1. Introduction

In recent years, the Camassa-Holm equation [4],

\[ u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{R} \]  

(1.1)

which models the propagation of shallow water waves has attracted considerable attention from a large number of researchers, and two remarkable properties of (1.1) were found. The first one is that the equation possesses the solutions in the form of peaked solitons or ‘peakons’ [4, 8]. The peakon \( u(t, x) = ce^{-|x-ct|}, c \neq 0 \) is smooth except at its crest and the tallest among all waves of the fixed energy. It is a feature observed for the traveling waves of largest amplitude which solves the governing equations for water waves [9, 10, 29, 33]. The other remarkable property is that the equation has breaking waves [4, 11]; that is, the solution remains bounded while its slope becomes unbounded in finite time. After wave breaking the solutions can be continued uniquely as either global conservative [2] or global dissipative solutions [3].

The Camassa-Holm equation also admits many integrable multicomponent generalizations. The most popular one is

\[
\begin{align*}
m_t - A u_x + um_x + 2u_x m + \rho \rho_x &= 0 \\
\rho_t + (\rho u)_x &= 0 \\
m &= u - u_{xx}
\end{align*}
\]  

(1.2)

Notice that the C-H equation can be obtained via the obvious reduction \( \rho \equiv 0 \) and \( A = 0 \). System (1.2) was derived in [27], where \( \rho(t, x) \) is related to the free surface elevation from the equilibrium (or scalar density), and \( A \geq 0 \) characterizes
a linear underlying shear flow. Recently, Constantin-Ivanov [12] and Ivanov [23] established a rigorous justification of the derivation of system (1.2). Mathematical properties of the system have been also studied further in many works, for example [1, 6, 7, 14, 15, 19, 22, 26, 28]. Chen, Liu and Zhang [6] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher, Lechtenfeld, and Yin [14] investigated local well-posedness for the two-component Camassa-Holm system with initial data \((u_0, \rho_0 - 1) \in H^s \times H^{s-1}\) with \(s \geq 2\) by applying Kato’s theory [24] and provided some precise blow-up scenarios for strong solutions to the system. The local well-posedness is improved by Gui and Liu [20] to the Besov Spaces (especially in the Sobolev space \(H^s \times H^{s-1}\) with \(s > 3/2\)), and they showed that the finite time blow-up is determined by either the slope of the first component \(u\) or the slope of the second component \(\rho\) [8, 14]. The blow-up criterion is made more precise in [25] where Liu and Zhang showed that the wave breaking in finite time only depends on the slope of \(u\). This blow-up criterion is improved to the lowest Sobolev spaces \(H^s \times H^{s-1}\) with \(s > 3/2\) [19].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. We are interested in the effect of the weakly dissipative term on the two-component Camassa-Holm equation. Wu, Escher and Yin have investigated the blow-up phenomena, the blow-up rate of the strong solutions of the weakly dissipative CH equation [31] and DP equation [30]. Inspired by the above results, in this paper, we investigate the following generalized weakly dissipative two-component Camassa-Holm system

\[
\begin{align*}
    u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \lambda(u - u_{xx}) + \rho \rho_x = 0, \\
    \rho_t + (\rho u)_x = 0, \\
    u(0, x) = u_0(x), \rho(0, x) = \rho_0(x), \\
    u(t, x) = u(t, x + 1), \rho(t, x) = \rho(t, x + 1), \\
    t > 0, x \in \mathbb{R},
\end{align*}
\]  

or equivalently,

\[
\begin{align*}
    m_t - Au_x + \sigma(u m_x + 2u x m) + 3(1 - \sigma)u u_x + \lambda m + \rho \rho_x = 0, \\
    \rho_t + (\rho u)_x = 0, \\
    m = u - u_{xx}, \quad t > 0, x \in \mathbb{R},
\end{align*}
\]

where \(\lambda m = \lambda(I - \partial_{xx})u\) is the weakly dissipative term, \(\lambda \geq 0\) and \(A\) are constants, and \(\sigma\) is a new free parameter. When \(A = 0, \lambda = 0\) and \(\rho = 1\), Guan and Yin have obtained a new result of the existence of the strong solution and some new blow-up results [16]. Meanwhile, they have proved the global existence of the weak solution about the two-component CH equation [17]. Henry investigates the infinite propagation speed of the solution for a two-component CH equation [21].

Similar to [12, 14], we can use the method of Besov spaces together with the transport equation theory to show that system (1.4) is locally well-posedness in \(H^s \times H^{s-1}\) with \(s > 3/2\). The two equations for \(u\) and \(\rho\) are of a transport structure \(\partial_t f + v \partial_x f = g\). It is well known that most of the available estimates require \(v\) to have some level of regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as \(v\) belongs to \(L^1(0, T; \text{Lip})\). More specially, \(u\) and \(\rho\) are “transported” along directions of \(\sigma u\) and \(u\) respectively. Then, the
solution can be estimated in a Gronwall way involving \( \| u_x \|_{L^\infty} \). Hence, one can use these estimates to derive a criterion which says if \( \int_0^T \| u_x(\tau) \|_{L^\infty} \, d\tau < \infty \), then solutions can be extended further in time. Compared with the result in [5], we find that the equation [1.4] has the same blow-up rate when the blow-up occurs. This fact shows that the blow-up rate of equation [1.4] is not affected by the weakly dissipative term. But the occurrence of blow-up of equation [1.4] is affected by the dissipative parameter \( \lambda \).

The basic elementary framework is as follows. Section 2 gives the local well-posedness of system [1.4] and a wave-breaking criterion, which implies that the wave breaking only depends on the slope of \( u \), not the slope of \( \rho \). Section 3 improves the blow-up criterion with a more precise conditions. Section 4 determine the use these estimates to derive a criterion which says if solution can be estimated in a Gronwall way involving \( \| u_x \|_{L^\infty} \).

Notation. Throughout this paper, we identity periodic function spaces over the unit \( S \) in \( \mathbb{R}^2 \), i.e. \( S = \mathbb{R}/\mathbb{Z} \).

2. Formation of singularities for \( \sigma \neq 0 \)

We consider the following generalized weakly dissipative two-component Camassa-Holm system:

\[
\begin{align*}
  u_t - u_{1xx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \lambda(u - u_{xx}) + \rho \rho_x &= 0, \\
  \rho_t + (\rho u)_x &= 0, \\
  u(0, x) &= u_0(x), \\
  \rho(0, x) &= \rho_0(x), \\
  u(t, x) &= u(t, x + 1), \\
  \rho(t, x) &= \rho(t, x + 1),
\end{align*}
\]

where \( \lambda \geq 0 \) and \( A \) are constants, and \( \sigma \) is a new free parameter. System (2.1) can be written in the “transport” form

\[
\begin{align*}
  u_t + \sigma uu_x &= -\partial_x G * (-Au + \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2} \rho^2) - \lambda u, \\
  \rho_t + (\rho u)_x &= 0, \\
  u(0, x) &= u_0(x), \\
  \rho(0, x) &= \rho_0(x), \\
  u(t, x) &= u(t, x + 1), \\
  \rho(t, x) &= \rho(t, x + 1),
\end{align*}
\]

where \( G(x) := \frac{\cosh(x - |x| - \frac{1}{2})}{2 \sinh(1/2)} \), \( x \in S \), and \( (1 - \partial_x^2)^{-1} f = G * f \) for all \( f \in L^2(S) \).

Applying the transport equation theory combined with the method of Besov spaces, one may follow the similar argument as in [20] to obtain the following local well-posedness result for the system (2.1). The proof is very similar to that of [20] Theorem 1.1 and is omitted.

**Theorem 2.1.** Assume \( (u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S) \) with \( s > 3/2 \), then there exist a maximal time \( T = T((u_0, \rho_0 - 1)_{H^s \times H^{s-1}}) > 0 \) and a unique solution \( (u, \rho - 1) \) of equation (2.1) in \( C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}) \) with initial data \( (u_0, \rho_0) \). Moreover, the solution depends continuously on the initial data, and \( T \) is independent of \( s \).
Lemma 2.2 ([26]). Let $0 < s < 1$. Suppose that $f_0 \in H^s$, $g \in L^1([0, T]; H^s)$, $v, v_\tau \in L^1([0, T]; L^\infty)$, and that $f \in L^\infty([0, T]; H^s) \cap C([0, T]; S')$ solves the one-dimensional linear transport equation
\[ \partial_t f + v \partial_x f = g \]
\[ f(0, x) = f_0(x) \]
then $f \in C([0, T]; H^s)$. More precisely, there exists a constant $C$ depending only on $s$ such that
\[ \| f(t) \|_{H^s} \leq \| f_0 \|_{H^s} + C \left( \int_0^t \| g(\tau) \|_{H^s} d\tau + \int_0^t \| f(\tau) \|_{H^s} V'(\tau) d\tau \right), \]
then
\[ \| f(t) \|_{H^s} \leq e^{CV(t)} (\| f_0 \|_{H^s} + C \int_0^t \| g(\tau) \|_{H^s} d\tau), \]
where $V(t) = \int_0^t (\| v(\tau) \|_{L^\infty} + \| v_\tau(\tau) \|_{L^\infty}) d\tau$.

We may use [19, Lemma 2.1] to handle the regularity propagation of solutions to (2.1). In addition, Lemma 2.2 was proved using the Littlewood-Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument as in [19], we can obtain the following blow-up criterion.

Theorem 2.3. Let $\sigma \neq 0$, $(u, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^\sigma(S) \times H^{\sigma - 1}(S)$ with $s > 3/2$, and $T$ be the maximal time of existence. Then
\[ T < \infty \Rightarrow \int_0^T \| u_x(\tau) \|_{L^\infty} d\tau = \infty. \]

Regarding the finite time blow-up, we consider the trajectory equation of the system (2.1),
\[ \frac{d q(t, x)}{dt} = u(t, q(t, x)), \quad t \in [0, T) \]
\[ q(0, x) = x, \quad x \in S. \]
where $u \in C^1([0, T]; H^{s-1})$ is the first component of the solution $(u, \rho)$ to (2.1) with initial data $(u_0, \rho_0) \in H^\sigma(S) \times H^{\sigma - 1}(S)$ with $s > 3/2$, and $T > 0$ is the maximal time of the existence. Applying Theorem 2.1, we know that $q(t, \cdot) : S \to S$ is the diffeomorphism for every $t \in [0, T)$, and
\[ q_x(t, x) = \exp \left( \int_0^t u_x(\tau, q(\tau, x)) d\tau \right) > 0, \quad \forall (t, x) \in [0, T) \times S. \]
Hence, the $L^\infty$-norm of any function $v(t, \cdot) \in L^\infty, t \in [0, T)$ is preserved under the diffeomorphism $q(t, \cdot)$ with $t \in [0, T)$; that is, $\| v(t, \cdot) \|_{L^\infty} = \| v(t, q(t, \cdot)) \|_{L^\infty}$.

Lemma 2.4 ([11]). Let $T > 0$ and $v \in C^1([0, T); H^1(R))$, then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with $m(t) := \inf_{x \in \mathbb{R}} v_x(t, x) = v_x(t, \xi(t))$. The function $m(t)$ is absolutely continuous on $(0, T)$ with
\[ \frac{dm(t)}{dt} = v_{tx}(t, \xi(t)) \quad \text{a.e. on } (0, T). \]

Lemma 2.5. Assume $(u_0, \rho_0 - 1) \in H^\sigma(S) \times H^{\sigma - 1}(S)$ with $s > 3/2$, and $(u, \rho)$ is the solution of system (2.1), then $\|(u, \rho - 1)\|_{H^1 \times L^2}^2 \leq \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2$. 
Proof. Multiplying the first equation in (2.1) by \(u\) and using integration by parts gives
\[
\frac{d}{dt} \int_S (u^2 + u_x^2) dx + 2\int_S u\rho_x dx = 0.
\]
Rewriting the second equation in (2.1) in the form \((\rho - 1)_t + \rho_x u + \rho u_x = 0\), and multiplying by \((\rho - 1)\) and using integration by parts, we have
\[
\frac{d}{dt} \int_S (\rho - 1)^2 dx + 2\int_S u\rho_x dx - 2\int_S u\rho_x dx + 2\int_S u_x^2 dx - 2\int_S u_x\rho dx = 0.
\]
Combining the above equalities, we have
\[
\frac{d}{dt} \int_S (u^2 + u_x^2) dx + \int_S (\rho - 1)^2 dx + 2\int_S u\rho_x dx - 2\int_S u\rho_x dx + 2\int_S u^2 dx - 2\int_S u_x^2 dx + 2\int_S u_x\rho dx = 0.
\]
So we have
\[
\int_S (u^2 + u_x^2) dx + (\rho - 1)^2 dx + 2\int_S u\rho_x dx - 2\int_S u\rho_x dx + 2\int_S u^2 dx - 2\int_S u_x^2 dx + 2\int_S u_x\rho dx = 0.
\]
Since \(2\int_0^t (u^2 + u_x^2) d\tau \geq 0\), we obtain
\[
\|(u, \rho - 1)\|^2_{H^1 \times L^2} = \int_S (u^2 + u_x^2) dx + (\rho - 1)^2 dx \leq \|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}.
\]
The proof is complete.

Lemma 2.6 ([32]). (1) For all \(f \in H^1(S)\), we have
\[
\max_{x \in [0, 1]} f^2(x) \leq \frac{e + 1}{2(e - 1)} \|f\|^2_1,
\]
where \(\frac{e + 1}{2(e - 1)}\) is the best constant.

(2) For all \(f \in H^3(S)\), we have
\[
\max_{x \in [0, 1]} f^2(x) \leq c \|f\|^2_1,
\]
where the possible best constant \(c \in (1, \frac{13}{12}]\), and the best constant is \(\frac{e + 1}{2(e - 1)}\).

Lemma 2.7. If \(f \in H^3(S)\), then
\[
\max_{x \in [0, 1]} f^2_x(x) \leq \frac{1}{12} \|f\|^2_{H^2(S)}.
\]

Proof. From [32] Theorem 2.1, the Fourier expansion of \(f(x)\) can be written as
\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx).
\]
Then
\[
f_x(x) = -\sum_{n=1}^{\infty} (2n\pi a_n \sin(2\pi nx)).
\]
Using that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, we have
\[\max_{x \in S} f_x^2(x) \leq \left( \sum_{n=1}^{\infty} |2n\pi a_n| \right)^2 = \left( \sum_{n=1}^{\infty} (2n\pi)^2 |a_n| \frac{1}{2n\pi} \right)^2 \leq \sum_{n=1}^{\infty} ((2n\pi)^2 |a_n|)^2 \sum_{n=1}^{\infty} \frac{1}{(2n\pi)^2} \leq \frac{1}{24} \sum_{n=1}^{\infty} (16n^4 \pi^4 a_n^2) = \frac{1}{12} \int_{S} f_{xx} dx \leq \frac{1}{12} \| f \|^2_{H^2(S)}.
\]
The proof is complete. □

Applying the above lemmas and the method of characteristics, we may carry out the estimates along the characteristics $q(t, x)$ which captures $\sup_{x \in S} u_x(t, x)$ and $\inf_{x \in S} u_x(t, x)$.

**Lemma 2.8.** Let $\sigma \neq 0$ and $(u, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and $T$ be the maximal time of existence.

(1) When $\sigma > 0$, we have
\[\sup_{x \in S} u_x(t, x) \leq \| u_0x \|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \|\rho_0\|^2_{L^\infty} + C_1^2} \quad \text{where the constants are defined as follows:}
\]
\[C_1 = \sqrt{\frac{5(e+1)}{2(e-1)} + \left( \frac{1+\lambda^2}{2} + \frac{(e+1)(3-\sigma)}{e-1} \right) \|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}}, \quad \text{(2.8)}
\]
(2) When $\sigma < 0$, we have
\[\inf_{x \in S} u_x(t, x) \geq -\| u_0x \|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - C_2^2} \quad \text{where the constants are defined as follows:}
\]
\[C_2 = \sqrt{\frac{5(e+1)}{2(e-1)} + \left( \frac{A^2}{2} + \frac{(5-\sigma)(e+3-\sigma)}{2(e-1)} \right) \|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}}, \quad \text{(2.9)}
\]
Proof. The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s \geq 3$. Thus, we take $s = 3$ in the proof. Here we may assume that $u_0 \neq 0$. Otherwise, the results become trivial.

Differentiating the first equation in (2.2) with respect to $x$ and using the identity $-\partial_x^2 G * f = f - G * f$, we have
\[u_{tx} + \sigma uu_{xx} + \frac{\sigma u_x^2}{2} = \frac{1}{2} \rho^2 + \frac{3-\sigma}{2} u^2 + A\partial_x^2 G * u - G * \left( \frac{\sigma u_x^2}{2} + \frac{3-\sigma}{2} u^2 + \frac{1}{2} \rho^2 \right) - \lambda u_x.
\]
(2.10)
(1) When $\sigma > 0$, using Lemma 2.4 and the fact that
$$\sup_{x \in S} [v_x(t, x)] = - \inf_{x \in S} [-v_x(t, x)],$$
we can consider $\tilde{m}(t)$ and $\eta(t)$ as
$$\tilde{m}(t) := u_x(t, \eta(t)) = \sup_{x \in S} (u_x(t, x)), \quad t \in [0, T).$$  (2.11)
This gives
$$u_x(t, \eta(t)) = 0 \quad \text{a.e. on } t \in [0, T)$$  (2.12)
Take the trajectory $q(t, x)$ defined in (2.4). We know that $q(t, \cdot) : S \to S$ is a
diffeomorphism for every $t \in [0, T)$, then there exists $x_1(t) \in S$ such that
$$q(t, x_1(t)) = \eta(t), \quad t \in [0, T).$$  (2.13)
Let
$$\tilde{\zeta}(t) = \rho(t, q(t, x_1)), \quad t \in [0, T).$$  (2.14)
Then along the trajectory $q(t, x_1(t))$, equation (2.10) and the second equation of
(2.1) become
$$\tilde{m}'(t) = -\frac{\sigma}{2} \tilde{m}^2(t) - \lambda \tilde{m}(t) + \frac{1}{2} \tilde{\zeta}^2(t) + f(t, q(t, x_1))$$  (2.15)
where
$$f = \frac{3 - \sigma}{2} u^2 + A \partial_x^2 G * u - G * \left( \frac{\sigma}{2} u_x^2 + \frac{3 - \sigma}{2} u^2 \right) - \frac{1}{2} G * (\rho - 1)$$  (2.16)
Since $\partial_x^2 G * u = \partial_x G * \partial_x u$, we have
$$f \leq \frac{3 - \sigma}{2} u^2 + A \partial_x G * \partial_x u - G * \left( \frac{3 - \sigma}{2} u^2 \right) - \frac{1}{2} G * (\rho - 1)$$
Based on the following formulas:
$$\frac{3 - \sigma}{2} u_x^2 \leq \frac{3 - \sigma}{2}, \quad \frac{e + 1}{2(e - 1)} \|u\|_{H^1}^2,$$
$$A |\partial_x G * \partial_x u| \leq A \|G_x\|_{L^2} \|u_x\|_{L^2} \leq \frac{e + 1}{2(e - 1)} + \frac{1}{4} A^2 \|u_x\|_{L^2}^2,$$
$$|G * \left( \frac{\sigma}{2} u_x^2 \right)| \leq \|G_x\|_{L^\infty} \left( \frac{\sigma}{2} u_x^2 \right) \|L^1 \| \leq \frac{e + 1}{2(e - 1)} \cdot \frac{\sigma}{2} \|u_x\|_{L^2}^2,$$
$$|G * \left( \frac{3 - \sigma}{2} u^2 \right)| \leq \|G_x\|_{L^\infty} \frac{3 - \sigma}{2} u^2 \|L^1 \| \leq \frac{e + 1}{2(e - 1)} \cdot \frac{3 - \sigma}{2} \|u\|_{L^2}^2,$$
$$\frac{1}{2} |G * 1| \leq \frac{1}{4} \|G\|_{L^\infty} \leq \frac{e + 1}{4(e - 1)},$$
$$|G * (\rho - 1)| \leq \|G\|_{L^2} \|\rho - 1\|_{L^2} \leq \frac{e + 1}{2(e - 1)} + \frac{1}{4} \|\rho - 1\|_{L^2}^2,$$
\[
\frac{1}{2} |G \ast (\rho - 1)^2| \leq \frac{1}{2} \|G\|_{L^\infty} \| (\rho - 1)^2 \|_{L^1} \leq \frac{e + 1}{4(e - 1)} \| \rho - 1 \|_{L^2}^2,
\]

from the above inequalities and Lemma 2.5, we obtain an upper bound of \( f \),
\[
f \leq \frac{5(e + 1)}{4(e - 1)} + \frac{1}{4} \| \rho - 1 \|_{L^2}^2 + \left( \frac{A^2}{4} + \frac{(e + 1)|3 - \sigma|}{2(e - 1)} \right) \| u \|_{H^1}^2.
\]

Similarly, we obtain a lower bound of \( f \),
\[
-f \leq \frac{\sigma - 3}{2} u^2 + A|\partial_x G \ast \partial_x u| + |G \ast \left( \frac{\sigma}{2} u_x^2 + \frac{3 - \sigma}{2} u^2 \right)| + \frac{1}{2} |G \ast 1| + |G \ast (\rho - 1)| + \frac{1}{2} |G \ast (\rho - 1)^2|
\]
\[
\leq \frac{5(e + 1)}{4(e - 1)} \frac{e}{2(e - 1)} \| \rho - 1 \|_{L^2}^2 \left( \frac{A^2}{4} + \frac{(e + 1)(|\sigma| + 2|3 - \sigma|)}{4(e - 1)} \right) \| u \|_{H^1}^2.
\]

Combining (2.17) and (2.18), we obtain
\[
|f| \leq \frac{5(e + 1)}{4(e - 1)} \left( \frac{A^2}{4} + \frac{2e + (e + 1)(|\sigma| + 2|3 - \sigma|)}{4(e - 1)} \right) \| (u_0, \rho_0 - 1) \|_{H^1 \times L^2}^2. \tag{2.19}
\]

Since \( s \geq 3 \), it follows that \( u \in C^s_0(S) \) and
\[
\inf_{x \in S} u_x(t, x) \leq 0, \quad \sup_{x \in S} u_x(t, x) \geq 0, \quad t \in [0, T). \tag{2.20}
\]

Hence, we obtain
\[
\bar{m}(t) > 0 \quad \text{for} \quad t \in [0, T). \tag{2.21}
\]

From the second equation in (2.15), we have
\[
\bar{\zeta}(t) = \bar{\zeta}(0) e^{- \int_0^t \bar{m}(x) dx}, \tag{2.22}
\]
\[
|\rho(t, q(t, x_1))| = |\bar{\zeta}(t)| \leq |\bar{\zeta}(0)| \leq \| \rho \|_{L^\infty}.
\]

For any given \( x \in S \), we define
\[
P_1(t) = \bar{m}(t) - \| u_{0x} \|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\| \rho \|_{L^\infty}^2 + C_1^2}{\sigma}}.
\]

Notice that \( P_1(t) \) is a \( C^1 \)-function in \( [0, T) \) and satisfies
\[
P_1(0) = \bar{m}(0) - \| u_{0x} \|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\| \rho \|_{L^\infty}^2 + C_1^2}{\sigma}} \leq \bar{m}(0) - \| u_{0x} \|_{L^\infty} \leq 0.
\]

Next, we claim that
\[
P_1(t) \leq 0 \quad \text{for} \quad t \in [0, T). \tag{2.23}
\]

If not, then suppose that there is a \( t_0 \in [0, T) \) such that \( P_1(t_0) > 0 \). Define \( t_1 = \max \{ t < t_0 : P_1(t) = 0 \} \), then \( P_1(t_1) = 0, P_1'(t_1) \geq 0 \). That is,
\[
\bar{m}(t_1) = \| u_{0x} \|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\| \rho \|_{L^\infty}^2 + C_1^2}{\sigma}}, \quad \bar{m}'(t_1) = P_1'(t_1) \geq 0.
\]
On the other hand, we have

\[ m'(t_1) = -\frac{\sigma}{2} m^2(t_1) - \lambda m(t_1) + \frac{1}{2} \zeta^2(t_1) + f(t_1, q(t_1, x_1)) \]

\[ \leq -\frac{\sigma}{2} \left( \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|^2_{L^\infty} + C_1^2}{\sigma}} + \frac{\lambda}{\sigma} \right)^2 \]

\[ + \frac{\lambda^2}{2\sigma} + \frac{1}{2} \|\rho_0\|^2_{L^\infty} + \frac{1}{2} C_2^2 < 0. \]

This yields a contraction. Thus, \( P_1(t) \leq 0 \) for \( t \in [0, T) \). Since \( x \) is chosen arbitrarily, we obtain (2.6).

(2) When \( \sigma < 0 \), we have a finer estimate

\[ -f \leq -A(\partial_x G * \partial_x u) + G * \frac{3 - \sigma}{2} u^2 + \frac{1}{2} (G * 1) + G * (\rho - 1) + \frac{1}{2} G * (\rho - 1)^2 \]

\[ \leq A|\partial_x G * \partial_x u| + |G * \frac{3 - \sigma}{2} u^2| + \frac{1}{2} |G * 1| + |G * (\rho - 1)| + \frac{1}{2} G * (\rho - 1)^2 | \]

\[ \leq \frac{5(\varepsilon + 1)}{4(\varepsilon - 1)} + \left( \frac{A^2}{4} + \frac{(5 - \sigma)e + 3 - \sigma}{4(\varepsilon - 1)} \right) \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2} = \frac{1}{2} C_2^2. \]

We consider the functions \( m(t) \) and \( \xi(t) \) in Lemma 2.4

\[ m(t) := \inf_{x \in S} \{u_x(t, x)\}, \quad t \in [0, T) \quad (2.25) \]

Then \( u_{xx}(t, \xi(t)) = 0 \) a.e. on \( t \in [0, T) \). Choose \( x_2(t) \in S \), such that \( q(t, x_2(t)) = \xi(t), t \in [0, T) \). Let \( \zeta(t) = \rho(t, q(t, x_2)), t \in [0, T) \). Along the trajectory \( q(t, x_2) \), equation (2.10) and the second equation of (2.1) become

\[ m'(t) = -\frac{\sigma}{2} m^2(t) - \lambda m(t) + \frac{1}{2} \zeta^2(t) + f(t, q(t, x_2)) \]

\[ \zeta'(t) = -\zeta(t)m(t). \]

Let \( P_2(t) = m(t) + \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}, \quad \forall x \in \mathbb{R} \). Then \( P_2(t) \) is a \( C^1 \)-function in \([0, T)\) and satisfies

\[ P_2(0) = m(0) + \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} \geq m(0) + \|u_{0x}\|_{L^\infty} \geq 0. \]

Now we claim that

\[ P_2(t) \geq 0 \quad \text{for } t \in [0, T). \quad (2.26) \]

Assume that there is a \( t_0 \in [0, T) \) such that \( P_2(t_0) < 0 \). Define \( t_2 = \max \{ t < t_0 : P_2(t) = 0 \} \); then \( P_2(t_2) = 0, P'_2(t_2) \leq 0 \). That is,

\[ m(t_2) = -\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}, \quad m'(t_2) = P'_2(t_2) \leq 0. \]

In addition, we have

\[ m'(t_2) = -\frac{\sigma}{2} m^2(t_2) - \lambda m(t_2) + \frac{1}{2} \zeta^2(t_2) + f(t_2, q(t_2, x_2)) \]

\[ \geq -\frac{\sigma}{2} (-\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} + \frac{\lambda}{\sigma})^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2} C_2^2 > 0. \]

This is a contradiction. Then we have \( P_2(t) \geq 0 \) for \( t \in [0, T) \), since \( x \) is chosen arbitrarily. \( \square \)
Now, we present the following estimates for $\|\rho\|_{L^\infty(S)}$, if $\sigma u_x$ is bounded from below.

**Lemma 2.9**\(^{(3)}\). Let $\sigma \neq 0$ and $(u, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and $T$ be the maximal time of the existence. If there is a $M \geq 0$ such that $\inf_{(t,x) \in [0,T] \times S} \sigma u_x \geq -M$, then we have following two statements.

1. If $\sigma > 0$, then $\|\rho(t,.)\|_{L^\infty(S)} \leq \|\rho_0\|_{L^\infty(S)} e^{Mt/\sigma}$.
2. If $\sigma < 0$, then $\|\rho(t,.)\|_{L^\infty(S)} \leq \|\rho_0\|_{L^\infty(S)} e^{Nt}$,

where $N = \|u_0x\|_L^\infty + (C_2/\sqrt{-\sigma})$ and $C_2$ is given in (2.24).

**Proof.** The proof of Lemma 2.9 is similar to that of [5, Proposition 3.8], so we omit it here. $\square$

From the above results, we can get the necessary and sufficient conditions for the blow-up of solutions.

**Theorem 2.10** (Wave-breaking criterion for $\sigma \neq 0$). Let $\sigma \neq 0$ and $(u, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and $T$ be the maximal time of the existence. Then the solution blows up in finite time if and only if

$$\lim_{t \to T, x \in S} \inf \sigma u_x(t, x) = -\infty. \quad (2.27)$$

**Proof.** Assume that $T < \infty$ and (2.27) is not valid, then there is some positive number $M > 0$, such that $\sigma u_x(t, x) \geq -M$, $\forall (t, x) \in [0, T) \times S$. From the above lemmas, we have $|u_x(t, x)| \leq C$, where $C = C(A, M, \sigma, \lambda, \|u_0, \rho_0 - 1\|_{H^s \times H^{s-1}})$. Thus, Theorem 2.3 implies that the maximal existence time $T = \infty$, which contradicts the assumption $T < \infty$.

On the other hand, the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ with $s > 1/2$ implies that if (2.27) holds, the corresponding solution blows up in finite time. The proof is complete. $\square$

### 3. Blow-up scenarios

**Theorem 3.1.** Let $\sigma > 0$ and $(u, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and $T$ be the maximal time of existence. Assume that there is some $x_0 \in S$ such that $\rho_0(x_0) = 0, u_0x(x_0) = \inf_{x \in S} u_0x(x)$ and

$$\|(u_0, \rho_0 - 1)\|^2_{H^s \times L^2} \leq \frac{8e - 10 - \frac{\lambda^2}{2\sigma}}{(18e - 1) - \frac{19}{(18A^2 + 19)e - (18A^2 + 17) + (2|3 - \sigma| + \sigma)(e + 1)}} \cdot 4(e - 1) \quad (3.1)$$

then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a $T$ such that

$$0 < T \leq \frac{2}{\sigma - \lambda} + \left(12\sigma(e - 1)(1 + |u_0x(x_0)|)\right)$$

$$\div \left(\frac{\sigma(3(2e - 40 - 324e - 324A^2e + 324A^2 + 306) - 36\lambda^2(e - 1) - (2|3 - \sigma| + \sigma)(e - 1)\|(u_0, \rho_0 - 1)\|^2_{H^s \times L^2}}{\sigma - \lambda}\right) \quad (3.2)$$

and that $\liminf_{t \to T} (\inf_{x \in S} u_x(t, x)) = -\infty$. 


By assumption (3.1), we obtain 
\[ \lambda_T \] in time, that is,

Here we also consider Proof.

Choose \( \xi(0) = x_0 \) and then \( \rho_0(\xi(0)) = \rho_0(x_0) = 0 \). Then by (3.3), we derive

\[ \rho(t, \xi(t)) = 0, \quad \forall t \in [0, T). \] (3.4)

Evaluating the result at \( x = \xi(t) \) and combining (3.4) with \( u_{xx}(t, \xi(t)) = 0 \), we have

\[ m'(t) = -\frac{\sigma}{2} m^2(t) - \lambda m(t) + \frac{3 - \sigma}{2} u^2(t, \xi(t)) + A(G_x \ast u_x)(t, \xi(t)) \]

\[ - G * \left( \frac{\sigma}{2} u_x^2 + \frac{3 - \sigma}{2} u^2 + \frac{1}{2} \rho^2 \right)(t, \xi(t)) \]

\[ = -\frac{\sigma}{2} m^2(t) - \lambda m(t) + f(t, q(t, x_2)) \]

\[ = -\frac{\sigma}{2} (m(t) + \frac{\lambda}{\sigma})^2 + \frac{3\lambda}{2\sigma} + f(t, q(t, x_2)) \] (3.5)

We modify the estimates:

\[ A|G_x \ast u_x| \leq A\|G_x\|_{L^2}\|u_x\|_{L^2} \leq \frac{1}{18} \frac{e + 1}{2(e - 1)} + \frac{9}{2} A^2 \|u_x\|^2_{L^2}. \]

\[ |G \ast (\rho - 1)| \leq \|G\|_{L^2} \|\rho - 1\|_{L^2} \leq \frac{1}{18} \frac{e + 1}{2(e - 1)} + \frac{9}{2} \|\rho - 1\|^2_{L^2}. \]

Similarly, we obtain the upper bound of \( f \) as

\[ f \leq \frac{10 - 8e}{18(e - 1)} + \frac{(18A^2 + 19)e - (18A^2 + 17) + (2|3 - \sigma| + \sigma)(e + 1)}{4(e - 1)} \]

\[ \times \left[ \|u_0, \rho_0 - 1\|_{H^1 \times L^2} \right]^2 := -C_3. \]

By assumption (3.1), we obtain \( \frac{\lambda^2}{2} - C_3 < 0 \) and

\[ m'(t) \leq -\frac{\sigma}{2} (m(t) + \frac{\lambda}{\sigma})^2 + \frac{3\lambda}{2\sigma} - C_3 \leq \frac{\lambda^2}{2\sigma} - C_3 < 0, \quad t \in [0, T). \] (3.6)

So \( m(t) \) is strictly decreasing in \( [0, T) \). If the solution \( (u, \rho) \) of (2.1) exists globally in time, that is, \( T = \infty \), we will show that it leads to a contradiction.

Let \( t_1 = \frac{2\sigma(1 + |u_0(x_0)|)}{2\sigma C_3 - \lambda^2} \). Integrating (3.6) over \( [0, t_1] \) gives

\[ m(t_1) = m(0) + \int_0^{t_1} m'(t)dt \leq |u_0(x_0)| + (\frac{\lambda^2}{2\sigma} - C_3)t_1 = -1. \] (3.7)

For \( t \in [t_1, T) \), we have \( m(t) \leq m(t_1) \leq -1 \). From (3.6), we have

\[ m'(t) \leq -\frac{\sigma}{2} (m(t) + \frac{\lambda}{\sigma})^2. \] (3.8)

Integrating over \([t_1, T)\), by (3.7), yields

\[ -\frac{1}{m(t) + \frac{\lambda}{\sigma} + 1} \leq -\frac{1}{m(t_1) + \frac{\lambda}{\sigma} + 1} \leq -\frac{\sigma}{2} (t - t_1), \quad t \in [t_1, T), \]
Theorem 3.2. Let $\sigma \neq 0$ and $(\mu, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and $T$ be the maximal time of the existence.

(1) When $\sigma > 0$, assume that there is an $x_0 \in S$ such that $\rho_0(x_0) = 0$, $u_{0x}(x_0) = \inf_{x \in S} u_{0x}(x)$ and $u_{0x}(x_0) < -\sqrt{\frac{\lambda^2}{\sigma^2} + \frac{C_2^2}{\sigma}} - \frac{\lambda}{\sigma}$, where $C_1$ is defined in (2.8). Then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a $T_1$ such that

$$0 < T_1 \leq -\frac{2(\lambda + \sigma u_{0x}(x_0))}{(\lambda + \sigma u_{0x}(x_0))^2 - (\lambda^2 + \sigma C_2^2)},$$

and

$$\liminf_{t \to T_1^-} \left\{ \inf_{x \in S} u_x(t, x) \right\} = -\infty.$$

(2) When $\sigma < 0$, assume that there is some $x_0 \in S$ such that $u_{0x}(x_0) > \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}$, where $C_2$ is defined in (2.9). Then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a $T_2$ such that

$$0 < T_2 \leq -\frac{2(\lambda + \sigma u_{0x}(x_0))}{(\lambda + \sigma u_{0x}(x_0))^2 - (\lambda^2 - \sigma C_2^2)},$$

and

$$\liminf_{t \to T_2^-} \left\{ \sup_{x \in S} u_x(t, x) \right\} = \infty.$$

Proof. (1) When $\sigma > 0$, using the upper bound of $f$ in (2.17) and (3.4), we have

$$m'(t) \leq -\frac{\sigma}{2} \left( m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2} C_1^2, \quad t \in [0, T).$$

By the assumption $m(0) = u_{0x}(x_0) < -\sqrt{\frac{\lambda^2}{\sigma^2} + \frac{C_2^2}{\sigma}} - \frac{\lambda}{\sigma}$, we have that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{\sigma(u_{0x}(x_0) + \frac{\lambda}{\sigma})^2} \left( \frac{\lambda^2}{2\sigma} + \frac{1}{2} C_1^2 \right) \in (0, \frac{1}{2}).$$

Since $m(t) < m(0) = u_{0x}(x_0) < -\frac{\lambda}{\sigma}$, it holds

$$m'(t) \leq -\frac{\sigma}{2} \left( m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2} C_1^2 \leq -\delta \sigma \left( m(t) + \frac{\lambda}{\sigma} \right)^2.$$

By a similar argument as in the proof of Theorem 3.1 we obtain

$$m(t) \leq \frac{\lambda + \sigma u_{0x}(x_0)}{\sigma + (\delta \sigma^2 u_{0x}(x_0) + \lambda \delta \sigma)t} - \frac{\lambda}{\sigma} \to -\infty \quad \text{as} \; t \to -\frac{1}{\lambda \delta + \delta \sigma u_{0x}(x_0)}.$$

Thus, we have $0 < T_1 \leq -\frac{1}{\lambda \delta + \delta \sigma u_{0x}(x_0)}$. 

So, $T \leq t_1 + \frac{2}{\sigma},$ which is a contradiction to $T = \infty$. Consequently, the proof is complete. \qed
Proof. We assume that \( q \) take the trajectory \( q(t, x_1) \) with \( x_1 \) defined in (2.13), then
\[
\ddot{m}(t) = -\frac{\sigma}{2} \dot{m}^2(t) - \lambda \dot{m}(t) + \frac{1}{2} \rho^2(t, \eta(t)) + f(t, q(t, x_1))
\]
\[
\geq -\frac{\sigma}{2} \left( \dot{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + f(t, q(t, x_1)).
\]
(3.9)

From the lower bound of \( f \) in (2.24), we obtain
\[
\ddot{m}(t) \geq -\frac{\sigma}{2} \left( \dot{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2} C_2^2, \quad t \in [0, T).
\]

By the assumption \( \ddot{m}(0) \geq u_{0x}(x_0) > \sqrt{\frac{\lambda^2}{\sigma^2} - C_2^2} - \frac{\lambda}{\sigma} \), we have that \( \ddot{m}(0) > 0 \) and \( \ddot{m}(t) \) is strictly increasing over \([0, T)\).

Set
\[
\theta = \frac{(\sigma u_{0x}(x_0) + \lambda)^2 - (\lambda^2 - \sigma C_2^2)}{2(\sigma u_{0x}(x_0) + \lambda)^2} \in (0, \frac{1}{2}).
\]

Since \( \ddot{m}(t) > \ddot{m}(0) \geq u_{0x}(x_0) > -\frac{\lambda}{\sigma} \), we obtain
\[
\ddot{m}(t) \geq -\frac{\sigma}{2} \left( \dot{m}(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2} C_2^2 \geq -\theta \sigma \left( \dot{m}(t) + \frac{\lambda}{\sigma} \right)^2.
\]

Similarly, we obtain
\[
\ddot{m}(t) \geq \frac{\lambda + \sigma u_{0x}(x_0)}{\sigma + (\theta \sigma^2 u_{0x}(x_0) + \lambda \theta \sigma) t} - \frac{\lambda}{\sigma} \rightarrow -\infty \quad \text{as} \ t \rightarrow -\frac{1}{\lambda \theta + \theta \sigma^2 u_{0x}(x_0)}.
\]

Therefore, \( 0 < T_2 \leq \frac{1}{\lambda \theta + \lambda \sigma^2 u_{0x}(x_0)} \). The proof is complete. \( \square \)

Remark. If \( \sigma = 3 \) and \( A = 0 \), then all solutions of system (2.1) with initial data \((u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)\) with \( s > 3/2 \) satisfying \( u_0 \neq 0 \) and \( \rho_0(x_0) = 0 \) for some \( x_0 \in S \), blow up in finite time.

4. Blow-up rate

**Theorem 4.1.** Let \( \sigma \neq 0 \). If \( T < \infty \) is the blow-up time of the solution \((u, \rho)\) to (2.1) with initial data \((u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)\), \( s > 3/2 \) satisfying the assumptions of Theorem 3.3. Then
\[
\lim_{t \to T^-} \left\{ \inf_{x \in S} u_x(t, x)(T - t) \right\} = -\frac{2}{\sigma}, \quad \sigma > 0,
\]
\[
\lim_{t \to T^-} \left\{ \sup_{x \in S} u_x(t, x)(T - t) \right\} = -\frac{2}{\sigma}, \quad \sigma < 0.
\]

**Proof.** We assume that \( s = 3 \) to prove the theorem.

(1) when \( \sigma > 0 \), from (3.5) we have
\[
\ddot{m}(t) = \frac{\sigma}{2} \left( m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + f(t, q(t, x)).
\]

From (2.19), note that
\[
M = \frac{5(e + 1)}{4(e - 1)} + \left( \frac{A^2}{2} + 2e + (e + 1)(|\sigma| + 2|3 - \sigma|) \right) \|(u_0, \rho_0 - 1)\|^2_{H^2 \times L^2},
\]
(4.4)

Then
\[
-\frac{\sigma}{2} \left( m(t) + \frac{\lambda}{\sigma} \right)^2 - \frac{\lambda^2}{2\sigma} - M \leq \ddot{m}(t) \leq -\frac{\sigma}{2} \left( m(t) + \frac{\lambda}{\sigma} \right)^2 + \frac{\lambda^2}{2\sigma} + M.
\]
(4.5)
Choose $\varepsilon \in (0, \frac{\sigma}{2})$, since $\lim_{t \to T^{-}}(m(t) + \frac{\lambda}{\sigma}) = -\infty$, there is some $t_{0} \in (0, T)$, such that $m(t_{0}) + \frac{\lambda}{\sigma} < 0$ and $(m(t_{0}) + \frac{\lambda}{\sigma})^{2} > \frac{1}{\varepsilon^{2}}(\frac{\lambda^2}{2\sigma} + M)$. Since $m$ is locally Lipschitz, it follows that $m$ is absolutely continuous. We deduce that $m$ is decreasing on $[t_{0}, T)$ and

$$
(m(t) + \frac{\lambda}{\sigma})^{2} > \frac{1}{\varepsilon^{2}}\left(\frac{\lambda^2}{2\sigma} + M\right), \quad t \in [t_{0}, T).
$$

(4.6)

Combining (4.5) with (4.6), we have

$$
\frac{\sigma}{2} - \varepsilon \leq \frac{d}{dt}\left(\frac{1}{m(t) + \frac{\lambda}{\sigma}}\right) \leq \frac{\sigma}{2} + \varepsilon, \quad t \in [t_{0}, T).
$$

(4.7)

Integrating over $(t, T)$ with $t \in [t_{0}, T)$ and noticing that $\lim_{t \to T^{-}}(m(t) + \frac{\lambda}{\sigma}) = -\infty$, we obtain

$$
\left(\frac{\sigma}{2} - \varepsilon\right)(T - t) \leq -\frac{1}{m(t) + \frac{\lambda}{\sigma}} \leq \left(\frac{\sigma}{2} + \varepsilon\right)(T - t).
$$

Since $\varepsilon \in (0, \frac{\sigma}{2})$ is arbitrary, in view of the definition of $m(t)$, we have

$$
\lim_{t \to T^{-}}\{m(t)(T - t) + \frac{\lambda}{\sigma}(T - t)\} = -\frac{2}{\sigma};
$$

that is, $\lim_{t \to T^{-}}\{\inf_{x \in S} u_{x}(t, x)(T - t)\} = -\frac{2}{\sigma}$.

(2) When $\sigma < 0$, we consider the functions $\bar{m}(t)$ and $\eta(t)$ as defined in (2.11).

From (3.9) and (4.4), we have

$$
\bar{m}'(t) \geq -\frac{2}{\sigma}\left(\bar{m}(t) + \frac{\lambda}{\sigma}\right)^{2} + \frac{\lambda^{2}}{2\sigma} - M.
$$

Because $\bar{m}(t) \to \infty$ as $t \to T^{-}$, there is a $t_{1} \in (0, T)$, such that $\bar{m}(t_{1}) > \sqrt{\frac{\lambda^{2}}{2\sigma} - \frac{2M}{\sigma}} - \frac{\lambda}{\sigma} > 0$. Thus, we have that $\bar{m}'(t) > 0$ and $\bar{m}(t)$ is strictly increasing on $[t_{1}, T)$, and

$$
\bar{m}(t) > \bar{m}(t_{1}) > 0.
$$

(4.8)

By the transport equation for $\rho$, we have

$$
\frac{d\rho(t, \eta(t))}{dt} = -\bar{m}(t)\rho(t, \eta(t)).
$$

Then

$$
\rho(t, \eta(t)) = \rho(t_{1}, \eta(t_{1}))e^{-\int_{t_{1}}^{t} \bar{m}(\tau)d\tau}, \quad t \in [t_{1}, T).
$$

(4.9)

Combining (4.8) with (4.9) yields

$$
\rho^{2}(t, \eta(t)) \leq \rho^{2}(t_{1}, \eta(t_{1})), \quad t \in [t_{1}, T)
$$

(4.10)

From (3.9) and (4.10), we have

$$
-\frac{\sigma}{2}\left(\bar{m}(t) + \frac{\lambda}{\sigma}\right)^{2} - \frac{\lambda^{2}}{2\sigma}\rho^{2}(t_{1}, \eta(t_{1})) - M
$$

$$
\leq \bar{m}' \leq -\frac{\sigma}{2}\left(\bar{m}(t) + \frac{\lambda}{\sigma}\right)^{2} - \frac{\lambda^{2}}{2\sigma} + \frac{1}{\varepsilon^{2}}\rho^{2}(t_{1}, \eta(t_{1}))) + M.
$$

(4.11)

Choose $\varepsilon \in (0, -\frac{\sigma}{2})$, and pick a $t_{2} \in [t_{1}, T)$, such that

$$
\left(\bar{m}(t_{2}) + \frac{\lambda}{\sigma}\right)^{2} \geq \frac{1}{\varepsilon^{2}}\left(\frac{\lambda^{2}}{2\sigma} + M\right) - \frac{\lambda^{2}}{2\sigma}.
$$

(4.12)

From (4.11) and (4.12), we have

$$
\frac{\sigma}{2} - \varepsilon \leq \frac{d}{dt}\left(\frac{1}{\bar{m}(t) + \frac{\lambda}{\sigma}}\right) \leq \frac{\sigma}{2} + \varepsilon, \quad t \in [t_{2}, T).
$$

(4.13)
Integrating (4.13) over \([t, T]\) with \(t \in [t_2, T]\) and \(\lim_{t \to T^-} m(t) = \infty\) gives
\[
(\sigma - \varepsilon)(T - t) \leq -\frac{1}{\bar{m}(t)} \leq (\sigma + \varepsilon)(T - t).
\]

Since \(\varepsilon \in (0, -\frac{\sigma}{2})\) is arbitrary, in view of the definition of \(\bar{m}(t)\), we have
\[
\lim_{t \to T^-} \{\sup_{x \in S} u_x(t, x)(T - t)\} = -\frac{2}{\sigma}.
\]

This completes the proof of Theorem 4.1. \(\square\)

5. Existence of a global solution

In this section, we provide a sufficient condition for the global solution of system (2.1) in the case when \(0 < \sigma < 2\).

Lemma 5.1. Let \(0 < \sigma < 2\) and \((u, \rho)\) be the solution of (2.1) with initial data \((u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S), s > 3/2,\) and \(T\) be the maximal time of existence. Assume that \(\inf_{x \in S} \rho_0(x) > 0\).

(1) When \(0 < \sigma \leq 1\), it holds
\[
|\inf_{x \in S} u_x(t, x)| \leq \frac{1}{\inf_{x \in S} \rho_0(x)} C_4 e^{C_4 t},
\]
\[
|\sup_{x \in S} u_x(t, x)| \leq \frac{1}{\inf_{x \in S} \rho_0(x)} C_4 e^{\frac{\sigma}{2} t}.
\]

(2) When \(1 < \sigma < 2\), it holds
\[
|\inf_{x \in S} u_x(t, x)| \leq \frac{1}{\inf_{x \in S} \rho_0(x)} C_4 e^{\frac{\sigma}{2} t},
\]
\[
|\sup_{x \in S} u_x(t, x)| \leq \frac{1}{\inf_{x \in S} \rho_0(x)} C_4 e^{C_4 t},
\]
where constants \(C_3\) and \(C_4\) are defined as follows:
\[
C_3 = 1 + \frac{5(e + 1)}{4(e - 1)} \left(1 + \frac{A^2}{4} + \frac{2e + (e + 1)(|\sigma| + 2|3 - \sigma|)}{4(e - 1)}\right)\|u_0\|_{\dot{H}^{1/2}}^2 + \|\rho_0 - 1\|_{H^{3/2}}^2,
\]
\[
C_4 = 1 + \|u_0 x\|_{L^\infty}^2 + \|\rho_0\|_{L^\infty}^2.
\]

Proof. A density argument indicates that it suffices to prove the desired results for \(s \geq 3\). Since \(s \geq 3\), we have \(u \in C_0^1(S)\) and
\[
\inf_{x \in S} u_x(t, x) < 0, \quad \sup_{x \in S} u_x(t, x) > 0, \quad t \in [0, T).
\]

(1) First we will derive the estimate for \(|\inf_{x \in S} u_x(t, x)|\). Define \(m(t)\) and \(\xi(t)\) as in (2.25), and consider along the characteristics \(q(t, x_2(t))\). Then
\[
m(t) \leq 0 \quad \text{for} \quad t \in [0, T).
\]
Let \(\zeta(t) = \rho(t, \xi(t))\) and evaluating (2.10) and the second equation of system (2.1) at \((t, \xi(t))\), we have
\[
m'(t) = -\frac{\sigma}{2} m^2(t) - \lambda m(t) + \frac{1}{2} \xi^2(t) + f(t, q(t, x_2(t))
\]
\[
\zeta'(t) = -\zeta(t) m(t),
\]
where $f$ is defined in (2.10). The second equation above implies that $\zeta(t)$ and $\zeta(0)$ are of the same sign.

Next we construct a Lyapunov function for our system as in [13]. Since here we have a free parameter $\sigma$, we could not find a uniform Lyapunov function. Instead, we split the case $0 < \sigma \leq 1$ and the case $1 < \sigma < 2$. From the assumption of the theorem, we know that $\zeta(0) = \rho(0, \xi(0)) > 0$. When $0 < \sigma \leq 1$, we define the Lyapunov function

$$\omega_1(t) = \zeta(0)\zeta(t) + \frac{\zeta(0)}{\zeta(t)}(1 + m^2(t)),$$

which is always positive for $t \in [0, T)$. Differentiating $\omega_1(t)$ and using (5.2) gives

$$\omega_1'(t) = \zeta(0)\zeta'(t) - \frac{\zeta(0)}{\zeta(t)}(1 + m^2(t))\zeta'(t) + \frac{2\zeta(0)}{\zeta(t)}m(t)m'(t)$$

$$= -\zeta(0)\zeta(t)m(t) - \frac{\zeta(0)}{\zeta(t)}(1 + m^2(t))(-\zeta(t)m(t))$$

$$+ \frac{2\zeta(0)}{\zeta(t)}m(t)(-\frac{\sigma}{2}m^2(t) - \lambda m(t) + \frac{1}{2}\zeta^2(t) + f)$$

$$= (1 - \sigma)\zeta(0)\zeta(t) m^3(t) + \zeta(0)\zeta(t) m(t) - 2\lambda\zeta(0)\zeta(t) m^2(t) + \frac{2\zeta(0)}{\zeta(t)}m(t)f$$

$$\leq \frac{\zeta(0)}{\zeta(t)}m(t) + \frac{2\zeta(0)}{\zeta(t)}m(t)f$$

$$\leq \frac{\zeta(0)}{\zeta(t)}(1 + m^2(t))(1 + |f|) \leq C_3\omega_1(t),$$

where

$$C_3 = 1 + \frac{5(e + 1)}{4(e - 1)} + \left(\frac{A^2}{4} + \frac{2e + (e + 1)(|\sigma| + 2|3 - \sigma|)}{4(e - 1)}\right)\|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}.$$

This gives

$$\omega_1(t) \leq \omega_1(0)e^{C_3t} = (\zeta^2(0) + 1 + m^2(0))e^{C_3t}$$

$$\leq (1 + \|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2)e^{C_3t} =: C_4e^{C_3t},$$

(5.4)

where $C_4 = 1 + \|u_0\|_{H^1}^2 + \|\rho_0\|_{L^2}^2$.

Recalling that $\zeta(t)$ and $\zeta(0)$ are of the same sign, the definition of $\omega_1(t)$ implies $\zeta(t)\zeta(0) \leq \omega_1(t)$ and $|\zeta(0)||m(t)| \leq \omega_1(t)$. By (5.4), we obtain

$$|\inf_{x \in S} u_x(t, x)| = |m(t)| \leq \frac{\omega_1(t)}{|\zeta(0)|} \leq \frac{1}{\inf_{x \in S} \rho_0(x)}C_4e^{C_3t}, \quad \text{for } t \in [0, T).$$

When $1 < \sigma < 2$, we define the Lyapunov function

$$\omega_2(t) = \zeta^\sigma(0)\frac{\zeta^2(t) + 1 + m^2(t)}{\zeta^\sigma(t)}.$$

(5.5)

Then

$$\omega_2'(t) = \frac{2\zeta^\sigma(0)}{\zeta^\sigma(t)}m(t)(\frac{\sigma - 1}{2}\zeta^2(t) - \lambda m(t) + f + \frac{\sigma}{2})$$

$$\leq \frac{\zeta^\sigma(0)}{\zeta^\sigma(t)}(1 + m^2(t))(|f| + \frac{\sigma}{2}) \leq \frac{\zeta^\sigma(0)}{\zeta^\sigma(t)}(1 + m^2(t))(|f| + 1) \leq C_3\omega_2(t).$$

(5.6)
Thus, we obtain
\[
\omega_2(t) \leq \omega_2(0) e^{C_3 t} = (\zeta^2(0) + 1 + m^2(0)) e^{C_3 t} \\
\leq (1 + \|w_0\|_L^2 + \|\rho_0\|_L^2) e^{C_3 t} = C_4 e^{C_3 t}.
\]
Applying Young’s inequality \(ab \leq \frac{a^p}{p} + \frac{b^q}{q}\) to (5.5) with \(p = \frac{2}{\sigma}\) and \(q = \frac{2}{2-\sigma}\) yields
\[
\frac{\omega_2(t)}{\zeta^\sigma(0)} = \left(\frac{\zeta^2(t)}{\zeta^\sigma(0)}\right)^{\frac{1}{2}} + \left(\frac{1 + m^2}{\zeta^\sigma(0)}\right)^{\frac{1}{2}} \\
\geq \frac{\sigma}{2} \left(\frac{\zeta^2(t)}{\zeta^\sigma(0)}\right)^{\frac{1}{2}} + \frac{2 - \sigma}{2} \left(\frac{1 + m^2}{\zeta^\sigma(0)}\right)^{\frac{1}{2}} \\
\geq (1 + m^2)^{\frac{2-\sigma}{2}} \geq |m(t)|^{2-\sigma}.
\]
So we have
\[
\inf_{x \in S} u_x(t, x) \leq \left(\frac{\omega_2(t)}{\zeta^\sigma(0)}\right)^{\frac{1}{2-\sigma}} \leq \frac{1}{\inf_{x \in S} \rho_0^{\frac{2-\sigma}{2}}(x)} C_4 \frac{1}{\zeta^\sigma(0)} e^{C_3 t}.
\]
(2) Now, we estimate \(|\sup_{x \in S} u_x(t, x)|\). Consider \(\bar{m}(t), \eta(t), q(t, x_1)\) as in (2.11) and (2.13), and
\[
\bar{m}'(t) = -\sigma \bar{m}^2(t) - \lambda \bar{m}(t) + \frac{1}{2} \zeta^2(t) + f(t, \eta(t, x_1)) \\
\bar{\zeta}'(t) = -\bar{\zeta}(t) \bar{m}(t)
\]
for \(t \in [0, T]\), where \(\bar{\zeta}(t) = \rho(t, \eta(t))\). We know that
\[
\bar{m}(t) \geq 0 \quad \text{for} \quad t \in [0, T].
\]
When \(0 < \sigma \leq 1\), we define the Lyapunov function
\[
\bar{\omega}_1(t) = \bar{\zeta}^\sigma(0) \frac{\bar{\zeta}^2(t) + 1 + \bar{m}^2(t)}{\bar{\zeta}^\sigma(t)}.
\]
Then from (5.6) and (5.8), we have \(\bar{\omega}_1'(t) \leq C_3 \bar{\omega}_1(t)\), then \(\bar{\omega}_1(t) \leq C_4 e^{C_3 t}\). Hence, by a similar argument as before, we obtain
\[
\frac{\bar{\omega}_1(t)}{\bar{\zeta}^\sigma(0)} \geq |\bar{m}(t)|^{2-\sigma}.
\]
Then
\[
\sup_{x \in S} u_x(t, x) \leq \left(\frac{\bar{\omega}_1(t)}{\bar{\zeta}^\sigma(0)}\right)^{\frac{1}{2-\sigma}} \leq \frac{1}{\inf_{x \in S} \rho_0^{\frac{2-\sigma}{2}}(x)} C_4 \frac{1}{\bar{\zeta}^\sigma(0)} e^{C_3 t}, \quad t \in [0, T].
\]
When \(1 < \sigma < 2\), consider the Lyapunov function
\[
\bar{\omega}_2(t) = \bar{\zeta}(0) \bar{\zeta}(t) + \bar{\zeta}(0) (1 + \bar{m}^2(t))
\]
From (5.3) and (5.8), we have \(\bar{\omega}_2'(t) \leq C_3 \bar{\omega}_2(t)\) and \(\bar{\omega}_2(t) \leq C_4 e^{C_3 t}\). Therefore,
\[
\sup_{x \in S} u_x(t, x) \leq \left(\frac{\bar{\omega}_2(t)}{\bar{\zeta}(0)}\right)^{\frac{1}{2-\sigma}} \leq \frac{1}{\inf_{x \in S} \rho_0(x)} C_4 e^{C_3 t}, \quad t \in [0, T].
\]
The proof is complete.
\[\square\]
Theorem 5.2. Let $0 < \sigma < 2$ and $(u, \rho)$ be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^\sigma(S) \times H^{\sigma-1}(S)$, $s > 3/2$, and $T$ be the maximal time of existence. If $\inf_{x \in S} \rho_0(x) > 0$, then $T = +\infty$ and the solution $(u, \rho)$ is global.

Proof. Assume on the contrary that $T < +\infty$ and the solution blows up in finite time. It then follows from Theorem 2.3 that

$$\int_0^T \|u_x(t)\|_{L^\infty} \, dt = \infty. \quad (5.11)$$

However, from the assumptions of the theorem and Lemma 5.1 we have $|u_x(t, x)| < \infty$ for all $(t, x) \in [0, T) \times S$. This is a contradiction to (5.11). So $T = +\infty$, and it means that the solution $(u, \rho)$ is global. \qed

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