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DISTINCTION OF TURBULENCE FROM CHAOS – ROUGH DEPENDENCE ON INITIAL DATA

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ABSTRACT. This article presents a new theory on the nature of turbulence: when the Reynolds number is large, violent fully developed turbulence is due to "rough dependence on initial data" rather than chaos which is caused by "sensitive dependence on initial data"; when the Reynolds number is moderate, (often transient) turbulence is due to chaos. The key in the validation of the theory is estimating the temporal growth of the initial perturbations with the Reynolds number as a parameter. Analytically, this amounts to estimating the temporal growth of the norm of the derivative of the solution map of the Navier-Stokes equations, for which here I obtain an upper bound $e^{C\sqrt{tRe}+C_1 t}$. This bound clearly indicates that when the Reynolds number is large, the temporal growth rate can potentially be large in short time, i.e. rough dependence on initial data.

1. INTRODUCTION

For a long time, fluid dynamists have suspected that turbulence is "more than" chaos. Many chaoticians including the present author have believed that turbulence is "no more than" chaos in Navier-Stokes equations. A recent result [4] on Euler equations forced the present author to have to change mind.

The signature of chaos is "sensitive dependence on initial data"; here I want to address "rough dependence on initial data" which is very different from sensitive dependence on initial data. For solutions (of some system) that exhibit sensitive dependence on initial data, their initial small deviations usually amplify exponentially (with an exponent named Liapunov exponent), and it takes time for the deviations to accumulate to substantial amount (say order $O(1)$ relative to the small initial deviation). If ϵ is the initial small deviation, and σ is the Liapunov exponent, then the time for the deviation to reach 1 is about $\frac{1}{\sigma}\ln\frac{1}{\epsilon}$. On the other hand, for solutions that exhibit rough dependence on initial data, their initial small deviations can reach substantial amount instantly. Take the 3D or 2D Euler equations of fluids as the example, for any $t \neq 0$ (and small for local existence), the solution map that maps the initial condition to the solution value at time t is nowhere locally uniformly continuous and nowhere differentiable [4]. In such a case, any small deviation of the initial condition can potentially reach substantial amount instantly. My

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theory is that the high Reynolds number violent turbulence is due to such rough dependence on initial data, rather than sensitive dependence on initial data of chaos. When the Reynolds number is sufficiently large (the viscosity is sufficiently small), even though the solution map of the Navier-Stokes equations is still differentiable, but the derivative of the solution map should be potentially extremely large everywhere (of order $e^{C\sqrt{tRe}}$ as shown below) since the solution map of the Navier-Stokes equations approaches the solution map of the Euler equations when the viscosity approaches zero (the Reynolds number approaches infinity). Such everywhere large derivative of the solution map of the Navier-Stokes equations manifests itself as the development of violent turbulence in a short time. In summary, moderate Reynolds number turbulence is due to sensitive dependence on initial data of chaos, while large enough Reynolds number turbulence is due to rough dependence on initial data. This is an important new understanding on the nature of turbulence [14]. One may call this the new complexity of turbulence [16] [10].

In terms of phase space dynamics of dynamical systems, when the Reynolds number is very high, fully developed turbulence is not the result of a strange attractor, rather a result of super fast deviation amplifications (facilitated by the large derivative of turbulent solutions in their initial data). Strange attractor is a long time object, while the development of such violent turbulence is of short time. Such fully developed turbulence is maintained by constantly super fast deviation amplifications. When the Reynolds number is set to infinity, deviation amplification rate is infinity. So the dynamics of Euler equations is very close to a random process. In contrast, chaos in finite dimensional conservative systems often manifests itself as the so-called stochastic layers. Dynamics inside the stochastic layers has the long term sensitive dependence on initial data. When the Reynolds number is moderate, viscous diffusive term in Navier-Stokes equations is stronger, deviation amplification rate is moderate. At this stage, turbulence is basically chaos in Navier-Stokes equations [18, 8, 9, 7]. In some cases, strange attractor, homoclinic orbits, and bifurcation routes to chaos can be observed [18]. When the Reynolds number is lowered to its critical value, the initiator for the transition from the basic laminar flow is the linear instability of the laminar flow near the basic laminar flow (i.e. the basic laminar flow plus high spatial frequency deviations). This the resolution of the Sommerfeld (turbulence) paradox [15].

The type of rough dependence on initial data shared by the solution map of the Euler equations is difficult to find in finite dimensional systems. The solution map of the Euler equations is still continuous in initial data. Such a solution map (continuous, but nowhere locally uniformly continuous) does not exist in finite dimensions. This may be the reason that one usually finds chaos (sensitive dependence on initial data) rather than rough dependence on initial data in finite dimensions. If the solution map of some special finite dimensional system is nowhere continuous, then the dependence on initial data is rough, but may be too rough to have any realistic application. In infinite dimensions, irregularities of solution maps are quite common, e.g. in water wave equations [2, 3].

Even though the relation between Liapunov exponent and chaos (and instability) can be complicated [12], generically a positive Liapunov exponent is a good indicator of chaotic dynamics. In connection with turbulence, Liapunov exponent and its extensions have been studied [17, 1]. To distinguish that turbulence is exhibiting rough or sensitive dependence on initial data, one needs to study the derivative of the solution map.

2. Derivative of the solution map

Let S^t be the solution map which maps the initial value $u(0)$ to the solution's value $u(t)$ at time t. So for any fixed time t, S^t is a map defined on the phase space. The temporal growth of the norm of the derivative DS^t of the solution map S^t describes the amplification of the initial perturbation. The well-known Liapunov exponent is defined by DS^t:

$$\sigma = \lim_{t\to+\infty} \frac{1}{t} \ln \|DS^t\|.$$

A positive Liapunov exponent implies that nearby orbits deviate exponentially in time, i.e. sensitive dependence on initial data. The Liapunov exponent is a measure of long term temporal growth of the norm of the derivative DS^t. The temporal property of the norm of DS^t can of course be much more complicated than simple long term exponential growth. In particular, the norm of DS^t can be large in short time (i.e. super fast temporal growth). In such a case, the dynamics (described by S^t) exhibits short term unpredictability (i.e. rough dependence on initial data). One can define the following exponent

$$\eta = \lim_{t\to 0^+} \frac{1}{t^\alpha} \ln \|DS^t\|, \quad \text{where } \alpha > 0.$$

When η is large (e.g. approaching infinity as a parameter approaches a limit), one has short term unpredictability. In the case of Navier-Stokes equations to be studied later, η can potentially be as large as $C\sqrt{Re}$ with $\alpha = 1/2$.

3. Derivative estimate for Navier-Stokes equations

To verify the rough dependence on initial data for the solution map of the Navier-Stokes equations, we need to estimate the temporal growth of the norm of the derivative of the solution map of the Navier-Stokes equations. The Navier-Stokes equations are given by

$$u_t + \frac{1}{Re}\Delta u = -\nabla p - u\cdot\nabla u, \tag{3.1}$$

$$\nabla\cdot u = 0, \tag{3.2}$$

where u is the d-dimensional fluid velocity ($d = 2, 3$), p is the fluid pressure, and Re is the Reynolds number. Applying the Leray projection, one gets

$$u_t + \frac{1}{Re}\Delta u = -\mathbb{P}\left(u\cdot\nabla u\right). \tag{3.3}$$

The Leray projection is an orthogonal projection in $L^2(\mathbb{R}^d)$, given by

$$\mathbb{P}g = g - \nabla\Delta^{-1}\nabla\cdot g.$$

Setting the Reynolds number to infinity $Re = 0$, the Navier-Stokes equation (3.3) reduces to the Euler equation

$$u_t = -\mathbb{P}\left(u\cdot\nabla u\right). \tag{3.4}$$

Let $H^n(\mathbb{R}^d)$ be the Sobolev space of divergence free fields. By the local wellposedness result of Kato [5, 6], when $n > \frac{d}{2} + 1$ ($d = 2, 3$), for any $u \in H^n(\mathbb{R}^d)$, there is a neighborhood B and a short time $T > 0$, such that for any $v \in B$ there exists a unique solution to the Navier-Stokes equation (3.3) in $C^0([0,T]; H^n(\mathbb{R}^d))$; as $Re \to \infty$, this solution converges to that of the Euler equation (3.4) in the same space. For any $t \in [0,T]$, let S^t be the solution map:

$$S^t : B \mapsto H^n(\mathbb{R}^d), \quad S^t(u(0)) = u(t), \tag{3.5}$$

which maps the initial condition to the solution's value at time t. The solution map is continuous for both Navier-Stokes equation (3.3) and Euler equation (3.4) [5, 6]. A recent result of Inci [4] shows that for Euler equation (3.4) the solution map is nowhere differentiable. Then it is natural to theorize that the norm of the derivative of the solution map approaches infinity (at most places) as the Reynolds number approaches infinity. Estimating the temporal growth of the norm of the derivative of the solution map is a daunting task. The entire subject of hydrodynamic stability is a special case where the base solution (where the derivative of the solution map is taken) is steady. Below I obtain an upper bound on the temporal growth of the norm of the derivative of the solution map. I believe the upper bound is sharp, i. e. there is no smaller upper bound.

Theorem 3.1. *The norm of the derivative of the solution map of Navier-Stokes equation* (3.3) *has the upper bound*

$$\|DS^t(u(0))\| \le e^{C\sqrt{tRe}+C_1 t}, \tag{3.6}$$

where

$$C = \frac{8}{\sqrt{2e}} \max_{\tau\in[0,T]} \|u(\tau)\|_n, \quad C_1 = 4 \max_{\tau\in[0,T]} \|u(\tau)\|_n = \frac{\sqrt{2e}}{2} C.$$

Proof. Applying the method of variation of parameters, one converts the Navier-Stokes equation (3.3) into the integral equation

$$u(t) = e^{\frac{t}{Re}\Delta} u(0) - \int_0^t e^{\frac{t-\tau}{Re}\Delta} \mathbb{P}\,(u \cdot \nabla u)\, d\tau. \tag{3.7}$$

Taking the differential in $u(0)$, one gets the differential form

$$du(t) = e^{\frac{t}{Re}\Delta} du(0) - \int_0^t e^{\frac{t-\tau}{Re}\Delta} \mathbb{P}\,(du \cdot \nabla u + u \cdot \nabla du)\, d\tau. \tag{3.8}$$

The norm of the derivative $DS^t(u(0)) = \partial u(t)/\partial u(0)$ is given by

$$\|DS^t(u(0))\| = \sup_{du(0)} \frac{\|du(t)\|_n}{\|du(0)\|_n}. \tag{3.9}$$

Applying the inequality

$$\|e^{\frac{t}{Re}\Delta} u\|_n \le \Big(\frac{1}{\sqrt{2e}}\sqrt{\frac{Re}{t}} + 1\Big)\|u\|_{n-1},$$

one gets

$$\begin{aligned}
&\|du(t)\|_n \\
&\le \|du(0)\|_n + 4 \max_{\tau\in[0,T]} \|u(\tau)\|_n \int_0^t \Big(\frac{\sqrt{Re}}{\sqrt{2e}}\frac{1}{\sqrt{t-\tau}} + 1\Big)\|du(\tau)\|_n d\tau.
\end{aligned}$$

Applying the Gronwall's inequality, one gets the estimate

$$\|du(t)\|_n \le e^{C\sqrt{tRe}+C_1 t}\|du(0)\|_n,$$

where

$$C = \frac{8}{\sqrt{2e}} \max_{\tau\in[0,T]} \|u(\tau)\|_n, \quad C_1 = 4 \max_{\tau\in[0,T]} \|u(\tau)\|_n = \frac{\sqrt{2e}}{2} C.$$

By (3.9),

$$\|DS^t(u(0))\| \le e^{C\sqrt{tRe}+C_1 t}.$$

$\square$

Remark 3.2. By Theorem 3.1, for any initial perturbation $\delta u(0)$, the deviation of the corresponding solutions can potentially amplifies according to

$$\|\delta u(t)\|_n \le e^{C\sqrt{tRe}\ +\ C_1 t}\|\delta u(0)\|_n.$$

When the Reynolds number is large, the amplification can potentially reach substantial amount in short time.

Remark 3.3. The same upper bound (3.6) also holds for the periodic boundary condition; i.e. when the Navier-Stokes equations (3.1)-(3.2) are defined on d-dimensional torus $\mathbb{T}^d$ instead of the whole space $\mathbb{R}^d$.

Remark 3.4. The beauty of the upper bound (3.6) can be revealed when the base solution (where the derivative of the solution map is taken) is steady. In such a case, one is dealing with hydrodynamic stability theory. The zero-viscosity limit of the eigenvalues of the linear Navier-Stokes equations at the steady state can be complicated [11]. In the zero-viscosity limit, some of the eigenvalues may persist to be the eigenvalues of the corresponding linear Euler equations [13]; some eigenvalues may condense into continuous spectra; and other eigenvalues may approach a set that is not in the spectra of the corresponding linear Euler equations. The $C_1 t$ exponent in (3.6) covers the growth induced by persistent unstable eigenvalues, while the $C\sqrt{tRe}$ exponent in (3.6) covers the growth induced by the rest eigenvalues. When Re is large, the $C\sqrt{tRe}$ can be large in short time. During such short time, stable eigenvalues do not imply “decay”. Even though its derivative does not exist, directional derivatives of the solution map of Euler equations can exist as shown by the existence of solutions to the well-known Rayleigh equations. The unbounded continuous spectrum [13] of the linear Euler equations leads to the nonexistence of the derivatives of the solution map of Euler equations. Figure 1 is the spectra of the 2D linear NS (Euler) operator at the shear $\omega = 2\cos x$ where ω is the vorticity and the spatial periodic domain is $[2\pi, \frac{2\pi}{0.7}]$. Figure 2 is the spectra of the 2D linear NS (Euler) operator at the cat's eye

$$\omega = 2\cos x + \cos y\ ,$$

where the spatial periodic domain is $[2\pi, 2\pi]$.

Remark 3.5. The upper bound (3.6) is sharp when the base solution (where the derivative of the solution map is taken) is the zero solution $u(t) = 0$. In this case,

$$\|DS^t(0)\| = 1,$$

and the upper bound (3.6) is also 1. In general, estimating the lower bound of $\|DS^t(u(0))\|$ may only be done on a case by case base for the base solutions. When the base solutions are steady, this is the theory of hydrodynamic instability.

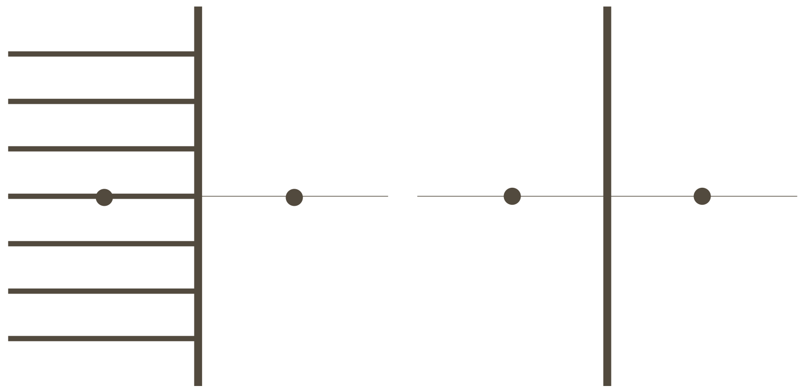

Figure 1. The spectra of the 2D linear NS operator (at a linear shear) where $\epsilon = 1/Re$, the isolated dots are the eigenvalues, and the bold face lines are unbounded continuous spectra.

4. Classical hydrodynamic instability – directional derivative

Classical hydrodynamic instability theory mainly focuses on the so-called linear instability of steady fluid flows. We can think that the linear instability theory is based on Taylor expansion of the solution map for Navier-Stokes equations (3.1)-(3.2). Let u^* be the steady flow (a fixed point in the phase space), v_0 be its initial perturbation, and $u^*+v(t)$ be the solution to the Navier-Stokes equations (3.1)-(3.2) with the initial condition $u^* + v_0$. According to Taylor expansion,

$$v(t) = dv(t) + d^2v(t) + \dots ,$$

where $dv(t)$ is the first differential in v_0 of the solution map at the steady flow u^*, similarly for $d^2v(t)$ etc.. Under the Euler dynamics, this expansion fails since the first differential does not exist [4]. The first differential satisfies the differential form

$$\begin{gathered} dv_t + \frac{1}{Re}\Delta dv = -\nabla dp - dv \cdot \nabla u^* - u^* \cdot \nabla dv, \\ \nabla \cdot dv = 0, \end{gathered} \tag{4.1}$$

where dp is the pressure differential. The linear instability refers to the instability of the differential form (4.1). In most cases studied, the steady flow u^* depends on only one spatial variable y (the so-called channel flow). This permits the following type solutions to the differential form,

$$dv(t) = \exp\{i(\sigma t + k_1 x + k_3 z)\} V(y), \tag{4.2}$$

where (x, y, z) are the spatial coordinates, σ is a complex parameter, and (k_1, k_3) are real parameters. One can view (4.2) as a single Fourier mode out of the Fourier transform of $dv(t)$. In the phase space of the dynamics, (4.2) is a directional differential with the specific direction specified by the (k_1, k_3) Fourier mode. $V(y)$ satisfies the well-known Orr-Sommerfeld equation (Rayleigh equation in the inviscid case $Re = \infty$). Even though the first differential $dv(t)$ does not exist in the inviscid case ((4.1) with $Re = \infty$), the directional differential (4.2) can exist with $V(y)$ solving the Rayleigh equation. The classical hydrodynamic instability theory mainly focuses on the studies of the directional differential (4.2).

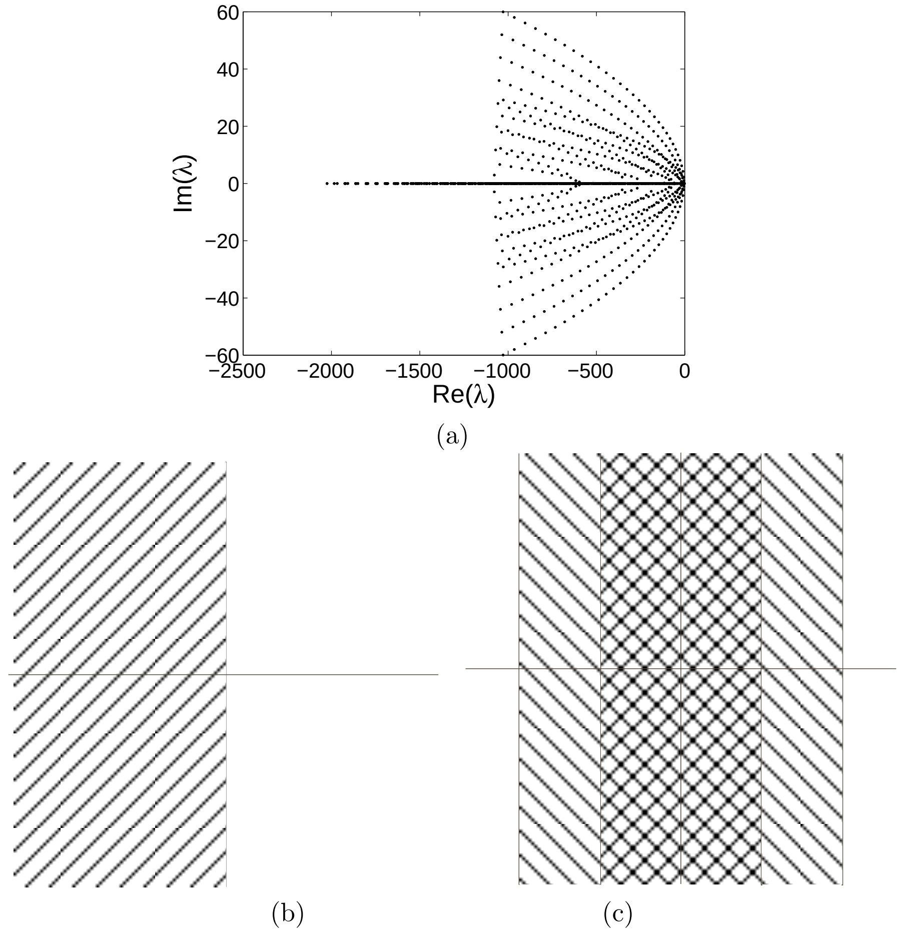

Figure 2. The spectra of the linear NS operator (at the cat's eye) where $\epsilon = 1/Re$, all the dots in (a) are eigenvalues, (b) is a half plane continuous spectrum, and (c) is the whole plane continuous spectrum.

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