OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR PERTURBED DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the oscillation of solutions to the nonlinear second-order differential equation
\[ \left( r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \right)' + P(t, x'(t))\psi(x(t)) + Q(t, x(t)) = 0. \]

We obtain sufficient conditions for the oscillation of all solutions to this equation.

1. Introduction

This article concerns the oscillation of solutions to the nonlinear second-order differential equation
\[ \left( r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \right)' + P(t, x'(t))\psi(x(t)) + Q(t, x(t)) = 0, \quad t \geq t_0 \] (1.1)
where \( r \in C^1(I, \mathbb{R}^+) \), \( P, Q \in C(I \times \mathbb{R}, \mathbb{R}) \), \( \psi(x) \in C(\mathbb{R}, \mathbb{R}^+) \), \( I = [T_0, \infty) \subset \mathbb{R} \), \( 0 < \psi(x) < \gamma \) and \( \alpha \) is a positive constant. Throughout this article, we assume the following conditions:

(E1) \( Q \in C(I \times \mathbb{R}, \mathbb{R}) \) and there exist \( f \in C^1(\mathbb{R}, \mathbb{R}) \) and a continuous function \( q(t) \) such that \( xf(x) > 0 \) and \( \frac{Q(t, x)}{f(x)} \geq q(t) \) for \( x \neq 0 \).

(E2) \( P \in C(I \times \mathbb{R}, \mathbb{R}) \) and there exists a continuous function \( p(t) \) such that \( \frac{P(t, x'(t))}{|x'(t)|^{\alpha-1}x'(t)} \geq p(t) \) for \( x' \neq 0 \).

We restrict our attention to solutions satisfying \( \sup\{|x(t)| : t \geq T\} > 0 \) for all \( T \geq T_0 \).

A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be non-oscillatory. If all solutions of (1.1) are oscillatory, (1.1) is called oscillatory.

The oscillatory behavior of solutions of second-order ordinary differential equations, including the existence of oscillatory and non-oscillatory solutions, has been

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the subject of intensive investigations; see for example [1]–[13]. Some criteria involve the behavior of the integral of alternating coefficients. In this article, we give general integral criteria for the oscillation of (1.1), which contain some of the results in the references as particular cases.

2. Main results

Let \( h(\cdot) \) and \( K(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) be continuous functions such that for each fixed \( t, s \), the function \( K(t, s, \cdot) \) is nondecreasing. Then there exists a solution to the integral equation

\[
v(t) = h(t) + \int_{t_0}^{t} K(t, s, v(s)) \, ds, \quad t \geq t_0.
\]

Furthermore there exists a “minimal solution” \( v \) in the sense that any solution \( y \) of this equation satisfies \( v(t) \leq y(t) \) for all \( t \geq t_0 \). See [1, p. 322].

Lemma 2.1. If \( v \) is the minimal solution of (2.1) and

\[
u(t) \geq h(t) + \int_{t_0}^{t} K(t, s, u(s)) \, ds, \quad t \geq t_0,
\]

then \( u(t) \geq v(t) \) for all \( t \geq t_0 \).

Similarly for a maximal solution \( u(t) \) of (2.1): if \( u(t) \leq h(t) + \int_{t_0}^{t} K(t, s, u(s)) \, ds \),

then \( u(t) \leq w(t) \) for all \( t \geq t_0 \).

Our main results reads as follows.

Theorem 2.2. Assume (E1), \( f'(x) \geq 0 \), \( p(t) \leq 0 \), \( q(t) > 0 \) and \( \int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} \, dt = \infty \). Also assume that there exists a positive function \( \rho(t) \) such that

\[
\int_{t_0}^{\infty} q(t) \rho(t) \, dt = \infty, \quad (2.2)
\]

\[
p(t) \rho(t) \geq r(t) p'(t). \quad (2.3)
\]

Then every solution of (1.1) is oscillatory.

Proof. For the shake of contradiction, suppose that (1.1) has a non-oscillatory solution \( x(t) \). Without loss of generality, suppose that it is an eventually positive solution (if it is an eventually negative solution, the proof is similar), that is, \( x(t) > 0 \) for all \( t \geq t_0 \). We consider the following three cases.

Case 1. Suppose that \( x'(t) \) is oscillatory. Then there exists \( t_1 \geq t_0 \) such that \( x'(t_1) = 0 \). From (1.1), we have

\[
[r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds\right)]'
\]

\[
= [r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)]' \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds\right)
\]

\[
+ p(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds\right)
\]

\[
=(-P(t, x'(t))\psi(x(t)) - Q(t, x(t))) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds\right)
\]
\[
+p(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \exp \left( \int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds \right) \\
\leq (-p(t)|x'(t)|^{\alpha-1}x'(t)\psi(x(t)) - q(t)f(x(t))) \exp \left( \int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds \right) \\
+ p(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \exp \left( \int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds \right) \\
= -q(t)f(x(t)) \exp \left( \int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds \right) < 0
\]

which implies that
\[
r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t) \exp \left( \int_{t_0}^{t} \frac{p(s)}{r(s)} \, ds \right) \\
< r(t_1)\psi(x(t_1))|x'(t_1)|^{\alpha-1}x'(t_1) \exp \left( \int_{t_0}^{t_1} \frac{p(s)}{r(s)} \, ds \right) = 0, \quad \forall t \geq t_1.
\]

It follows that \(x'(t) < 0\) for all \(t > t_1\), which contradicts to the assumption that \(x'(t)\) is oscillatory.

**Case 2.** Assume that \(x'(t) < 0\). From (1.1), we obtain
\[
-\left[r(t)\psi(x(t))|x'(t)|^{\alpha-1}x'(t)\right]' = \left[r(t)\psi(x(t))(-x'(t))\right]'
\]
\[
= P(t,x'(t))\psi(x(t)) + Q(t,x(t)) \\
\geq p(t)|x'(t)|^{\alpha-1}x'(t)\psi(x(t)) + q(t)f(x(t)) \\
= -p(t)(-x'(t))^{\alpha} \psi(x(t)) + q(t)f(x(t)) \geq 0
\]
then there exists an \(M > 0\) and a \(t_1 \geq t_0\), such that
\[
r(t)\psi(x(t))(-x'(t))^{\alpha} \geq M, \quad \forall t \geq t_1.
\] (2.4)

It follows that
\[
\gamma(-x'(t))^{\alpha} \geq \frac{M}{r(t)},
\]
\[
x(t) \leq -\int_{t_1}^{\infty} \left( \frac{M}{r(t)} \right)^{1/\alpha} \frac{1}{r^{1/\alpha}(t)} \, dt, \quad \forall t \geq t_1
\]
which implies \(\lim_{t \to -\infty} x(t) = -\infty\); this contradicts the assumption that \(x(t) > 0\).

**Case 3.** Suppose that \(x'(t) > 0\). Define \(w(t) = \rho(t)r(t)\psi(x(t))(x'(t))^{\alpha}\). Differentiating \(w(t)\) and using (1.1),
\[
w'(t) = [r(t)\psi(x(t))(x'(t))^{\alpha}]' \rho(t) + r(t)\psi(x(t))(x'(t))^{\alpha} \rho'(t), \quad \forall t \geq t_0.
\] (2.5)

Then we obtain
\[
\frac{w'(t)}{f(x(t))} = -\frac{P(t,x'(t))\psi(x(t))\rho(t)}{f(x(t))} - \frac{Q(t,x(t))\rho(t)}{f(x(t))} \\
+ \frac{\rho'(t)r(t)\psi(x(t))(x'(t))^{\alpha}}{f(x(t))}, \quad \forall t \geq t_0.
\]

Noticing that
\[
\left( \frac{w(t)}{f(x(t))} \right)' = \frac{w'(t)f(x(t)) - w(t)f'(x(t))x'(t)}{f^2(x(t))}
\]
Using (2.2), (2.3) and

\[ P(t, x(t)) \psi(x(t)) \rho(t) \]
\[ + \frac{\rho'(t) r(t) \psi(x(t))(x'(t))^\alpha}{f(x(t))} \]
\[ - w(t) f'(x(t)) x'(t) \frac{f(x(t))}{f^2(x(t))}, \quad \forall t \geq t_0. \]

Integrating the above from \( t_0 \) to \( t \), we obtain

\[ \frac{w(t)}{f(x(t))} = \frac{w(t_0)}{f(x(t_0))} - \int_{t_0}^{t} \left[ \frac{P(s, x'(s)) \psi(x(s)) \rho(s)}{f(x(s))} + \frac{Q(s, x(s)) \rho(s)}{f(x(s))} \right. \]
\[ - \frac{\rho'(s) r(s) \psi(x(s))(x'(s))^\alpha}{f(x(s))} \]
\[ + \frac{w(s) f'(x(s)) x'(s)}{f^2(x(s))} \bigg|_{s = t_0}^{s = t} ds, \]
\[ \frac{w(t)}{f(x(t))} \leq \frac{w(t_0)}{f(x(t_0))} - \int_{t_0}^{t} \left[ q(s) \rho(s) + \frac{(\rho(s) p(s) - \rho'(s) r(s))(x'(s))^\alpha}{f(x(s))} \psi(x(s)) \right. \]
\[ + \frac{w(s) f'(x(s)) x'(s)}{f^2(x(s))} \bigg|_{s = t_0}^{s = t} ds. \]

Using (2.2), (2.3) and \( x'(t) > 0 \), we have

\[ 0 \leq \lim_{t \to \infty} \frac{w(t)}{f(x(t))} = -\infty, \]

this is a contradiction. The proof is complete.

**Theorem 2.3.** Assume that \( f'(x) \geq 0 \) and \( \psi(x(t)) \equiv 1 \). Also assume that

\[ \rho_0(t) = \exp \left( \int_{t_0}^{t} \frac{p(s)}{r(s)} ds \right), \quad (2.6) \]
\[ \int_{t_0}^{\infty} \frac{dt}{(\rho_0(t) r(t))^{1/\alpha}} = \infty, \quad (2.7) \]

and \( \rho_0(t) \) satisfies (2.2). Then every solution of (1.1) is oscillatory.

**Proof.** Let \( x(t) \) be a non-oscillatory solution of (1.1). Without loss of generality, we assume that \( x(t) \) is eventually positive. Let \( w(t) = \rho_0(t) r(t) |x'(t)|^{\alpha - 1} x'(t) \). Then

\[ w(t) x'(t) = \rho_0(t) r(t) |x'(t)|^{\alpha - 1} x'(t) \geq 0 \quad \text{for} \quad t \geq t_0 \]

and

\[ w'(t) = (r(t) |x'(t)|^{\alpha - 1} x'(t))^\prime \rho_0(t) + r(t) |x'(t)|^{\alpha - 1} x'(t) \rho_0'(t) \quad \forall t \geq t_0. \quad (2.8) \]

In view of (2.1) and (2.3), we obtain

\[ w'(t) = (- P(t, x'(t)) - Q(t, x(t)) \rho_0(t) + |x'(t)|^{\alpha - 1} x'(t) p(t) \rho_0(t), \]
\[ w'(t) \leq (- p(t) |x'(t)|^{\alpha - 1} x'(t) - q(t) f(x(t))) \rho_0(t) + |x'(t)|^{\alpha - 1} x'(t) p(t) \rho_0(t), \quad (2.9) \]
\[ \frac{w'(t)}{f(x(t))} \leq - q(t) \rho_0(t) \quad \forall t \geq t_0. \]

Since

\[ \left( \frac{w(t)}{f(x(t))} \right)' = \frac{w'(t) f(x(t)) - w(t) f'(x(t)) x'(t)}{f^2(x(t))} \]
\[ \leq - q(t) \rho_0(t) - \frac{w(t) f'(x(t)) x'(t)}{f^2(x(t))} \quad \forall t \geq t_0, \]
integrating from \( t_0 \) to \( t \), we have
\[
- \frac{w(t)}{f(x(t))} \geq - \frac{w(t_0)}{f(x(t_0))} + \int_{t_0}^{t} q(s) \rho_0(s) ds + \int_{t_0}^{t} \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))} ds, \quad \forall t \geq t_0.
\]
By using (2.2), there exists a constant \( m > 0 \) and \( t_1 \geq t_0 \) such that
\[
- \frac{w(t_0)}{f(x(t_0))} + \int_{t_0}^{t} q(s) \rho_0(s) ds + \int_{t_0}^{t_1} \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))} ds \geq m \quad \forall t \geq t_0
\]
which means that
\[
- \frac{w(t)}{f(x(t))} \geq m + \int_{t_1}^{t} \frac{w(s)f'(x(s))x'(s)}{f^2(x(s))} ds.
\]
Because that \( x(t) \) is positive, (2.11) implies \(-w(t) > 0\), or equivalently \( x'(t) < 0 \).
Let
\[
u(t) = -w(t) = -\rho_0(t)r(t)|x'(t)|^{\alpha - 1}x'(t) = \rho_0(t)r(t)(-x'(t))^\alpha,
\]
thus (2.11) can be written as
\[
u(t) \geq m f(x(t)) + \int_{t_1}^{t} \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))} u(s) ds.
\]
Define
\[
K(t, s, u) = \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))} u.
\]
Then, for any fixed \( t \) and \( s \), \( K(t, s, u) \) is nondecreasing in \( u \). Let \( v(t) \) be the minimal solution of the equation
\[
v(t) = m f(x(t)) + \int_{t_1}^{t} \frac{f(x(t))f'(x(s))(-x'(s))}{f^2(x(s))} v(s) ds.
\]
Applying Lemma 2.1 we obtain
\[
u(t) \geq v(t) \quad \forall t \geq t_0.
\]
Dividing both sides of (2.15) by \( f(x(t)) \) and deriving both sides of (2.15),
\[
\left( \frac{v(t)}{f(x(t))} \right)' = \left( m + \int_{t_1}^{t} \frac{f'(x(s))(-x'(s))}{f^2(x(s))} v(s) ds \right)' = \frac{f'(x(t))(-x'(t))}{f^2(x(t))} v(t).
\]
On the other hand
\[
\left( \frac{v(t)}{f(x(t))} \right)' = \frac{v'(t)}{f(x(t))} + \frac{f'(x(t))(-x'(t))}{f^2(x(t))} v(t).
\]
Combining (2.17) and (2.18), it follows that
\[
v'(t) = 0.
\]
So \( v(t) = v(t_1) = m f(x(t_1)), t \geq t_0 \). From (2.16), we obtain
\[
-x'(t) \geq \left( m f(x(t_1)) \right)^{1/\alpha} \left( \rho_0(t)r(t) \right)^{1/\alpha}, \quad \forall t \geq t_1.
\]
Integrating both sides of this inequality above from \( t_1 \) to \( t \), we have
\[
-x'(t) + x(t_1) \geq \left( m f(x(t_1)) \right)^{1/\alpha} \int_{t_1}^{t} ds \left( \rho_0(s)r(s) \right)^{1/\alpha}.
\]
Letting \( t \to \infty \), and using (2.7), it follows that \( \lim_{t \to \infty} x(t) \leq -\infty \), which contradicts to that \( x(t) \) is eventually positive. The proof is complete.
In what follows, we always assume that $H(t) \in C^2(\mathbb{R}; \mathbb{R})$ and it satisfies the following two conditions:

(H1) $H(t) > 0$ for all $t \geq t_0$, $H(t)$ is a bounded;
(H2) $H'(t) = h(t)$ is a bounded.

**Theorem 2.4.** Assume that $f'(x) \geq 0$, 
\[ \int_{t_0}^{f(\infty)} \frac{dt}{(r(t))^{1/\alpha}} = \infty, \quad \psi(x(t)) \equiv 1, \quad \text{and} \]
\[ p(t) \leq 0, \quad q(t) > 0, \quad (2.21) \]
or
\[ p(t) \leq 0, \quad q(t) \leq 0, \quad \lim_{t \to \infty} \frac{p(t)}{q(t)} = M > 0. \quad (2.22) \]

Suppose further that there exists a function $H(t)$ that satisfies (H1), (H2), and such that
\[ \int_{t_0}^{\infty} H(t) \varphi(t) dt = \infty, \quad (2.23) \]
\[ \limsup_{t \to \infty} v(t)r(t) < \infty, \quad (2.24) \]

where
\[ \varphi(t) = v(t)(q(t) - p(t)h(t) - (r(t)h(t))'), \quad (2.25) \]
\[ v(t) = \exp \left( \int_{t_0}^{t} \left( \frac{p(s)}{r(s)} - \frac{h(s)}{H(s)} \right) ds \right). \quad (2.26) \]

Then every solution of \([1.1]\) is oscillatory.

**Proof.** Assume to the contrary that \([1.1]\) has a non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_0$. Define
\[ u(t) = v(t)r(t) \left( \frac{|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} + h(t) \right). \quad (2.27) \]

Differentiating, we obtain
\[ u'(t) = \left( \frac{p(t)}{r(t)} - \frac{h(t)}{H(t)} \right) u(t) + v(t) \left[ - \frac{p(t,x'(t))}{f(x(t))} \right. \]
\[ - \frac{Q(t,x(t))}{f(x(t))} - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2f'(x(t))}{f^2(x(t))} + (r(t)h(t))' \bigg], \]
\[ u'(t) \leq \left( \frac{p(t)}{r(t)} - \frac{h(t)}{H(t)} \right) u(t) + v(t) \left[ - \frac{p(t)|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} - q(t) \right. \]
\[ - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2f'(x(t))}{f^2(x(t))} + (r(t)h(t))' \bigg] \]
\[ \leq p(t)v(t)h(t) - \frac{h(t)}{H(t)} u(t) - q(t)v(t) + v(t)(r(t)h(t))' \]
\[ = - \frac{h(t)}{H(t)} u(t) - v(t)[q(t) - p(t)h(t) - (r(t)h(t))'] \]
\[ u'(t) \leq - \frac{h(t)}{H(t)} u(t) - \varphi(t). \]

Multiplying by $H(t)$, it follows that
\[ \varphi(t)H(t) \leq -H(t)u'(t) - h(t)u(t). \quad (2.28) \]
We consider the following three cases.

**Case 1.** $u(t)$ is oscillatory. Then there exists a sequence $\{t_n\}, \, (n = 1, 2, \ldots)$, $t_n \to \infty$ as $n \to \infty$ and such that $u(t_n) = 0 \, (n = 1, 2, \ldots)$. Integrating both sides of (2.28) from $t_0$ to $t_n$, we obtain
\[
\int_{t_0}^{t_n} H(t)\varphi(t)\,dt \leq -\int_{t_0}^{t_n} H(t)u'(t)\,dt - \int_{t_0}^{t_n} h(t)u(t)\,dt
\]
\[
= -H(t_0)u(t_0) \mid_{t_0}^{t_n} - \int_{t_0}^{t_n} (H'(t)u(t) + h(t)u(t))\,dt
\]
\[
= H(t_0)u(t_0) - H(t_n)u(t_n) = H(t_0)u(t_0);
\]
that is,
\[
\lim_{t_n \to \infty} \int_{t_0}^{t_n} H(t)\varphi(t)\,dt \leq H(t_0)u(t_0),
\]
which contradicts (2.23).

**Case 2.** $u(t)$ is eventually positive. Integrating both sides of (2.28) from $t_0$ to $\infty$, we obtain
\[
\int_{t_0}^{\infty} H(t)\varphi(t)\,dt \leq H(t_0)u(t_0) - \lim_{t \to \infty} H(t)u(t) \leq H(t_0)u(t_0),
\]
which also contradicts (2.23).

**Case 3.** $u(t)$ is eventually negative. If $\limsup_{t \to \infty} u(t) > -\infty$, then there exists a sequence $\{t_n\}, \, (n = 1, 2, \ldots)$, that satisfies $\{t_n\} \to \infty$ as $n \to \infty$ and such that $\lim_{t_n \to \infty} u(t_n) = \limsup_{t \to \infty} u(t) = M_1 > -\infty$. Because $H(t)$ is a bounded function, then there exists a $M_2 > 0$ such that $H(t_n) \leq M_2, \, (n = 1, 2, \ldots)$. According to (2.28), we obtain
\[
\int_{t_0}^{t_n} H(t)\varphi(t)\,dt \leq H(t_0)u(t_0) - H(t_n)u(t_n) \leq H(t_0)u(t_0) - M_2 u(t_n).
\]
Using (2.23) and taking limit as $t_n \to \infty$, it is easy to show that
\[
\lim_{t_n \to \infty} \int_{t_0}^{t_n} H(t)\varphi(t)\,dt = \infty
\]
\[
\leq H(t_0)u(t_0) - \lim_{t_n \to \infty} H(t_n)u(t_n)
\]
\[
\leq H(t_0)u(t_0) - M_1 M_2 < \infty,
\]
which is obviously a contradiction.

If $\limsup_{t \to \infty} u(t) = -\infty$, then $\lim_{t \to \infty} u(t) = -\infty$. From the definition of $h(t)$, combining (2.24) and (2.27), it follows that $x'(t) < 0$ and
\[
\lim_{t \to \infty} (|x'(t)|^{\frac{1}{\alpha - 1}} x'(t)/f(x(t))) = -\infty,
\]
which implies that $\lim_{t \to \infty} ((-x'(t))^\alpha/f(x(t))) = \infty$. Owing to $p(t) \leq 0, \, q(t) \geq 0$, or $p(t) \leq 0, \, q(t) \leq 0$ and $\lim_{t \to \infty} (p(t)/q(t)) = M > 0$, using the similar method of the proof of Case 2 in Theorem 2.2 we will derive a contradiction. The proof is complete. \qed
Theorem 2.5. Assume that \( (2.24) \) holds, \( f'(x) \geq 0, \int_{t_0}^{\infty} \frac{dt}{(x(t))^{1/\alpha}} = \infty, \) and \( (2.21) \) or \( (2.22) \) hold. Suppose further that there exists a function \( H(t) \) that satisfies \( (H1), (H2) \), and such that

\[
\int_{t_0}^{\infty} H(t)\hat{\varphi}(t)dt = \infty, \tag{2.30}
\]

where

\[
\hat{\varphi}(t) = v(t)(q(t) + p(t)h(t) + (r(t)h(t))'), \tag{2.31}
\]

and \( v(t) \) is defined in \( (2.26) \). Then every solution of \( (1.1) \) is oscillatory when \( \psi(x(t)) \equiv 1 \).

Proof. For the sake of contradiction, let \( (1.1) \) have a non-oscillatory solution. Without loss of generality, we may assume that \( (1.1) \) has an eventually positive \( x(t) > 0 \) for all \( t \geq t_0 \). Define

\[
u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\alpha-1}x'(t)}{f(x(t))} - h(t)\right).
\]

The rest of proof is similar to Theorem 2.4 and is omitted. \( \square \)

Theorem 2.6. Assume \( (2.24) \), \( p(t) \leq 0, q(t) > 0, f'(x) \geq 0 \) and \( \int_{t_0}^{\infty} \frac{dt}{(x(t))^{1/\alpha}} = \infty \). Suppose further that there exists a function \( H(t) \) that satisfies \( (H1), (H2) \), and such that

\[
\int_{t_0}^{\infty} H(t)\phi(t)dt = \infty, \tag{2.32}
\]

where

\[
\phi(t) = v(t)(-p(t)h(t) - (r(t)h(t))'), \tag{2.33}
\]

where \( v(t) \) is defined in \( (2.26) \). Then every solution of \( (1.1) \) is oscillatory when \( \psi(x(t)) \equiv 1 \).

Proof. To the contrary, assume that \( (1.1) \) has a non-oscillatory solution \( x(t) \). Without loss of generality, we may assume that \( (1.1) \) has an eventually positive \( x(t) > 0 \) for all \( t \geq t_0 \). Define

\[
u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\alpha-1}x'(t)}{x(t)} + h(t)\right). \tag{2.34}
\]

We use (E1) and noting that \( xf(x) \geq 0 \) for \( x \neq 0 \), so \( \frac{f(x)}{x} \geq 0 \) for \( x \neq 0 \). Differentiating \( (2.34) \), we obtain

\[
u'(t) = \left(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\right)u(t) + v(t)\left[-\frac{P(t,x'(t))}{x(t)} - \frac{Q(t,x(t))}{x(t)} - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2}{x^2(t)} + (r(t)h(t))'\right]
\]

\[
\leq \left(\frac{p(t)}{r(t)} - \frac{h(t)}{H(t)}\right)u(t) + v(t)\left[-\frac{p(t)|x'(t)|^{\alpha-1}x'(t)}{x(t)} - \frac{q(t)f(x(t))}{x(t)} - \frac{r(t)|x'(t)|^{\alpha-1}(x'(t))^2}{x^2(t)} + (r(t)h(t))'\right]
\]

\[
\leq p(t)v(t)h(t) - \frac{h(t)}{H(t)}u(t) + v(t)(r(t)h(t))'
\]

\[
= -\frac{h(t)}{H(t)}u(t) - v(t)[-p(t)h(t) - (r(t)h(t))']
\]
\[
= -\frac{h(t)}{H(t)} u(t) - \phi(t).
\]

Multiplying by \( H(t) \), it follows that
\[
H(t)\phi(t) \leq -H(t)u'(t) - h(t)u(t).
\]
The rest of the proof is similar to Theorem 2.4, and it is omitted.

\textbf{Theorem 2.7.} Assume (2.24), \( p(t) \leq 0, q(t) > 0 \), \( f'(x) \geq 0 \) and
\[
\int_{t_0}^{\infty} \frac{dt}{(r(t))^{1/\alpha}} = \infty.
\]
Suppose further that there exists a function \( H(t) \) satisfying \((H1), (H2)\), and such that
\[
\int_{t_0}^{\infty} H(t)\tilde{\phi}(t)dt = \infty,
\]
where
\[
\tilde{\phi}(t) = v(t)(p(t)h(t) + (r(t)h(t))'),
\]
where \( v(t) \) is defined in (2.26). Then every solution of (1.1) is oscillatory when \( \psi(x(t)) \equiv 1 \).

\textbf{Proof.} For the sake of contradiction, assume that (1.1) has a non-oscillatory solution. Without loss of generality, we may assume that (1.1) has an eventually positive \( x(t) > 0 \) for all \( t \geq t_0 \). Define
\[
u(t) = v(t)r(t)\left(\frac{|x'(t)|^{\alpha-1}x''(t)}{x(t)} - h(t)\right).
\]
The rest of the proof is similar to Theorem 2.4 and it is omitted here.

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\textbf{References}


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