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MULTIPLE SOLUTIONS FOR PERTURBED p-LAPLACIAN BOUNDARY-VALUE PROBLEMS WITH IMPULSIVE EFFECTS

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Abstract. We establish the existence of three distinct solutions for a perturbed p-Laplacian boundary value problem with impulsive effects. Our approach is based on variational methods.

1. Introduction

In this work, we show the existence of at least three solutions for the nonlinear perturbed problem

$$
\begin{gathered}
-(\rho(x)\Phi_p(u'(x)))' + s(x)\Phi_p(u'(x)) = \lambda f(x,u(x)) + \mu g(x,u(x)) \quad \text{a.e. } x \in (a,b), \\
\alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0
\end{gathered}
\tag{1.1}
$$

with the impulsive conditions

$$
\Delta(\rho(x_j)\Phi_p(u'(x_j))) = I_j(u(x_j)), \quad j = 1,2,\dots,l \tag{1.2}
$$

where $a, b \in \mathbb{R}$ with $a < b$, $p > 1$, $\Phi_p(t) = |t|^{p-2}t$, $\rho, s \in L^\infty([a,b])$ with $\rho_0 := \operatorname{ess\,inf}_{x\in[a,b]} \rho(x) > 0$, $s_0 := \operatorname{ess\,inf}_{x\in[a,b]} s(x) > 0$, $\rho(a^+) = \rho(a) > 0$, $\rho(b^-) = \rho(b) > 0$, α_1, α_2, β_1, β_2 are positive constants, $f, g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $x_0 = a < x_1 < x_2 < \dots < x_l < x_{l+1} = b$,

$$
\Delta(\rho(x_j)\Phi_p(u'(x_j))) = \rho(x_j^+)\Phi_p(u'(x_j^+)) - \rho(x_j^-)\Phi_p(u'(x_j^-))
$$

where $z(y^+)$ and $z(y^-)$ denote the right and left limits of $z(y)$ at y, respectively, $I_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, \dots, l$ are continuous satisfying the condition $\sum_{j=1}^{p}(I_j(t_1) - I_j(t_2))(t_1 - t_2) \geq 0$ for every $t_1, t_2 \in \mathbb{R}$, λ is a positive parameter and μ is a non-negative parameter.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in mechanical systems with impact, biological systems such as heart beats, population dynamics, theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry, engineering, control theory and so on. For the

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background, theory and applications of impulsive differential equations, we refer the interest readers to [3, 4, 10, 13, 14, 16, 19, 21].

Existence and multiplicity of solutions for impulsive differential equations have been studied by several authors and, for an overview on this subject, we refer the reader to the papers [1, 2, 15, 18, 23, 24, 26, 27]. For instance, Tian and Ge in [23], using variational methods, have studied the existence of at least two positive solutions for the nonlinear impulsive boundary-value problem

$$\begin{gathered}
-(\rho(t)\Phi_p(u'(t)))' + s(t)\Phi_p(u'(t)) = f(t,u(t)) \quad \text{a.e. } t \neq t_i,\ t \in (a,b), \\
\Delta(\rho(t_i)\Phi_p(u'(t_j))) = I_i(u(t_i)), \quad i = 1,2,\dots,l \\
\alpha u'(a) - \beta u(a) = A, \quad \gamma u'(b) + \sigma u(b) = B,
\end{gathered}$$

where $a, b \in \mathbb{R}$ with $a < b$, $p > 1$, $\Phi_p(t) = |t|^{p-2}t$, $\rho, s \in L^\infty([a,b])$ with $\operatorname{ess\,inf}_{t\in[a,b]} \rho(t) > 0$, $\operatorname{ess\,inf}_{t\in[a,b]} s(t) > 0$, $0 < \rho(a), \rho(b) < +\infty$, $A \leq 0$, $B \geq 0$, α, β, γ, σ are positive constants, $I_i \in C([0,+\infty),\ [0,+\infty))$ for $i = 1,\dots,l$, $f \in C([a,b]\times[0,+\infty),[0,+\infty))$, $f(t,0) \neq 0$ for $t \in [a,b]$, $t_0 = a < t_1 < t_2 \dots < t_l < t_{l+1} = b$, $\Delta(\rho(t_i)\Phi_p(u'(t_i))) = \rho(t_i^+)\Phi_p(u'(t_i^+)) - \rho(t_i^-)\Phi_p(u'(t_i^-))$ where $x(t_i^+)$ (respectively $x(t_i^-)$) denotes the right limit (respectively left limit) of $x(t)$ at $t = t_i$ for $i = 1,\dots,l$. Also, Tain and Ge in [24] have studied the existence of positive solutions to the linear and nonlinear Sturm-Liouville impulsive problem by using variational methods. In fact they have generalized the results of [18, 23]. In [1], Bai and Dai by using critical point theory, some criteria have obtained to guarantee that the impulsive problem

$$\begin{gathered}
-(\rho(t)\Phi_p(u'(t)))' + s(t)\Phi_p(u'(t)) = \lambda f(t,u(t)) \quad \text{a.e. } t \neq t_i,\ t \in (a,b), \\
\Delta(\rho(t_i)\Phi_p(u'(t_j))) = I_i(u(t_i)), \quad i = 1,2,\dots,l \\
\alpha u'(a) - \beta u(a) = A, \quad \gamma u'(b) + \sigma u(b) = B,
\end{gathered}$$

where $a, b \in \mathbb{R}$ with $a < b$, $p > 1$, $\Phi_p(t) = |t|^{p-2}t$, $\rho, s \in L^\infty([a,b])$ with $\operatorname{ess\,inf}_{t\in[a,b]} \rho(t) > 0$, $\operatorname{ess\,inf}_{t\in[a,b]} s(t) > 0$, $0 < \rho(a), \rho(b) < +\infty$, λ is a positive parameter, A, B are constant, α, β, γ, σ are positive constants, $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a continuous function, $I_i : \mathbb{R} \to \mathbb{R}$ for $i = 1,\dots,l$ are continuous functions, $t_0 = a < t_1 < t_2 \dots < t_l < t_{l+1} = b$, $\Delta(\rho(t_i)\Phi_p(u'(t_i))) = \rho(t_i^+)\Phi_p(u'(t_i^+)) - \rho(t_i^-)\Phi_p(u'(t_i^-))$ where $x(t_i^+)$ (respectively $x(t_i^-)$) denotes the right limit (respectively left limit) of $x(t)$ at $t = t_i$ for $i = 1,\dots,l$, has at least one solution, two solutions and infinitely many solutions when the parameter lies in different intervals. In particular, in [2], Bai and Dai, employing a three critical points theorem due to Ricceri [20] have ensured the existence of at least three solutions for (1.1)-(1.2) in the case $\mu = 0$.

In this article, motivated by [2], employing a three critical points theorem obtained in [5] which we recall in the next section (Theorem 2.1), we ensure the existence of at least three weak solutions for the problem (1.1)-(1.2). We explicitly observe that in [2], $\mu = 0$ and no exact estimate of λ for which the problem (1.1)-(1.2) admits multiple solutions is ensured. The aim of this work is to establish precise values of λ and μ for which the problem (1.1)-(1.2) admits at least three weak solutions.

Theorem 2.1 has been used for establishing the existence of at least three solutions for eigenvalue problems in the papers [6, 7, 8, 12]. Fora review on the subject, we refer the reader to [11].

2. Preliminaries

Our main tool is the following three critical points theorem.

Theorem 2.1 ([5, Theorem 2.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that*

(a1) $\frac{1}{r} \sup_{\Phi(x)\le r} \Psi(x) < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(a2) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Let $X := W^{1,p}([a, b])$ equipped with the norm

$$\|u\| := \Big(\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx\Big)^{1/p}$$

which is equivalent to the usual one. The following lemma is useful for proving our main result.

Lemma 2.2 ([23, Lemma 2.6]). *Let $u \in X$. Then*

$$\|u\|_\infty = \max_{x\in[a,b]} |u(x)| \le M\|u\| \tag{2.1}$$

where

$$M = 2^{1/q} \max\Big\{\frac{1}{(b-a)^{1/p} s_0^{1/p}}, \frac{(b-a)^{1/p}}{\rho_0^{1/p}}\Big\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

By a classical solution of the problem (1.1)-(1.2), we mean a function $u \in \{u(x) \in X : \rho(x)\Phi_p(u')(.) \in W^{1,\infty}(x_j, x_{j+1}),\ j = 0, 1, \dots, l\}$ such that u satisfies (1.1)-(1.2). We say that a function $u \in X$ is a weak solution of the problem (1.1)-(1.2) if

$$\begin{aligned}
&\int_a^b \rho(x)\Phi_p(u'(x))v'(x)dx + \int_a^b s(x)\Phi_p(u(x))v(x)dx \\
&+ \rho(a)\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)v(a) + \rho(b)\Phi_p\Big(\frac{\beta_2 u(b)}{\beta 1}\Big)v(b) + \sum_{j=1}^{l} I_j(u(x_j))v(x_j) \\
&- \lambda \int_a^b f(x, u(x))v(x)dx - \mu \int_a^b g(x, u(x))v(x)dx = 0
\end{aligned}$$

for every $v \in X$.

For the sake of convenience, in the sequel, we define

$$\begin{gathered}
F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [a,b] \times \mathbb{R}, \\
G(x,t) = \int_0^t g(x,\xi)d\xi \quad \text{for all } (x,t) \in [a,b] \times \mathbb{R}, \\
C_1 = \frac{M^p}{p}\Big(\frac{\rho(a)\alpha_2^{p-1}}{\alpha_1^{p-1}} + \frac{\rho(b)\beta_2^{p-1}}{\beta_1^{p-1}}\Big)
\end{gathered}$$

$$C_2=\frac{1}{p}-\sum_{j=1}^{l}\frac{b_j}{\gamma_j+1}M^{\gamma_j+1},$$
$$C_3=\frac{1}{p}+\sum_{j=1}^{l}\frac{b_j}{\gamma_j+1}M^{\gamma_j+1},$$
$$C_4=\sum_{j=1}^{l}\Big(a_jM+\frac{b_j}{\gamma_j+1}M^{\gamma_j+1}\Big).$$

For given constants $\delta_1,\ \delta_2,\ \eta_1$ and η_2 put

$$K_1:=\Big((b-a)\Big(\frac{\delta_1}{\eta_1}+\frac{\delta_2}{\eta_2}\Big)+\frac{\alpha_1}{\alpha_2}\delta_1+\frac{\beta_1}{\beta_2}\delta_2\Big)/\Big((b-a)\Big(\frac{1}{\eta_1}+\frac{1}{\eta_2}-1\Big)\Big),$$
$$K_2:=|\delta_1|^p\int_a^{a+\frac{b-a}{\eta_1}}\rho(x)dx+|K_1|^p\int_{a+\frac{b-a}{\eta_1}}^{b-\frac{b-a}{\eta_2}}\rho(x)dx+|\delta_2|^p\int_{b-\frac{b-a}{\eta_1}}^{b}\rho(x)dx,$$
$$K_3=\max\Big\{\frac{\alpha_1}{\alpha_2}|\delta_1|,\ \Big(\frac{b-a}{\eta_1}+\frac{\alpha_1}{\alpha_2}\Big)|\delta_1|,\ \Big(\frac{b-a}{\eta_2}+\frac{\beta_1}{\beta_2}\Big)|\delta_2|,\ \frac{\beta_1}{\beta_2}|\delta_2|\Big\},$$
$$K_4:=(C_1+C_3)\Big(K_2+K_3^p\int_a^b s(x)dx\Big)+C_4\Big(K_2+K_3^p\int_a^b s(x)dx\Big)^{1/p},$$
$$h_1(x)=\delta_1\Big(x+\frac{\alpha_1}{\alpha_2}-a\Big),\ h_2(x)=K_1\Big(x-a-\frac{b-a}{\eta_1}\Big)+\delta_1\Big(\frac{b-a}{\eta_1}+\frac{\alpha_1}{\alpha_2}\Big),$$
$$h_3(x)=\delta_2\Big(x-\frac{\beta_1}{\beta_2}-b\Big),$$

and

$$K^F:=\int_a^{a+\frac{b-a}{\eta_1}}F(x,h_1(x))dx+\int_{a+\frac{b-a}{\eta_1}}^{b-\frac{b-a}{\eta_2}}F(x,h_2(x))dx+\int_{b-\frac{b-a}{\eta_2}}^{b}F(x,h_3(x))dx.$$

In this article, we assume throughout, and without further mention, that the following condition holds:

(A1) The impulsive functions I_j have sublinear growth, i.e., there exist constants $a_j>0$, $b_j>0$, and $\gamma_j\in[0,p-1)$ for $j=1,2,\dots,l$ such that
$$|I_j(t)|\le a_j+b_j|t|^{\gamma_j}\quad\text{for very } t\in\mathbb{R},\ j=1,2,\dots,l.$$

Moreover, set $G^\theta:=\int_\Omega\max_{|t|\le\theta}G(x,t)dt$ for all $\theta>0$, and $G_\eta:=\inf_{\Omega\times[0,\eta]}G$ for all $\eta>0$. If g is sign-changing, then clearly, $G^\theta\ge0$ and $G_\eta\le0$.

A special case of our main results is the following theorem, whose proof we delay until the end of the paper.

Theorem 2.3. *Assume that $C_2':=\frac{1}{p}-\sum_{j=1}^{l}\frac{b_j}{\gamma_j+1}2^{\frac{\gamma_j+1}{q}}>0$. Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function. Put $F(t)=\int_0^t f(\xi)d\xi$ for each $t\in\mathbb{R}$. Suppose that*
$$\liminf_{\xi\to0}\frac{F(\xi)}{\frac{C_2'}{2^{p/q}}\xi^p-\frac{C_4'}{2^{1/q}}\xi}=\limsup_{\xi\to+\infty}\frac{F(\xi)}{\frac{C_2'}{2^{p/q}}\xi^p-\frac{C_4'}{2^{1/q}}\xi}=0$$
where
$$C_4':=\sum_{j=1}^{l}\Big(a_j2^{1/q}+\frac{b_j}{\gamma_j+1}2^{\frac{\gamma_j+1}{q}}\Big).$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[0,1]} \int_0^t g(x,s)ds}{\frac{C_2'}{2^{p/q}}t^p - \frac{C_4'}{2^{1/q}}t} < +\infty,$$

there exists $\delta^*_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem

$$\begin{gathered} -(\Phi_p(u'(x)))' + \Phi_p(u'(x)) = \lambda f(u(x)) + \mu g(x, u(x)) \quad a.e.\ x \in (0,1), \\ u'(0^+) - u(0) = 0, \quad u'(1^-) + u(1) = 0 \end{gathered}$$

with the impulsive conditions

$$\Delta(\rho(x_j)\Phi_p(u'(x_j))) = I_j(u(x_j)), \quad j = 1, 2, \dots, l$$

admits at least three weak solutions.

We need the following proposition in the proof our main result.

Proposition 2.4. *Let $T : X \to X^*$ be the operator defined by*

$$\begin{aligned} T(u)v = \int_a^b \rho(x)\Phi_p(u'(x))h'(x)dx + \int_a^b s(x)\Phi_p(u(x))h(x)dx \\ + \rho(a)\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)h(a) + \rho(b)\Phi_p\Big(\frac{\beta_2 u(b)}{\beta 1}\Big)h(b) \\ + \sum_{j=1}^{l} I_j(u(x_j))v(x_j) \end{aligned}$$

for every $u, h \in X$. Then T admits a continuous inverse on X^*.

Proof. For any $u \in X \setminus \{0\}$,

$$\begin{aligned} &\lim_{\|u\|\to\infty} \frac{\langle T(u), u\rangle}{\|u\|} \\ &= \lim_{\|u\|\to\infty} \Big(\frac{\int_a^b \rho(x)\Phi_p(u'(x))u'(x)dx + \int_a^b s(x)\Phi_p(u(x))u(x)dx}{\|u\|} \\ &\quad + \frac{\rho(a)\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)u(a) + \rho(b)\Phi_p\Big(\frac{\beta_2 u(b)}{\beta 1}\Big)u(b) + \sum_{j=1}^{l} I_j(u(x_j))u(x_j)}{\|u\|} \Big) \\ &= \lim_{\|u\|\to\infty} \Big(\frac{\int_a^b \rho(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx}{\|u\|} \\ &\quad + \frac{\rho(a)\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)u(a) + \rho(b)\Phi_p\Big(\frac{\beta_2 u(b)}{\beta 1}\Big)u(b) + \sum_{j=1}^{l} I_j(u(x_j))u(x_j)}{\|u\|} \Big) \\ &= \lim_{\|u\|\to\infty} \frac{\|u\|^p + \rho(a)\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)u(a) + \rho(b)\Phi_p\Big(\frac{\beta_2 u(b)}{\beta 1}\Big)u(b)}{\|u\|} \\ &\quad + \frac{\sum_{j=1}^{l} I_j(u(x_j))u(x_j)}{\|u\|} = \infty. \end{aligned}$$

Thus, the map T is coercive.

For any $u \in X$ and $v \in X$, we have

$$
\begin{aligned}
&\langle T(u)-T(v), u-v\rangle \\
&= \int_a^b \Big(\rho(x)(\Phi_p(u'(x))-\Phi_p(v'(x)))(u'(x)-v'(x)) \\
&\quad + s(x)(\Phi_p(u(x))-\Phi_p(u(x)))(u(x)-v(x))\Big)dx \\
&\quad + \rho(a)(\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)-\Phi_p\Big(\frac{\alpha_2 v(a)}{\alpha_1}\Big))(u(a)-v(a)) + \rho(b)(\Phi_p\Big(\frac{\beta_2 u(b)}{\beta_1}\Big) \\
&\quad - \Phi_p\Big(\frac{\beta_2 v(b)}{\beta_1}\Big))(u(b)-v(b)) + \sum_{j=1}^{l}(I_j(u(x_j))-I_j(v(x_j)))(u(x_j)-v(x_j)).
\end{aligned}
$$

Hence, from our assumptions on the data, we have

$$
\begin{aligned}
\langle T(u)-T(v), u-v\rangle \geq \int_a^b &\Big(\rho(x)(\Phi_p(u'(x))-\Phi_p(v'(x)))(u'(x)-v'(x)) \\
&+ s(x)(\Phi_p(u(x))-\Phi_p(u(x)))(u(x)-v(x))\Big)dx.
\end{aligned}
$$

Now, taking into account [22, (2.)], there exist c_p, $d_p > 0$ such that

$$
\begin{aligned}
&\langle T(u)-T(v), u-v\rangle \\
&\geq \begin{cases} c_p \int_a^b \Big(\rho(x)|u'(x)-v'(x)|^p + s(x)|u(x)-v(x)|^p\Big)dx & \text{if } p \geq 2, \\ d_p \int_a^b \Big(\frac{\rho(x)|u'(x)-v'(x))|^2}{(|u'(x)|+|v'(x)|)^{2-p}} + \frac{s(x)|u(x)-v(x))|^2}{(|u(x)|+|v(x)|)^{2-p}}\Big)dx & \text{if } 1 < p < 2. \end{cases}
\end{aligned} \tag{2.2}
$$

At this point, if $p \geq 2$, then it follows that

$$
\langle T(u)-T(v), u-v\rangle \geq c_p\|u-v\|^p,
$$

so T is uniformly monotone. By [25, Theorem 26.A (d)], T^{-1} exists and is continuous on X^*. On the other hand, if $1 < p < 2$, by Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_a^b s(x)|u(x)-v(x)|^p dx \\
&\leq \Big(\int_a^b \frac{s(x)|u(x)-v(x)|^2}{(|u(x)|+|v(x)|)^{2-p}}dx\Big)^{p/2}\Big(\int_a^b s(x)(|u(x)|+|v(x)|)^p dx\Big)^{(2-p)/2} \\
&\leq \Big(\int_a^b \frac{s(x)|u(x)-v(x)|^2}{(|u(x)|+|v(x)|)^{2-p}}dx\Big)^{p/2} 2^{\frac{(p-1)(2-p)}{2}}\Big(\int_a^b s(x)(|u(x)|^p+|v(x)|^p)dx\Big)^{\frac{2-p}{2}} \\
&\leq 2^{\frac{(p-1)(2-p)}{2}}\Big(\int_a^b \frac{s(x)|u(x)-v(x)|^2}{(|u(x)|+|v(x)|)^{2-p}}dx\Big)^{p/2}\big(\|u\|+\|v\|\big)^{(2-p)p/2}.
\end{aligned} \tag{2.3}
$$

Similarly, one has

$$
\begin{aligned}
&\int_a^b \rho(x)|u'(x)-v'(x)|^p dx \\
&\leq 2^{\frac{(p-1)(2-p)}{2}}\Big(\int_a^b \frac{\rho(x)|u'(x)-v'(x)|^2}{(|u'(x)|+|v'(x)|)^{2-p}}dx\Big)^{p/2}\big(\|u\|+\|v\|\big)^{(2-p)p/2}.
\end{aligned} \tag{2.4}
$$

Then, relation (2.2) together with (2.3) and (2.4), yields

$$
\langle T(u)-T(v), u-v\rangle
$$

$$
\begin{aligned}
&\geq \frac{2^{\frac{(p-1)(2-p)}{2}}d_p}{(\|u\|+\|v\|)^{2-p}}\Big(\Big(\int_a^b \rho(x)|u'(x)-v'(x)|^p dx\Big)^{2/p} \\
&\quad + \Big(\int_a^b s(x)|u(x)-v(x)|^p dx\Big)^{2/p}\Big) \\
&\geq \frac{2^{p-2}d_p}{(\|u\|+\|v\|)^{2-p}}\Big(\int_a^b \rho(x)|u'(x)-v'(x)|^p dx + \int_a^b s(x)|u(x)-v(x)|^p dx\Big)^{2/p} \\
&= 2^{p-2}d_p \frac{\|u-v\|^2}{(\|u\|+\|v\|)^{2-p}}.
\end{aligned}
$$

Thus, T is strictly monotone. By [25, Theorem 26.A (d)], T^{-1} exists and is bounded. Moreover, given $g_1, g_2 \in X^*$, by the inequality

$$
\langle T(u)-T(v), u-v\rangle \geq 2^{p-2}d_p \frac{\|u-v\|^2}{(\|u\|+\|v\|)^{2-p}},
$$

choosing $u = T^{-1}(g_1)$ and $v = T^{-1}(g_2)$ we have

$$
\|T^{-1}(g_1) - T^{-1}(g_2)\| \leq \frac{1}{2^{p-2}d_p}(\|T^{-1}(g_1)\| + \|T^{-1}(g_2)\|)^{2-p}\|g_1 - g_2\|_{X^*}.
$$

So T^{-1} is locally Lipschitz continuous and hence continuous. This completes the proof. $\square$

3. Main results

To introduce our result, we fix three constants $\theta > 0$, δ_1 and δ_2 such that

$$
\frac{K_4}{K^F} < \frac{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta}{\int_a^b \sup_{|t|\leq\theta} F(x,t)dx}
$$

and taking

$$
\lambda \in \Lambda := \Big]\frac{K_4}{K^F}, \ \frac{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta}{\int_a^b \sup_{|t|\leq\theta} F(x,t)dx}\Big[,
$$

we set

$$
\delta_{\lambda,g} := \min\Big\{\frac{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta - \lambda\int_a^b \sup_{|t|\leq\theta} F(x,t)dx}{G^\theta}, \ \frac{K_4 - \lambda K^F}{(b-a)G_\eta}\Big\} \tag{3.1}
$$

and

$$
\overline{\delta}_{\lambda,g} := \min\Big\{\delta_{\lambda,g}, \ \frac{1}{\max\{0, (b-a)\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{\frac{C_2}{M^p}t^p - \frac{C_4}{M}t}\}}\Big\}, \tag{3.2}
$$

where we define $\frac{r}{0} = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$
\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{\frac{C_2}{M^p}t^p - \frac{C_4}{M}t} \leq 0,
$$

and $G_\eta = G^\theta = 0$.

Now, we formulate our main result.

Theorem 3.1. *Assume that $C_2 > 0$ and there exist constants δ_1 and δ_2, and positive constants θ, η_1 and η_2 with $\delta_1^2 + \delta_2^2 \neq 0$, $\eta_1 + \eta_2 < \eta_1\eta_2$ and*

$$K_2^{1/p} > \frac{\theta}{M} > (\frac{C_4}{C_1})^{1/(p-1)}$$

such that

(A2) $\frac{\int_a^b \sup_{|t|\leq\theta} F(x,t)dx}{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta} < \frac{K^F}{K_4}$;

(A3) $\limsup_{|t|\to+\infty} \frac{\sup_{x\in[a,b]} F(x,t)}{\frac{C_2}{M^p}t^p - \frac{C_4}{M}t} \leq 0.$

Then, for each

$$\lambda \in \Lambda := \Big]\frac{K_4}{K^F}, \frac{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta}{\int_a^b \sup_{|t|\leq\theta} F(x,t)dx}\Big[$$

and for every L^1-Caratéodory function $g : [a,b]\times\mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]} G(x,t)}{\frac{C_2}{M^p}t^p - \frac{C_4}{M}t} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.2) such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g}[$, the problem (1.1)-(1.2) admits at least three distinct weak solutions in X.

Proof. To apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^{l}\int_0^{u(x_j)} I_j(t)dt + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p, ,$$

$$\Psi(u) = \int_a^b [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx.$$

Now we show that the functionals Φ and Ψ satisfy the required conditions. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b [f(x,u(x)) + \frac{\mu}{\lambda}g(x,u(x))]v(x)dx,$$

for every $v \in X$, as well as, is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $u \in X$, let $u_n \to u$ weakly in X as $n \to +\infty$. Then we have u_n converges uniformly to u on $[a,b]$ as $n \to +\infty$ (see [25]). Since f and g are L^1-Carathéodory functions, f and g are continuous in $\mathbb{R}$ for every $x \in [a,b]$. So $f(x,u_n) + \frac{\mu}{\lambda}g(x,u_n) \to f(x,u) + \frac{\mu}{\lambda}g(x,u)$ strongly as $n \to +\infty$, from which follows $\Psi'(u_n) \to \Psi'(u)$ strongly as $n \to +\infty$. Thus we have established that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by Proposition 26.2 of [25]. Moreover, Φ is continuously differentiable and whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)v = &\int_a^b \rho(x)\Phi_p(u'(x))v'(x)dx + \int_a^b s(x)\Phi_p(u(x))v(x)dx\\ &+ \rho(a)\Phi_p\Big(\frac{\alpha_2 u(a)}{\alpha_1}\Big)v(a) + \rho(b)\Phi_p\Big(\frac{\beta_2 u(b)}{\beta 1}\Big)v(b) + \sum_{j=1}^{l} I_j(u(x_j))v(x_j)\end{aligned}$$

for every $v \in X$, while Proposition 2.4 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, let for fixed $u \in X$, assume $u_n \to u$ weakly in X as $n \to +\infty$. The continuity and convexity of $\|u\|^p$ imply $\|u\|^p$ is sequentially weakly lower semicontinuous, which combining the continuity of I_j for $j = 1, \dots, l$ yields that

$$\begin{aligned}&\lim_{n\to+\infty}\Big(\frac{1}{p}\|u_n\|^p + \sum_{j=1}^{l}\int_0^{u_n(x_j)} I_j(t)dt + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u_n(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u_n(b)|^p\Big)\\ &\geq \frac{1}{p}\|u\|^p + \sum_{j=1}^{l}\int_0^{u(x_j)} I_j(t)dt + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p,\end{aligned}$$

namely

$$\liminf_{n\to+\infty}\Phi(u_n) \geq \Phi(u)$$

which means Φ is sequentially weakly lower semicontinuous. Clearly, the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$. Put $r = \frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta$ and

$$w(x) = \begin{cases} h_1(x), & x \in [a, a + \frac{b-a}{\eta_1}),\\ h_2(x), & x \in [a + \frac{b-a}{\eta_1}, b - \frac{b-a}{\eta_1}],\\ h_3(x), & x \in (a + \frac{b-a}{\eta_1}, b].\end{cases} \tag{3.3}$$

It is easy to see that $w \in X$ and, in particular, in view of

$$\int_a^b \rho(x)|w'(x)|^p dx = K_2 \quad \text{and} \quad 0 \leq \int_a^b s(x)|w(x)|^p dx \leq K_3^p \int_a^b s(x)dx,$$

we have

$$\|w\| \leq \Big(K_2 + K_3^p\int_a^b s(x)dx\Big)^{1/p},$$

which in conjunction with the inequality

$$\Phi(u) \leq (C_1 + C_3)\|u\|^p + C_4\|u\| \tag{3.4}$$

for all $u \in X$ (see[2]), yields

$$\Phi(w) \leq K_4. \tag{3.5}$$

Moreover, by the same reasoning as given given in the proof [2, Lemma 5], using (3.5), from the condition

$$K_2^{1/p} > \frac{\theta}{M} > \big(\frac{C_4}{C_1}\big)^{1/(p-1)}$$

one has $0 < r < \Phi(w)$. Taking (2.1) into account, by the same arguing as given in the proof [2, Lemma 5] we have

$$\Phi^{-1}(]-\infty, r]) \subseteq \{u \in X; \|u\|_\infty \leq \theta\},$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_a^b [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx\\ &\leq \int_a^b \sup_{|t|\leq\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta.\end{aligned}$$

On the other hand, from the definition of Ψ, we infer

$$\begin{aligned}\Psi(w) &= \int_a^b F(x,w(x))dx + \frac{\mu}{\lambda}\int_a^b G(x,w(x))dx \\ &= K^F + \frac{\mu}{\lambda}\int_a^b G(x,w(x))dx \\ &\geq K^F + (b-a)\frac{\mu}{\lambda}\inf_{[a,b]]\times[0,\eta]} G \\ &= K^F + (b-a)\frac{\mu}{\lambda}G_\eta.\end{aligned}$$

Therefore, owing to Assumption (A2) and (3.5), we have

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_a^b[F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx}{r} \\ &\leq \frac{\int_a^b \sup_{|t|\leq\theta}F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{K^F + \frac{\mu}{\lambda}\int_a^b G(x,w(x))dx}{K_4} \\ &\geq \frac{\int_a^b F(x,w(x))dx + (b-a)\frac{\mu}{\lambda}G_\eta}{K_4}.\end{aligned} \tag{3.7}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta - \lambda\int_a^b \sup_{|t|\leq\theta}F(x,t)dx}{G^\theta},$$

which means

$$\frac{\int_a^b \sup_{|t|\leq\theta}F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{K_4 - \lambda K^F}{(b-a)G_\eta},$$

and this means

$$\frac{K^F + (b-a)\frac{\mu}{\lambda}G_\eta}{K_4} > \frac{1}{\lambda}.$$

Then

$$\frac{\int_a^b \sup_{|t|\leq\theta}F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\frac{C_2}{M^p}\theta^p - \frac{C_4}{M}\theta} < \frac{1}{\lambda} < \frac{K^F + (b-a)\frac{\mu}{\lambda}G_\eta}{K_4}. \tag{3.8}$$

Hence from (3.6)-(3.8), the condition (a1) of Theorem 2.1 is verified.

Finally, since $\mu < \bar{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{|t|\to\infty} \frac{\sup_{x\in[a,b]}G(x,t)}{\frac{C_2}{M^p}t^p - \frac{C_4}{M}t} < l$$

and $\mu l < M^p$. Therefore, there exists a function $h \in L^1([a,b])$ such that

$$G(x,t) \leq l(\frac{C_2}{M^p}t^p - \frac{C_4}{M}t) + h(x) \quad \text{for all } x\in[a,b] \text{ and for all } t\in\mathbb{R}.$$

Now, fix $0 < \epsilon < \frac{M^p}{\lambda} - \frac{\mu l}{\lambda}$. From (A3) there is a function $h_\epsilon \in L^1([a,b])$ such that

$$F(x,t) \leq \epsilon(\frac{C_2}{M^p}t^p - \frac{C_4}{M}t) + h_\epsilon(x) \quad \text{for all } x \in [a,b] \text{and for all } t \in \mathbb{R}.$$

Using (3.4), it follows that, for each $u \in X$,

$$\begin{aligned}
&\Phi(u) - \lambda\Psi(u) \\
&= \frac{1}{p}\|u\|^p + \sum_{j=1}^{l}\int_0^{u(x_j)} I_j(t)dt + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p \\
&\quad - \lambda\int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx \\
&\geq (C_2 - \lambda\epsilon\frac{C_2}{M^p} - \mu l\frac{C_2}{M^p})\|u\|^p - (C_4 + \lambda\epsilon\frac{C_4}{M} + \mu l\frac{C_4}{M})\|u\| - \lambda\|h_\epsilon\|_1 - \mu\|h\|_1,
\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty,$$

which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a2) of Theorem 2.1 is satisfied. Since, from (3.6) and (3.8),

$$\lambda \in \Big]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\Big[,$$

Theorem 2.1, with $\overline{x} = w$, assures the existence of three critical points for the functional $\Phi - \lambda\Psi$, and the proof is complete. □

Here, we exhibit an example whose construction is motivated by [2, Example 1], in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.2. Consider the problem

$$\begin{gathered}
-((x+3)|u'(x)|u'(x))' + (2x+2)|u(x)|u(x) = \lambda f(x,u(x)) + \mu g(x,u(x)) \\
\text{a.e. } x \in (1,2), \\
u'(1^+) - u(1) = 0, \quad u'(2^-) + u(2) = 0, \\
\Delta((x_1+3)|u'(x_1)|u'(x_1) = -(\frac{1}{12} + \frac{5}{24}|u(x_1)|^{3/2}), \quad x_1 \in (1,2)
\end{gathered} \tag{3.9}$$

where

$$f(x,t) = \begin{cases} x(3t^2 - 2t) & \text{if } (x,t) \in [1,2]\times(-\infty,1], \\ xt & \text{if } (x,t) \in [1,2]\times[1,+\infty). \end{cases}$$

$g(x,t) = e^{x-t}t^3$ for all $x \in [1,2]$ and $t \in \mathbb{R}$, and $I_1(u(x_1)) = -(\frac{1}{12} + \frac{5}{24}|u(x_1)|^{3/2})$ satisfying the condition $(|v(x_1)|^{3/2} - |u(x_1)|^{3/2})(u(x_1) - v(x_1)) \geq 0$ for all $u, v \in W^{1,3}([1,2])$. A direct calculation shows

$$F(x,t) = \begin{cases} x(t^3 - t^2) & \text{if } (x,t) \in [1,2]\times(-\infty,1], \\ \frac{x}{2}(t^2 - 1) & \text{if } (x,t) \in [1,2]\times[1,+\infty). \end{cases}$$

In view of Lemma 2.2, $M = 1$. Choose $\eta_1 = \eta_2 = 4$, $\delta_1 = 1$, $\delta_2 = -1$ and $\theta = 1$. We observe that $C_1 = 3$, $C_2 = \frac{1}{4}$, $C_3 = 5/12$, $C_4 = 1/6$, $K_1 = 0$, $K_2 = 9/4$, $K_3 = 5/4$,

$K_4 \approx \frac{1}{12\times 2.011\times 10^{-3}}$, $K^F \approx 3.125\times 10^{-1}$ and $\int_1^2 \sup_{|t|\le\theta} F(x,t)dx \le 0$. So, since

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in[1,2]} F(x,t)}{\frac{t^3}{4}-\frac{t}{6}} = 0,$$

we see that all assumptions of Theorem 3.1 are satisfied. Hence, for each $\lambda > \frac{\frac{1}{12\times 2.011\times 10^{-3}}}{3.125\times 10^{-1}}$ and every $\mu \ge 0$ (since $g_\infty = 0$), the problem (3.9) has at least three solutions in $W^{1,3}([1,2])$.

The following example illustrates the result in Theorem 2.3.

Example 3.3. Consider the problem

$$\begin{gathered} -(|u'(x)|u'(x))' + |u(x)|u(x) = \lambda e^{-u(x)} u^2(x)(3-u(x)) + \mu e^{x-u(x)^+}(u(x)^+)^\gamma, \\ \text{a.e. } x\in(0,1) \\ u'(0^+) - u(0) = 0, \quad u'(1^-) + u(1) = 0, \\ \Delta((x_1+3)|u'(x_1)|u'(x_1) = -(\frac{1}{12} + \frac{5}{24}|u(x_1)|^{3/2}), \quad x_1\in(0,1) \end{gathered} \tag{3.10}$$

where $u^+ = \max\{u,0\}$, $I_1(u(x_1)) = -(\frac{1}{12}+\frac{5}{24}|u(x_1)|^{3/2})$ satisfying the condition $(|v(x_1)|^{3/2} - |u(x_1)|^{3/2})(u(x_1)-v(x_1)) \ge 0$ for all $u,v\in W^{1,3}([1,2])$ and γ is a positive real number. It is obvious that $C_2' = 1/4$ and $C_4' = 1/6$. Also a direct calculation shows $F(t) = e^{-t}t^3$ for all $t\in\mathbb{R}$. So, one has

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\frac{1}{16}\xi^3 - \frac{1}{6\sqrt[3]{4}}\xi} = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\frac{1}{16}\xi^3 - \frac{1}{6\sqrt[3]{4}}\xi} = 0.$$

Hence, using Theorem 2.3, there is $\lambda^* > 0$ such that, since $g_\infty = 0$, for each $\lambda > \lambda^*$ and $\mu \ge 0$, the problem (3.10) admits at least three solutions.

Proof of Theorem 2.3. Fix $\lambda > \lambda^* := \frac{K_4'}{K'^F}$ for some constants δ_1 and δ_2, and positive constants η_1 and η_2 with $\delta_1^2 + \delta_2^2 \ne 0$, $\eta_1 + \eta_2 < \eta_1\eta_2$ where

$$\begin{aligned} K_4' &:= (C_1' + C_3')\Big(\frac{|\delta_1|^p}{4} + \frac{5^p}{2^{p+1}}(|\delta_1| + |\delta_2|)^p + \frac{|\delta_2|^p}{4} + (\frac{5}{4}\max\{|\delta_1|,|\delta_2|\})^p\Big) \\ &\quad + C_4'\Big(\frac{|\delta_1|^p}{4} + \frac{5^p}{2^{p+1}}(|\delta_1| + |\delta_2|)^p + \frac{|\delta_2|^p}{4} + (\frac{5}{4}\max\{|\delta_1|,|\delta_2|\})^p\Big)^{1/p} \end{aligned}$$

where $C_1' := \frac{2^p}{p}$ and $C_3' = \frac{1}{p} + \sum_{j=1}^l \frac{b_j}{\gamma_j+1} 2^{\frac{\gamma_j+1}{q}}$, and

$$\begin{aligned} K'^F &:= \int_0^{1/4} F(|\delta_1|(x+1))dx + \int_{1/4}^{3/4} F\Big(-\frac{5}{2}(|\delta_1|+|\delta_2|)(x-\frac{1}{4}) + \frac{5|\delta_1|}{4}\Big)dx \\ &\quad + \int_{3/4}^1 F(|\delta_2|(x-2))dx. \end{aligned}$$

Recalling that

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\frac{C_2'}{2^{p/q}}\xi^p - \frac{C_4'}{2^{1/q}}\xi} = 0,$$

there is a sequence $\{\theta_n\} \subset]0,+\infty[$ such that $\lim_{n\to\infty}\theta_n = 0$ and

$$\lim_{n\to\infty} \frac{\sup_{|\xi|\le\theta_n} F(\xi)}{\frac{C_2'}{2^{p/q}}\theta_n^p - \frac{C_4'}{2^{1/q}}\theta_n} = 0.$$

Indeed, one has

$$\lim_{n\to\infty} \frac{\sup_{|\xi|\le\theta_n} F(\xi)}{\frac{C_2'}{2^{p/q}}\theta_n^p - \frac{C_4'}{2^{1/q}}\theta_n} = \lim_{n\to\infty} \frac{F(\xi_{\theta_n})}{\frac{C_2}{2^{p/q}}\xi_{\theta_n}^p - \frac{C_4'}{2^{1/q}}\xi_{\theta_n}} \frac{\frac{C_2'}{2^{p/q}}\xi_{\theta_n}^p - \frac{C_4'}{2^{1/q}}\xi_{\theta_n}}{\frac{C_2'}{2^{p/q}}\theta_n^p - \frac{C_4'}{2^{1/q}}\theta_n} = 0,$$

where $F(\xi_{\theta_n}) = \sup_{|\xi|\le\theta_n} F(\xi)$. Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup_{|\xi|\le\overline{\theta}} F(\xi)}{\frac{C_2'}{2^{p/q}}\overline{\theta}^p - \frac{C_4'}{2^{1/q}}\overline{\theta}} < \min\Big\{\frac{K'^F}{(b-a)K_4'};\ \frac{1}{(b-a)\lambda}\Big\}$$

and

$$\Big(\frac{|\delta_1|^p}{4} + \frac{5^p}{2^{p+1}}(|\delta_1| + |\delta_2|)^p + \frac{|\delta_2|^p}{4}\Big)^{1/p} > \frac{\overline{\theta}}{2^{1/q}} > (\frac{C_4'}{C_1'})^{1/(p-1)}.$$

The conclusion follows by using Theorem 3.1 with $\eta_1 = \eta_2 = 4$. □

Remark 3.4. The methods used here can be applied studying discrete boundary value problems as in [9], and also non-smooth variational problems as in [17].

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