that

$$\limsup_{t\to\infty} |x(\varphi)(t)| = \frac{1}{\alpha}UC_0 + \beta,$$

where $\beta \in \mathbb{R}^m$. Therefore, for any given $\varepsilon \in \mathbb{R}^m$ with $\varepsilon > 0$, there exists a large $t_1 \in [0,\infty)_{\mathbb{T}}$ such that

$$|x(\varphi)(t)| \le \frac{1}{\alpha}UC_0 + \beta + \varepsilon \quad \text{for all } t \in [t_1 - \tau, \infty)_{\mathbb{T}}. \tag{3.6}$$

Let

$$U(\varepsilon) = \frac{1}{\alpha}UC_0 + \beta + \varepsilon$$

and $x(t)$ for $x(\varphi)(t)$. According to (2.1) we see that when $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned} x(t) &= e_{-\alpha}(t,t_1)x(t_1) \\ &\quad + \int_{t_1}^{t} e_{-\alpha}(t,\sigma(s))[(A(s)+\alpha I)x(s) + F(s,x(s),x(g(s)))) + C(s)]\Delta s, \end{aligned}$$

and this, together with (3.6), yields

$$\begin{aligned} |x(t)| &\le e_{-\alpha}(t,t_1)|x(t_1)| + \int_{t_1}^{t} e_{-\alpha}(t,\sigma(s))\Delta s[(A_0 + A_1 + A_2)U(\varepsilon) + C_0] \\ &= e_{-\alpha}(t,t_1)|x(t_1)| + \frac{1}{\alpha}[1 - e_{-\alpha}(t,t_1)][(A_0 + A_1 + A_2)U(\varepsilon) + C_0], \end{aligned}$$

which, with the aid of $e_{-\alpha}(t,t_1) = e_{-\alpha}(t,0)e_{-\alpha}(0,t_1) \to 0$ as $t \to \infty$ (see assumption (H1)), results in

$$\frac{1}{\alpha}UC_0 + \beta \le \frac{1}{\alpha}[(A_0 + A_1 + A_2)U(\varepsilon) + C_0],$$

namely

$$\big(1 - \frac{A_0 + A_1 + A_2}{\alpha}\big)\beta \le \frac{A_0 + A_0 + A_2}{\alpha}\varepsilon. \tag{3.7}$$

Note that assumption (H4) and Lemma 2.6 imply

$$\big(1 - \frac{A_0 + A_1 + A_2}{\alpha}\big)^{-1} \ge 0.$$

Consequently, by (3.7) we see that

$$\beta \le \big(I - \frac{A_0 + A_1 + A_2}{\alpha}\big)^{-1}\frac{A_0 + A_1 + A_2}{\alpha}\varepsilon,$$

which yields $\beta \le 0$. Now from (3.6) we show that $\mathbb{S}$ is a global attractor of (1.1). The proof is complete. $\square$

Remark 3.3. The proof of the first part in Theorem 3.2 shows that the solution $x(\varphi)$ of (1.1) satisfies that

$$|x(\varphi)(t)| \le \mathbb{S} \quad \text{for all } t \in [0,\infty)_{\mathbb{T}}$$

when $\varphi \in \mathbb{S}$, where $\mathbb{S}$ is defined as in Theorem 3.2. In other words, $\mathbb{S}$ is an invariant set of (1.1). Accordingly, our Theorem 3.2 covers the conclusions in [14, Theorems 1 and 2].

Note that if the functions $A(t)$, $C(t)$ and $F(t,x,y)$ satisfy $A(t)+\alpha I \ge 0$, $C(t) \ge 0$ and $F(t,x,y) \ge 0$ for all $t \in \mathbb{T}$ and $x, y \in \mathbb{R}^m$, then from (2.1) we have

$$x(\varphi)(t) \ge e_{-\alpha}(t,0)\varphi(0) \to 0 \text{ as } t \to \infty.$$

Hence, by (3.6) we see that, for any given positive vector $\varepsilon \in \mathbb{R}^m$ there exists a positive number $T \in \mathbb{T}$ such that

$$-\varepsilon \le x(\varphi)(t) \le \frac{1}{\alpha} U C_0 + \varepsilon \quad \text{for } t \in [T,\infty)_{\mathbb{T}}.$$

Now the following conclusion is clear.

Theorem 3.4. *Assume that in* (1.1)*, the functions* $A(t)$*,* $C(t)$ *and* $F(t,x,y)$ *satisfy* $A(t)+\alpha I \ge 0$*,* $C(t) \ge 0$ *and* $F(t,x,y) \ge 0$ *for all* $t \in \mathbb{T}$ *and* $x, y \in \mathbb{R}^m$*. Then the set*

$$\mathbb{S}_0 = \big\{ s \in \mathbb{R}^m : 0 \le s \le \frac{1}{\alpha} U C_0 \big\}$$

is a global attractor of (1.1)*.*

Theorem 3.5. *System* (1.1) *is extremely stable.*

Proof. Let $x_1 = x(\varphi_1)$ and $x_2 = x(\varphi_2)$ be any two solutions of (1.1) and $z(t) = x_1(t) - x_2(t)$. Then, Theorem 3.2 implies that $z(t)$ is bounded. Thus we may assume that

$$\limsup_{t\to\infty} |z(t)| = \widetilde{z}. \tag{3.8}$$

This means that, for any positive vector $\varepsilon \in \mathbb{R}^m$, there exists a large $t_1 \in [0,\infty)_{\mathbb{T}}$ such that

$$|z(t)| \le \widetilde{z} + \varepsilon, \quad t \in [t_1 - \tau, \infty)_{\mathbb{T}}. \tag{3.9}$$

In addition, $z(t)$ must satisfy

$$z^{\Delta}(t) = A(t)z(t) + F(t, x_1(t), x_1(g(t))) - F(t, x_2(t), x_2(g(t))).$$

Similar to (2.1), it follows that when $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned} z(t) = e_{-\alpha}(t,t_1)z(t_1) &+ \int_{t_1}^{t} e_{-\alpha}(t,\sigma(s))(A(s)+\alpha I)z(s)\Delta s \\ &+ \int_{t_1}^{t} e_{-\alpha}(t,\sigma(s))[F(s,x_1(s),x_1(g(s))) - F(s,x_2(s),x_2(g(s)))]\Delta s, \end{aligned}$$

which, associated with (3.9), implies that when $t \in [t_1,\infty)_{\mathbb{T}}$,

$$|z(t)| \le e_{-\alpha}(t,t_1)|z(t_1)| + \frac{1}{\alpha}[1 - e_{-\alpha}(t,t_1)](A_0 + A_1 + A_2)(\widetilde{z} + \varepsilon). \tag{3.10}$$

Invoking (3.8), from (3.10) we have

$$\widetilde{z} \le \frac{A_0 + A_1 + A_2}{\alpha}(\widetilde{z} + \varepsilon)$$

and then

$$\widetilde{z} \le \Big(I - \frac{A_0 + A_1 + A_2}{\alpha}\Big)^{-1} \frac{A_0 + A_1 + A_2}{\alpha}\varepsilon. \tag{3.11}$$

Since $\varepsilon > 0$ is arbitrary, by (3.8) and (3.11) we see that $\lim_{t\to\infty} z(t) = 0$, which completes our proof. □

4. Existence of periodic solutions

In this section, we consider the existence of periodic solutions for (1.1). To do this, we set

$$G(t, x(t)) = A(t)x(t) + F(t, x(t), x(g(t))) + C(t). \tag{4.1}$$

For a solution $x(\varphi)$ of (1.1) and $t \in [0,\infty)_{\mathbb{T}}$, the delay function $x_t(\varphi) : [-\tau, 0]_{\mathbb{T}} \to \mathbb{R}^m$ is defined by $x_t(\varphi)(\theta) = x(\varphi)(t + \theta)$ for $\theta \in [-\tau, 0]_{\mathbb{T}}$.

Theorem 4.1. *Suppose that $\omega \ge \tau$, $A(t) = A(t + \omega)$, $C(t) = C(t + \omega)$, $F(t,\cdot,\cdot) = F(t + \omega,\cdot,\cdot)$ and $g(t + \omega) = g(t) + \omega$ for all $t \in [0,\infty)_{\mathbb{T}}$. Then (1.1) has a unique ω-periodic solution.*

Proof. Let $u : C[-\tau, 0]_{\mathbb{T}} \to C[-\tau, 0]_{\mathbb{T}}$ be defined by

$$u(\varphi) = x_\omega(\varphi), \varphi \in C[-\tau, 0]_{\mathbb{T}}.$$

Then, Theorem 3.1 implies that u is continuous on $C[-\tau, 0]_{\mathbb{T}}$. Since the solution of (1.1) is existent and unique, by the periodicity of A, C and F with respect to t we obtain that

$$x(\varphi)(t + \omega) = x(x_\omega(\varphi))(t), t \in [0,\infty)_{\mathbb{T}}, \tag{4.2}$$

where $x(x_\omega(\varphi))$ represents the solution of (1.1) through $x_\omega(\varphi)$. From (4.2) we have $x_\omega(x_\omega(\varphi)) = x_{2\omega}(\varphi)$ and hence $u^2(\varphi) = x_{2\omega}(\varphi)$. In general, for any positive integer n, it follows from the mathematical induction that $u^n(\varphi) = x_{n\omega}(\varphi)$.

(i) First, we show that u maps any bounded subset $\mathbb{E} \subset C[-\tau, 0]_{\mathbb{T}}$ into a relatively compact set. We note first that Theorem 3.2 implies that there exists $M_1 \in \mathbb{R}^m$ such that for any $\varphi \in \mathbb{E}$, the solution $x(\varphi)$ of (1.1) satisfies $|x(\varphi)(t)| \le M_1$ for all $t \in \mathbb{T}$. Then, by assumption (H3), there exists $M_2 \in \mathbb{R}^m$ so that $|G(t, x(\varphi)(t))| \le M_2$ for all $t \in \mathbb{T}$ and $\varphi \in \mathbb{E}$, where G is defined by (4.1). Subsequently, for all $\varphi \in \mathbb{E}$ we have

$$\begin{aligned} |u^\Delta(\varphi)(\theta)| &= |x^\Delta(\varphi)(\omega + \theta)| \\ &= |G(\omega + \theta, x(\varphi)(\omega + \theta))| \le M_2,\ \theta \in [-\tau, 0]_{\mathbb{T}}, \end{aligned} \tag{4.3}$$

where Δ denotes the derivative with respect to θ, and the hypothesis $\omega \ge \tau$ has been imposed. Now invoking the Mean Value Theorem [3, Theorem 1.14], we have from (4.3) that

$$|u(\varphi)(\theta_1) - u(\varphi)(\theta_2)| \le M_2|\theta_1 - \theta_2| \text{ for all } \varphi \in \mathbb{S},$$

which means that $u(\mathbb{E})$ is equi-continuous. Since $u(\mathbb{E})$ is bounded, Lemma 2.5 implies that $u(\mathbb{E})$ is relatively compact and hence u is completely continuous.

(ii) Next, we show that u has a fixed point. To this end, we recall that, by Theorem 3.2,

$$\big\{s \in \mathbb{R}^m : |s| \le \frac{1}{\alpha} UC_0\big\}$$

is a global attractor of (1.1). Then, for a given $\varepsilon_0 \in \mathbb{R}^m$ with $\varepsilon_0 > 0$, and any $\varphi \in C[-\tau, 0]_{\mathbb{T}}$, there exists $\widetilde{T} = \widetilde{T}(\varepsilon_0, \varphi)$ such that

$$|x(\varphi)(t)| \le \frac{1}{\alpha} UC_0 + \varepsilon_0 \text{ for all } t \in [\widetilde{T},\infty)_{\mathbb{T}}. \tag{4.4}$$

Taking an integer $n = n(\varepsilon_0, \varphi)$ with $n\omega \geq \widetilde{T} + \tau$, it follows from (4.4) that

$$|x(\varphi)(n\omega + \theta)| \leq \frac{1}{\alpha} UC_0 + \varepsilon_0 \quad \text{for all } \theta \in [-\tau, 0]_{\mathbb{T}}. \tag{4.5}$$

Let

$$\mathbb{E} = \big\{\varphi \in C[-\tau, 0]_{\mathbb{T}} : |\varphi(\theta)| \leq \frac{1}{\alpha} UC_0 + \varepsilon_0, \theta \in [-\tau, 0]_{\mathbb{T}}\big\}. \tag{4.6}$$

Then (4.5) implies $x_{n\omega}(\varphi) \in \mathbb{E}$ and hence $u^n(\varphi) \in \mathbb{E}$. By Lemma 2.7 we see that u has a fixed point $\varphi_0 \in \mathbb{E}$.

(iii) Note that the solution $x(\varphi_0)$ of (1.1) through φ_0 satisfies

$$x(\varphi_0)(t) = x(u(\varphi_0))(t) = x(x_\omega(\varphi_0))(t) = x(\varphi_0)(t + \omega) \quad \text{for all } t \in [0, \infty)_{\mathbb{T}},$$

where we have invoked (4.2). Hence $x(\varphi_0)$ is ω-periodic. Further, Theorem 3.5 implies that this periodic solution of (1.1) is unique. The proof is complete. ☐

Note that under the conditions in Theorem 3.4, the set

$$\big\{s \in \mathbb{R}^m : 0 \leq s \leq \frac{1}{\alpha} UC_0\big\}$$

is a global attractor of (1.1). Then, if we replace $\mathbb{E}$ as in (4.6) by

$$\mathbb{E}_0 = \big\{\varphi \in C[-\tau, 0]_{\mathbb{T}} : -\varepsilon_0 \leq \varphi(\theta) \leq \frac{1}{\alpha} UC_0 + \varepsilon_0, \theta \in [-\tau, 0]_{\mathbb{T}}\big\},$$

then, in a similar way as in the proof of Theorem 4.1 we see that u has a fixed point $\varphi_0 \in \mathbb{E}_0$ for which $x(\varphi_0)(t)$ is a unique ω-periodic solution of (1.1). Since $\varepsilon_0 > 0$ is arbitrary, $\varphi_0(\theta) \geq 0$ for all $\theta \in [-\tau, 0]_{\mathbb{T}}$. Therefore, from (2.1) we have

$$x(\varphi_0)(t) \geq 0 \;\text{ and }\; x(\varphi_0)(t) \;\text{ is not identically zero on }\; \mathbb{T}$$

provided $C(t) \geq 0$ is not identically zero on $\mathbb{T}$. In this case we refer to the periodic solution $x(\varphi_0)$ as positive. Now we extract the essence and obtain the following result.

Theorem 4.2. *Under the conditions in Theorems 3.4 and 4.1, if $C(t)$ is not identically zero on $\mathbb{T}$, then system* (1.1) *has a unique positive ω-periodic solution.*

Now we see that under the conditions in Theorem 4.1, system (1.1) possesses a unique periodic solution $x(\varphi_0)$. On the other hand, by Theorem 3.5 we have

$$x(\varphi)(t) - x(\varphi_0)(t) \to 0 \quad \text{as } t \to \infty$$

for any solution $x(\varphi)$ of (1.1). That is, $x(\varphi)$ is an asymptotical periodic solution of (1.1). Furthermore, $x(\varphi_0)(t) \equiv 0$ in case $C(t) \equiv 0$ on $\mathbb{T}$. As a consequence, the following result is clear.

Theorem 4.3. *Under the conditions in Theorem 4.1, each solution of* (1.1) *is asymptotically periodic. Specially, if $C(t) \equiv 0$ on $\mathbb{T}$, then each solution $x(t)$ of* (1.1) *converges to zero as $t \to \infty$.*

Remark 4.4. As a final remark, our results for periodicity cover those in [14, Theorem 3] and improve the conclusion in [15, Theorem 14]. Indeed, the authors in [15] only obtained an existence criterion of periodic solutions and ignored the uniqueness and asymptotical property.

5. Examples

Let us consider the following two examples to better understand our results.

Example 5.1. Let $\mathbb{T} = [-1,\infty) \subset \mathbb{R}$ and consider the equation of Nicholson blowflies type [5, Example 1]

$$x'(t) = -\frac{1}{2}x(t) + \frac{\lambda}{2}|x(t-1)|\big(\alpha + \beta e^{-x(t-1)}\big) + c(t), \quad t \in [-1,\infty), \tag{5.1}$$

where α, β and λ are positive constants, and $c(t) \geq 0$ is a 1-periodic function.

(i) For the case $c(t) \equiv 0$, to invoke Theorem 1.1 we first calculate

$$f_0 = \alpha + \beta, \quad f_\infty = \alpha, \quad A = 1, \quad B = e^{-1/2}.$$

Then, for any α and β satisfying

$$\frac{1}{B(\alpha+\beta)} \geq \frac{1}{A\alpha},$$

Theorem 1.1 is invalid since $(\frac{1}{B(\alpha+\beta)}, \frac{1}{A\alpha})$ is empty. Now we turn to impose our criteria under the same conditions as above. To this end, we take $F(t,x_1,x_2) = \frac{\lambda}{2}|x_2|(\alpha + \beta e^{-x_2})$ and

$$A_1 = 0, \quad A_2 = \frac{\lambda(\alpha+\beta)}{2}.$$

Then

$$|F(t,x_1,x_2) - F(t,y_1,y_2)| \leq A_2|x_2 - y_2|$$

and hence, for

$$\lambda < \frac{1}{\alpha+\beta}, \tag{5.2}$$

Theorem 4.3 implies that each solution $x(t)$ of (5.1) converges to zero as $t \to \infty$.

(ii) For the case $c(t) \geq 0$ and not identically zero, as long as the positive constants α, β and λ fulfill the relation (5.2), our Theorems 4.2 and 4.3 imply that (5.1) has a unique positive 1-periodic solution and others are asymptotically periodic.

Example 5.2. Let $\mathbb{T} = \cup_{k=1}^{\infty}[4k-8, 4k-6]$ and $\alpha = 1/3$. Then $1 - \alpha\mu(t) > 0$ for all $t \in \mathbb{T}$, $\mathbb{T}$ is a 4-periodic time scale and $e_{-\alpha}(t,0) \to 0$ as $t \to \infty$. Now we consider the delay dynamic system

$$x^{\Delta}(t) = \begin{pmatrix} -1/3 & 0 \\ 0 & -1/3 \end{pmatrix} x(t) + \begin{pmatrix} 1/4 & 1 \\ 0 & 1/4 \end{pmatrix} x(t-4) + \begin{pmatrix} \sin(\pi t/2) \\ \cos(\pi t/2) \end{pmatrix}, \quad t \in \mathbb{T} \tag{5.3}$$

and take

$$A_0 = A_1 = 0, \quad A_2 = \begin{pmatrix} 1/4 & 1 \\ 0 & 1/4 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, assumptions (H1)–(H4) are satisfied. Now by Theorems 4.1 and 4.3, system (5.3) has a unique 4-periodic solutions and others are asymptotically periodic.

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References

[1] R. Agarwal, M. Bohner, D. O'Regan, A. Peterson; *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. 141 (2002), 1-26.

[2] M. Bohner, A. Peterson; *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] M. Bohner, A. Peterson, Eds.; *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] S. S. Cheng, G. Zhang; *Existence of positive periodic solutions for non-autonomous functional differential equations*, Electron. J. Differ. Equ. 59 (2001), 1–8.

[5] D. Franco, E. Liz, P. J. Torres; *Existence of periodic solutions for functional equations with periodic delay*, Indian J. Pure Appl. Math. 38 (2007), 1–10.

[6] J. Hale; *Theory of Functional Differential Equations (Second edition)*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York-Heidelberg, 1977.

[7] A. R. Horn, C. R. Johnson; *Matrix Analysis*, Cambridge University Press, London, 1990.

[8] W. A. Horn; *Some fixed point theorems for compact maps and flows in Banach spaces*, Trans. Amer. Math. Soc. 149 (1970), 391–404.

[9] Q. K. Kong, D. X. Piao; *Periodic solutions of impulsive neutral dynamic equations on time scales, Dynamic systems and applications*, Vol. 5, 258-262, Dynamic, Atlanta, GA, 2008.

[10] B. Lisena; *Periodic solutions of logistic equations with time delay*, Appl. Math. Lett. 20 (2007), 1070–1074.

[11] C. G. Philos, I. K. Purnaras; *On the behavior of the solutions to periodic linear delay differential and difference equations*, J. Math. Anal. Appl. 322 (2006), 847–863.

[12] M. Pituk; *Asymptotic behavior of a nonhomogeneous linear recurrence system*, J. Math. Anal. Appl. 267 (2002), 626–642.

[13] Y. N. Raffoul; *Stability and periodicity in discrete delay equations*, J. Math. Anal. Appl. 324 (2006), 1356–1362.

[14] H. Zhao; *Invariant set and attractor of nonautonomous functional differential systems*, J. Math. Anal. Appl. 282 (2003), 437–443.

[15] Z. Q. Zhu, S. S. Cheng; *Stability analysis for multistep computational schemes*, Comput. Math. Appl. 55 (2008), 2753–2761.

[16] Z. Q. Zhu, Q. R. Wang; *Existence of nonoscillatory solutions to neutral dynamic equations on time scales*, J. Math. Anal. Appl. 335 (2007), 751–762.

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