Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 113, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LOWER BOUNDS FOR THE BLOW-UP TIME OF NONLINEAR PARABOLIC PROBLEMS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. In this article, we find a lower bound for the blow-up time of solutions to some nonlinear parabolic equations under Robin boundary conditions in bounded domains of $\mathbb{R}^n$.

1. Introduction

In this article, we consider the nonlinear initial-boundary value problem

$$\begin{gathered} (b(u))_t = \nabla \cdot (g(u)\nabla u) + f(u), \quad x \in \Omega,\ t > 0 \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \Omega \end{gathered} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n, n \geq 3$, is a bounded domain with smooth boundary, ν is the outward normal vector to $\partial\Omega$, γ is a positive constant and $u_0(x) \in C^1(\overline{\Omega})$ is the initial value. We assume that f is a nonnegative $C(\mathbb{R}^+)$ function and the nonnegative functions g and b satisfy

$$\begin{gathered} g \in C^1(\mathbb{R}^+), \quad g(s) \geq g_m > 0, \quad g'(s) \leq 0, \quad \forall s > 0, \\ b \in C^2(\mathbb{R}^+), \quad 0 < b'(s) \leq b'_M, \quad b''(s) \leq 0, \quad \forall s > 0, \end{gathered} \tag{1.2}$$

where g_m and b'_M are positive constants.

The reader is referred to [1, 3, 4, 5, 6, 8] for results on bounds for blow-up time in nonlinear parabolic problems. Ding [2] studied problem (1.1) under assumptions (1.2) and derived conditions on the data which imply blow-up or the global existence of solutions. In addition, Ding obtained a lower bound for the blow-up time when $\Omega \subseteq \mathbb{R}^3$ is a bounded convex domain. Here we obtain a lower bound for the blow-up time for (1.1) in general bounded domains $\Omega \subseteq \mathbb{R}^n, n \geq 3$.

2. A lower bound for the blow-up time

In this section we find a lower bound for the blow-up time T in an appropriate measure. The idea of the proof of the following theorem comes from [1].

2000 *Mathematics Subject Classification.* 35K55, 35B44.
Key words and phrases. Parabolic equation; Robin boundary condition; blow-up; lower bound.
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Submitted June 6, 2013. Published April 16, 2014.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$, and let the functions f, g, b satisfy*

$$0 < f(s) \leq cg(s)\Big(\int_0^s \frac{b'(y)}{g(y)}dy\Big)^{p+1}, \quad s > 0, \tag{2.1}$$

for some constants $c > 0$ and $p \geq 1$. If $u(x,t)$ is a nonnegative classical solution to problem (1.1), which becomes unbounded in the measure

$$\Phi(t) = \int_\Omega \Big(\int_0^{u(x,t)} \frac{b'(y)}{g(y)}dy\Big)^{2k} dx,$$

where k is a parameter restricted by the condition

$$k > \max\big\{p(n-2), 1\big\}, \tag{2.2}$$

then T is bounded from below by

$$\int_{\Phi(0)}^{+\infty} \frac{d\xi}{k_1 + k_2\xi^{\frac{3(n-2)}{3n-8}} + k_3\xi^{\frac{2n-3}{2(n-2)}}}, \tag{2.3}$$

where k_1, k_2 and k_3 are positive constants which will be determined later in the proof.

Proof. To simplify our computations we define

$$v(s) = \int_0^s \frac{b'(y)}{g(y)}dy, \quad s > 0. \tag{2.4}$$

Hence,

$$\begin{aligned}
\frac{d\Phi}{dt} &= \frac{d}{dt}\int_\Omega v^{2k}\,dx = 2k\int_\Omega v^{2k-1}\frac{b'(u)}{g(u)}u_t\,dx \\
&= 2k\int_\Omega v^{2k-1}\frac{(b(u))_t}{g(u)}\,dx \\
&= 2k\int_\Omega v^{2k-1}\frac{1}{g(u)}\Big[\nabla\cdot(g(u)\nabla u) + f(u)\Big]\,dx \\
&= -2k(2k-1)\int_\Omega v^{2k-2}v'(u)|\nabla u|^2\,dx + 2k\int_\Omega v^{2k-1}\frac{g'(u)}{g(u)}|\nabla u|^2\,dx \\
&\quad - 2k\gamma\int_{\partial\Omega} v^{2k-1}u\,ds + 2k\int_\Omega v^{2k-1}\frac{f(u)}{g(u)}\,dx \\
&\leq -2k(2k-1)\int_\Omega v^{2k-2}\frac{b'(u)}{g(u)}|\nabla u|^2\,dx + 2k\int_\Omega v^{2k-1}\frac{f(u)}{g(u)}\,dx,
\end{aligned}$$

where in the above inequality we used $u \geq 0$ and $g'(u) \leq 0$ from (1.2). From (2.4), we have

$$|\nabla u|^2 = \Big(\frac{g(u)}{b'(u)}\Big)^2|\nabla v|^2. \tag{2.5}$$

By (1.2), (2.5), and (2.1) we have

$$\begin{aligned}
\frac{d\Phi}{dt} &\leq -2k(2k-1)\int_\Omega v^{2k-2}\frac{g(u)}{b'(u)}|\nabla v|^2\,dx + 2k\int_\Omega v^{2k-1}\frac{f(u)}{g(u)}\,dx \\
&\leq -\frac{2(2k-1)g_m}{kb'_M}\int_\Omega |\nabla v^k|^2\,dx + 2kc\int_\Omega v^{2k+p}\,dx.
\end{aligned} \tag{2.6}$$

From (2.2), Hölder, and Young inequalities, we infer

$$\begin{aligned} \int_\Omega v^{2k+p}\,dx &\le |\Omega|^{m_1}\Big(\int_\Omega v^{\frac{k(2n-3)}{n-2}}\,dx\Big)^{m_2} \\ &\le m_1|\Omega| + m_2\int_\Omega v^{\frac{k(2n-3)}{n-2}}\,dx, \end{aligned} \tag{2.7}$$

where

$$m_1 = \frac{k(2n-3)-(n-2)(2k+p)}{k(2n-3)}, \quad m_2 = \frac{(n-2)(2k+p)}{k(2n-3)}.$$

From (2.7) and the Cauchy-Schwartz inequality we have:

$$\begin{aligned} \int_\Omega v^{\frac{k(2n-3)}{n-2}}\,dx &\le \Big(\int_\Omega v^{2k}\,dx\Big)^{1/2}\Big(\int_\Omega v^{\frac{2k(n-1)}{n-2}}\,dx\Big)^{1/2} \\ &\le \Big(\int_\Omega v^{2k}\,dx\Big)^{\frac34}\Big(\int_\Omega (v^k)^{\frac{2n}{n-2}}\,dx\Big)^{1/4}. \end{aligned} \tag{2.8}$$

Applying the Sobolev inequality (see [7]) to the last term in (2.8), for $n>3$, we obtain

$$\begin{aligned} \|v^k\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{n}{2(n-2)}} &\le (c_s)^{\frac{n}{2(n-2)}}\|v^k\|_{W^{1,2}(\Omega)}^{\frac{n}{2(n-2)}} \\ &\le (c_s)^{\frac{n}{2(n-2)}}\Big(\|\nabla v^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}} + \|v^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}}\Big) \end{aligned} \tag{2.9}$$

In the case, $n=3$, we have

$$\begin{aligned} \|v^k\|_{L^{\frac{2n}{n-2}}(\Omega)}^{\frac{n}{2(n-2)}} &\le (c_s)^{\frac{n}{2(n-2)}}\|v^k\|_{W^{1,2}(\Omega)}^{\frac{n}{2(n-2)}} \\ &\le 2^{\frac{4-n}{2(n-2)}}(c_s)^{\frac{n}{2(n-2)}}\Big(\|\nabla v^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}} + \|v^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}}\Big). \end{aligned} \tag{2.10}$$

Here, c_s is the best constant in the Sobolev inequality.

By inserting (2.9) in (2.8) for $n>3$ and (2.10) in (2.8) for $n=3$, we have

$$\begin{aligned} &\int_\Omega v^{\frac{k(2n-3)}{n-2}}\,dx \\ &\le c_0\Big(\int_\Omega v^{2k}\,dx\Big)^{\frac34}\Big(\|\nabla v^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}} + \|v^k\|_{L^2(\Omega)}^{\frac{n}{2(n-2)}}\Big) \\ &= c_0\Big(\int_\Omega v^{2k}\,dx\Big)^{\frac34}\Big(\int_\Omega |\nabla v^k|^2\,dx\Big)^{\frac{n}{4(n-2)}} + c_0\Big(\int_\Omega v^{2k}\,dx\Big)^{\frac{2n-3}{2(n-2)}}, \end{aligned} \tag{2.11}$$

where

$$c_0 = \begin{cases} 2^{\frac{4-n}{2(n-2)}}(c_s)^{\frac{n}{2(n-2)}}, & \text{for } n=3, \\ (c_s)^{\frac{n}{2(n-2)}}, & \text{for } n>3. \end{cases}$$

Now, using Young's inequality we obtain

$$\begin{aligned} &\int_\Omega v^{\frac{k(2n-3)}{n-2}}\,dx \\ &\le \frac{c_0^{\frac{4(n-2)}{3n-8}}(3n-8)}{4(n-2)\epsilon^{\frac{n}{3n-8}}}\Phi^{\frac{3(n-2)}{3n-8}} + \frac{n\epsilon}{4(n-2)}\int_\Omega |\nabla v^k|^2\,dx + c_0\Phi^{\frac{2n-3}{2(n-2)}}, \end{aligned} \tag{2.12}$$

where ϵ is a positive constant to be determined later. Substituting (2.12) into (2.7) yields

$$\begin{aligned} 2kc\int_\Omega v^{2k+p}\,dx \le 2kcm_2\Big\{\frac{(3n-8)}{4(n-2)\epsilon^{\frac{n}{3n-8}}}c_0^{\frac{4(n-2)}{3n-8}}\Phi^{\frac{3(n-2)}{3n-8}} + \frac{n\epsilon}{4(n-2)}\int_\Omega |\nabla v^k|^2\,dx \\ + c_0\Phi^{\frac{2n-3}{2(n-2)}}\Big\} + 2kcm_1|\Omega|. \end{aligned}$$

By inserting the last inequality in (2.6), we have

$$\frac{d\Phi}{dt} \le \Big(-\frac{2(2k-1)g_m}{kb'_M} + \frac{nkcm_2\epsilon}{2(n-2)}\Big)\int_\Omega |\nabla v^k|^2\,dx + k_1 + k_2\Phi^{\frac{3(n-2)}{3n-8}} + k_3\Phi^{\frac{2n-3}{2(n-2)}},$$

where

$$k_1 = 2kcm_1|\Omega|,\quad k_2 = \frac{2kcm_2(c_0)^{\frac{4(n-2)}{3n-8}}(3n-8)}{4(n-2)\epsilon^{\frac{n}{3n-8}}},\quad k_3 = 2kcc_0m_2.$$

For

$$\epsilon = \frac{4(n-2)(2k-1)g_m}{nk^2cm_2b'_M},$$

the above inequality becomes

$$\frac{d\Phi}{dt} \le k_1 + k_2\Phi^{\frac{3(n-2)}{3n-8}} + k_3\Phi^{\frac{2n-3}{2(n-2)}}.$$

Thus,

$$\frac{d\Phi}{k_1 + k_2\Phi^{\frac{3(n-2)}{3n-8}} + k_3\Phi^{\frac{2n-3}{2(n-2)}}} \le dt. \tag{2.13}$$

We integrate from 0 to t to obtain

$$\int_{\Phi(0)}^{\Phi(t)} \frac{d\xi}{k_1 + k_2\xi^{\frac{3(n-2)}{3n-8}} + k_3\xi^{\frac{2n-3}{2(n-2)}}} \le t,$$

where

$$\Phi(0) = \int_\Omega \Big(\int_0^{u_0(x)} \frac{b'(y)}{g(y)}dy\Big)^{2k}dx.$$

Passing to the limit as $t \to T^-$, we conclude that

$$\int_{\Phi(0)}^{+\infty} \frac{d\xi}{k_1 + k_2\xi^{\frac{3(n-2)}{3n-8}} + k_3\xi^{\frac{2n-3}{2(n-2)}}} \le T.$$

The proof is complete. □

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