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CAUCHY PROBLEM FOR A GENERALIZED WEAKLY DISSIPATIVE PERIODIC TWO-COMPONENT CAMASSA-HOLM SYSTEM

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Abstract. In this article, we study a generalized weakly dissipative periodic two-component Camassa-Holm system. We show that this system can exhibit the wave-breaking phenomenon and determine the exact blow-up rate of strong solution to the system. In addition, we establish a sufficient condition for having a global solution.

1. Introduction

In recent years, the Camassa-Holm equation [4],

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{R} \tag{1.1}$$

which models the propagation of shallow water waves has attracted considerable attention from a large number of researchers, and two remarkable properties of (1.1) were found. The first one is that the equation possesses the solutions in the form of peaked solitons or 'peakons' [4, 8]. The peakon $u(t,x) = ce^{-|x-ct|}$, $c \neq 0$ is smooth except at its crest and the tallest among all waves of the fixed energy. It is a feature observed for the traveling waves of largest amplitude which solves the governing equations for water waves [9, 10, 29, 33]. The other remarkable property is that the equation has breaking waves [4, 11]; that is, the solution remains bounded while its slope becomes unbounded in finite time. After wave breaking the solutions can be continued uniquely as either global conservative [2] or global dissipative solutions [3].

The Camassa-Holm equation also admits many integrable multicomponent generalizations. The most popular one is

$$\begin{gathered} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0 \\ \rho_t + (\rho u)_x = 0 \\ m = u - u_{xx} \end{gathered} \tag{1.2}$$

Notice that the C-H equation can be obtained via the obvious reduction $\rho \equiv 0$ and $A = 0$. System (1.2) was derived in [27], where $\rho(t,x)$ is related to the free surface elevation from the equilibrium (or scalar density), and $A \geq 0$ characterizes

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a linear underlying shear flow. Recently, Constantin-Ivanov [12] and Ivanov [23] established a rigorous justification of the derivation of system (1.2). Mathematical properties of the system have been also studied further in many works, for example [1, 6, 7, 14, 15, 19, 22, 26, 28]. Chen, Liu and Zhang [6] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher, Lechtenfeld, and Yin [14] investigated local well-posedness for the two-component Camassa-Holm system with initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s \geq 2$ by applying Kato's theory [24] and provided some precise blow-up scenarios for strong solutions to the system. The local well-posedness is improved by Gui and Liu [20] to the Besov Spaces (especially in the Sobolev space $H^s \times H^{s-1}$ with $s > 3/2$), and they showed that the finite time blow-up is determined by either the slope of the first component u or the slope of the second component ρ [8, 14]. The blow-up criterion is made more precise in [25] where Liu and Zhang showed that the wave breaking in finite time only depends on the slope of u. This blow-up criterion is improved to the lowest Sobolev spaces $H^s \times H^{s-1}$ with $s > 3/2$ [19].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. We are interested in the effect of the weakly dissipative term on the two-component Camassa-Holm equation. Wu, Escher and Yin have investigated the blow-up phenomena, the blow-up rate of the strong solutions of the weakly dissipative CH equation [31] and DP equation [30]. Inspired by the above results, in this paper, we investigate the following generalized weakly dissipative two-component Camassa-Holm system

$$\begin{gathered} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \lambda(u - u_{xx}) + \rho\rho_x = 0, \\ t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), \rho(t,x) = \rho(t,x+1), \quad t \geq 0,\ x \in \mathbb{R}, \end{gathered} \tag{1.3}$$

or equivalently,

$$\begin{gathered} m_t - Au_x + \sigma(um_x + 2u_x m) + 3(1-\sigma)uu_x + \lambda m + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \\ m = u - u_{xx}, \end{gathered} \tag{1.4}$$

where $\lambda m = \lambda(I - \partial_{xx})u$ is the weakly dissipative term, $\lambda \geq 0$ and A are constants, and σ is a new free parameter. When $A = 0$, $\lambda = 0$ and $\rho = 1$, Guan and Yin have obtained a new result of the existence of the strong solution and some new blow-up results [16]. Meanwhile, they have proved the global existence of the weak solution about the two-component CH equation [17]. Henry investigates the infinite propagation speed of the solution for a two-component CH equation [21].

Similar to [12, 14], we can use the method of Besov spaces together with the transport equation theory to show that system (1.4) is locally well-posedness in $H^s \times H^{s-1}$ with $s > 3/2$. The two equations for u and ρ are of a transport structure $\partial_t f + v\partial_x f = g$. It is well known that most of the available estimates require v to have some level of regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as v belongs to $L^1(0, T; Lip)$. More specially, u and ρ are "transported" along directions of σu and u respectively. Then, the solution can be estimated in a Gronwall way involving $\|u_x\|_{L^\infty}$. Hence, one can use these estimates to derive a criterion which says if $\int_0^T \|u_x(\tau)\|_{L^\infty} d\tau < \infty$, then solutions can be extended further in time. Compared with the result in [5], we find that the equation (1.4) has the same blow-up rate when the blow-up occurs. This fact shows that the blow-up rate of equation (1.4) is not affected by the weakly dissipative term. But the occurrence of blow-up of equation (1.4) is affected by the dissipative parameter λ.

The basic elementary framework is as follows. Section 2 gives the local well-posedness of system (1.4) and a wave-breaking criterion, which implies that the wave breaking only depends on the slope of u, not the slope of ρ. Section 3 improves the blow-up criterion with a more precise conditions. Section 4 determine the exact blow-up rate of strong solutions of system (1.4). Finally, section 5 provides a sufficient condition for global solutions.

Notation. Throughout this paper, we identity periodic function spaces over the unit S in $\mathbb{R}^2$, i.e. $S = R/Z$.

2. Formation of singularities for $\sigma \neq 0$

We consider the following generalized weakly dissipative two - component Camassa - Holm system:

$$
\begin{gathered}
u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \lambda(u - u_{xx}) + \rho\rho_x = 0, \\
t > 0,\ x \in \mathbb{R}, \\
\rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\
u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x), \\
u(t,x) = u(t,x+1), \quad \rho(t,x) = \rho(t,x+1),
\end{gathered} \tag{2.1}
$$

where $\lambda \geq 0$ and A are constants, and σ is a new free parameter.

System (2.1) can be written in the "transport" form

$$
\begin{gathered}
u_t + \sigma uu_x = -\partial_x G * (-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2) - \lambda u \quad t > 0,\ x \in \mathbb{R} \\
\rho_t + (\rho u)_x = 0 \quad t > 0,\ x \in \mathbb{R} \\
u(0,x) = u_0(x), \rho(0,x) = \rho_0(x), \quad x \in \mathbb{R} \\
u(t,x) = u(t,x+1), \rho(t,x) = \rho(t,x+1), \quad t \geq 0,\ x \in \mathbb{R}
\end{gathered} \tag{2.2}
$$

where $G(x) := \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(1/2)}$, $x \in S$, and $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(S)$.

Applying the transport equation theory combined with the method of Besov spaces, one may follow the similar argument as in [20] to obtain the following local well-posedness result for the system (2.1). The proof is very similar to that of [20, Theorem 1.1] and is omitted.

Theorem 2.1. *Assume $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$ with $s > 3/2$, then there exist a maximal time $T = T(\|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}) > 0$ and a unique solution $(u, \rho - 1)$ of equation (2.1) in $C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$ with initial data (u_0, ρ_0). Moreover, the solution depends continuously on the initial data, and T is independent of s.*

Lemma 2.2 ([26]). *Let $0 < s < 1$. Suppose that $f_0 \in H^s$, $g \in L^1([0,T];H^s)$, $v, v_x \in L^1([0,T];L^\infty)$, and that $f \in L^\infty([0,T];H^s) \cap C([0,T);S')$ solves the one-dimensional linear transport equation*

$$
\begin{gathered}
\partial_t f + v\partial_x f = g \\
f(0,x) = f_0(x)
\end{gathered}
$$

then $f \in C([0,T];H^s)$. More precisely, there exists a constant C depending only on s such that

$$
\|f(t)\|_{H^s} \le \|f_0\|_{H^s} + C\Big(\int_0^t \|g(\tau)\|_{H^s} d\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau\Big),
$$

then

$$
\|f(t)\|_{H^s} \le e^{CV(t)}(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau),
$$

where $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty}) d\tau$.

We may use [19, Lemma 2.1] to handle the regularity propagation of solutions to (2.1). In addition, Lemma 2.2 was proved using the Littlewood-Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument as in [19], we can obtain the following blow-up criterion.

Theorem 2.3. *Let $\sigma \neq 0$, (u,ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$ with $s > 3/2$, and T be the maximal time of existence. Then*

$$
T < \infty \Rightarrow \int_0^t \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{2.3}
$$

Regarding the finite time blow-up, we consider the trajectory equation of the system (2.1),

$$
\begin{gathered}
\frac{dq(t,x)}{dt} = u(t, q(t,x)), \quad t \in [0,T) \\
q(0,x) = x, \quad x \in S,
\end{gathered} \tag{2.4}
$$

where $u \in C^1([0,T);H^{s-1})$ is the first component of the solution (u,ρ) to (2.1) with initial data $(u_0,\rho_0) \in H^s(S) \times H^{s-1}(S)$ with $s > 3/2$, and $T > 0$ is the maximal time of the existence. Applying Theorem 2.1, we know that $q(t,\cdot) : S \to S$ is the diffeomorphism for every $t \in [0,T)$, and

$$
q_x(t,x) = \exp\Big(\int_0^t u_x(\tau, q(\tau,x)) d\tau\Big) > 0, \quad \forall (t,x) \in [0,T) \times S. \tag{2.5}
$$

Hence, the L^∞-norm of any function $v(t,\cdot) \in L^\infty, t \in [0,T)$ is preserved under the diffeomorphism $q(t,\cdot)$ with $t \in [0,T)$; that is, $\|v(t,\cdot)\|_{L^\infty} = \|v(t,q(t,\cdot))\|_{L^\infty}$.

Lemma 2.4 ([11]). *Let $T > 0$ and $v \in C^1([0,T);H^1(R))$, then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with $m(t) := \inf_{x\in\mathbb{R}}[v_x(t,x)] = v_x(t,\xi(t))$. The function $m(t)$ is absolutely continuous on $(0,T)$ with*

$$
\frac{dm(t)}{dt} = v_{tx}(t,\xi(t)) \quad \text{a.e. on } (0,T).
$$

Lemma 2.5. *Assume $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$ with $s > 3/2$, and (u,ρ) is the solution of system (2.1), then $\|(u, \rho - 1)\|^2_{H^1 \times L^2} \le \|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}$.*

Proof. Multiplying the first equation in (2.1) by u and using integration by parts gives

$$\frac{d}{dt}\int_S (u^2+u_x^2)dx + 2\lambda\int_S (u^2+u_x^2)dx + 2\int_S u\rho\rho_x dx = 0$$

Rewriting the second equation in (2.1) in the form $(\rho-1)_t + \rho_x u + \rho u_x = 0$, and multiplying by $(\rho-1)$ and using integration by parts, we have

$$\frac{d}{dt}\int_S (\rho-1)^2 dx + 2\int_S u\rho\rho_x dx - 2\int_S u\rho_x dx + 2\int_S u_x\rho^2 dx - 2\int_S u_x\rho dx = 0.$$

Combining the above equalities, we have

$$\begin{gathered}\frac{d}{dt}\int_S (u^2+u_x^2+(\rho-1)^2)dx + 2\lambda\int_S (u^2+u_x^2)dx = 0,\\ \frac{d}{dt}\int_S (u^2+u_x^2+(\rho-1)^2+2\lambda\int_0^t (u^2+u_x^2)d\tau)dx = 0.\end{gathered}$$

So we have

$$\begin{aligned}&\int_S (u^2+u_x^2+(\rho-1)^2+2\lambda\int_0^t (u^2+u_x^2)d\tau)dx\\ &= \int_S (u_0^2+u_{0x}^2+(\rho_0-1)^2)dx = \|(u_0,\rho_0-1)\|^2_{H^1\times L^2}.\end{aligned}$$

Since $2\lambda\int_0^t (u^2+u_x^2)d\tau \ge 0$, we obtain

$$\|(u,\rho-1)\|^2_{H^1\times L^2} = \int_S (u^2+u_x^2+(\rho-1)^2)dx \le \|(u_0,\rho_0-1)\|^2_{H^1\times L^2}.$$

The proof is complete. □

Lemma 2.6 ([32]). *(1) For all $f\in H^1(S)$, we have*

$$\max_{x\in[0,1]} f^2(x) \le \frac{e+1}{2(e-1)}\|f\|_1^2,$$

where $\frac{e+1}{2(e-1)}$ is the best constant.

(2) For all $f\in H^3(S)$, we have

$$\max_{x\in[0,1]} f^2(x) \le c\|f\|_1^2,$$

where the possible best constant $c\in(1,\frac{13}{12}]$, and the best constant is $\frac{e+1}{2(e-1)}$.

Lemma 2.7. *If $f\in H^3(S)$, then*

$$\max_{x\in[0,1]} f_x^2(x) \le \frac{1}{12}\|f\|^2_{H^2(S)}.$$

Proof. From [32, Theorem 2.1], the Fourier expansion of $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(2\pi nx).$$

Then

$$f_x(x) = -\sum_{n=1}^{\infty}(2n\pi a_n\sin(2\pi nx)).$$

Using that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, we have

$$\begin{aligned}
\max_{x\in S} f_x^2(x) &\le \Big(\sum_{n=1}^{\infty} |2n\pi a_n|\Big)^2 \\
&= \Big(\sum_{n=1}^{\infty} (2n\pi)^2|a_n|\frac{1}{2n\pi}\Big)^2 \\
&\le \sum_{n=1}^{\infty} ((2n\pi)^2|a_n|)^2 \sum_{n=1}^{\infty} (\frac{1}{2n\pi})^2 \\
&\le \frac{1}{24}\sum_{n=1}^{\infty} (16n^4\pi^4 a_n^2) \\
&= \frac{1}{12}\sum_{n=1}^{\infty} (8n^4\pi^4 a_n^2) \\
&= \frac{1}{12}\int_S f_{xx}^2 dx \le \frac{1}{12}\|f\|_{H^2(S)}^2.
\end{aligned}$$

The proof is complete. □

Applying the above lemmas and the method of characteristics, we may carry out the estimates along the characteristics $q(t,x)$ which captures $\sup_{x\in S} u_x(t,x)$ and $\inf_{x\in S} u_x(t,x)$.

Lemma 2.8. *Let $\sigma \neq 0$ and (u,ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S)\times H^{s-1}(S)$, $s > 3/2$, and T be the maximal time of existence.*

(1) When $\sigma > 0$, we have

$$\sup_{x\in S} u_x(t,x) \le \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}; \tag{2.6}$$

(2) When $\sigma < 0$, we have

$$\inf_{x\in S} u_x(t,x) \ge -\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}; \tag{2.7}$$

where the constants are defined as follows:

$$C_1 = \sqrt{\frac{5(e+1)}{2(e-1)} + (\frac{1+A^2}{2} + \frac{(e+1)|3-\sigma|}{e-1})\|(u_0,\rho_0-1)\|_{H^1\times L^2}^2}, \tag{2.8}$$

$$C_2 == \sqrt{\frac{5(e+1)}{2(e-1)} + (\frac{A^2}{2} + \frac{(5-\sigma)e+3-\sigma}{2(e-1)})\|(u_0,\rho_0-1)\|_{H^1\times L^2}^2}. \tag{2.9}$$

Proof. The local well-posedness theorem and a density argument imply that it suffices to prove the desired estimates for $s \ge 3$. Thus, we take $s = 3$ in the proof. Here we may assume that $u_0 \neq 0$. Otherwise, the results become trivial.

Differentiating the first equation in (2.2) with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$, we have

$$u_{tx} + \sigma u u_{xx} + \frac{\sigma}{2}u_x^2 = \frac{1}{2}\rho^2 + \frac{3-\sigma}{2}u^2 + A\partial_x^2 G * u - G * (\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2) - \lambda u_x. \tag{2.10}$$

(1) When $\sigma > 0$, using Lemma 2.4 and the fact that

$$\sup_{x\in S}[v_x(t,x)] = -\inf_{x\in S}[-v_x(t,x)],$$

we can consider $\bar{m}(t)$ and $\eta(t)$ as

$$\bar{m}(t) := u_x(t,\eta(t)) = \sup_{x\in S}(u_x(t,x)), \quad t\in[0,T). \tag{2.11}$$

This gives

$$u_{xx}(t,\eta(t)) = 0 \quad \text{a.e. on } t\in[0,T) \tag{2.12}$$

Take the trajectory $q(t,x)$ defined in (2.4). We know that $q(t,\cdot): S\to S$ is a diffeomorphism for every $t\in[0,T)$,then there exists $x_1(t)\in S$ such that

$$q(t,x_1(t)) = \eta(t), \quad t\in[0,T). \tag{2.13}$$

Let

$$\bar{\zeta}(t) = \rho(t,q(t,x_1)), \quad t\in[0,T). \tag{2.14}$$

Then along the trajectory $q(t,x_1(t))$, equation (2.10) and the second equation of (2.1) become

$$\begin{gathered}\bar{m}'(t) = -\frac{\sigma}{2}\bar{m}^2(t) - \lambda\bar{m}(t) + \frac{1}{2}\bar{\zeta}^2(t) + f(t,q(t,x_1))\\ \bar{\zeta}'(t) = -\bar{\zeta}(t)\bar{m}(t),\end{gathered} \tag{2.15}$$

where

$$f = \frac{3-\sigma}{2}u^2 + A\partial_x^2 G*u - G*\Big(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2\Big). \tag{2.16}$$

Since $\partial_x^2 G*u = \partial_x G*\partial_x u$, we have

$$\begin{aligned} f &= \frac{3-\sigma}{2}u^2 + A\partial_x G*\partial_x u - G*(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2) - \frac{1}{2}G*1 - G*(\rho-1)\\ &\quad - \frac{1}{2}G*(\rho-1)^2\\ &\le \frac{3-\sigma}{2}u^2 + A\partial_x G*\partial_x u - G*(\frac{3-\sigma}{2}u^2) - \frac{1}{2}G*1 - G*(\rho-1)\\ &\le \frac{|3-\sigma|}{2}u^2 + A|\partial_x G*\partial_x u| + |G*(\frac{3-\sigma}{2}u^2)| + \frac{1}{2}|G*1| + |G*(\rho-1)|. \end{aligned}$$

Based on the following formulas:

$$\begin{gathered} \frac{|3-\sigma|}{2}u^2 \le \frac{|3-\sigma|}{2}\cdot\frac{e+1}{2(e-1)}\|u\|_{H^1}^2,\\ A|\partial_x G*\partial_x u| \le A\|G_x\|_{L^2}\|u_x\|_{L^2} \le \frac{e+1}{2(e-1)} + \frac{1}{4}A^2\|u_x\|_{L^2}^2,\\ |G*(\frac{\sigma}{2}u_x^2)| \le \|G_x\|_{L^\infty}\|\frac{\sigma}{2}u_x^2\|_{L^1} \le \frac{e+1}{2(e-1)}\cdot\frac{\sigma}{2}\|u_x\|_{L^2}^2,\\ |G*(\frac{3-\sigma}{2}u^2)| \le \|G_x\|_{L^\infty}\|\frac{3-\sigma}{2}u^2\|_{L^1} \le \frac{e+1}{2(e-1)}\cdot\frac{|3-\sigma|}{2}\|u\|_{L^2}^2,\\ \frac{1}{2}|G*1| \le \frac{1}{2}\|G\|_{L^\infty} \le \frac{e+1}{4(e-1)},\\ |G*(\rho-1)| \le \|G\|_{L^2}\|\rho-1\|_{L^2} \le \frac{e+1}{2(e-1)} + \frac{1}{4}\|\rho-1\|_{L^2}^2, \end{gathered}$$

$$\frac{1}{2}|G*(\rho-1)^2| \le \frac{1}{2}\|G\|_{L^\infty}\|(\rho-1)^2\|_{L^1} \le \frac{e+1}{4(e-1)}\|\rho-1\|_{L^2}^2,$$

from the above inequalities and Lemma 2.5 we obtain an upper bound of f,

$$\begin{aligned} f &\le \frac{5(e+1)}{4(e-1)} + \frac{1}{4}\|\rho-1\|_{L^2}^2 + (\frac{A^2}{4} + \frac{(e+1)|3-\sigma|}{2(e-1)})\|u\|_{H^1}^2 \\ &\le \frac{5(e+1)}{4(e-1)} + (\frac{A^2+1}{4} + \frac{(e+1)|3-\sigma|}{2(e-1)})\|(u_0,\rho_0-1)\|_{H^1\times L^2}^2 \\ &= \frac{1}{2}C_1^2. \end{aligned} \tag{2.17}$$

Similarly, we obtain a lower bound of f,

$$\begin{aligned} -f &\le \frac{\sigma-3}{2}u^2 + A|\partial_x G*\partial_x u| + |G*(\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2)| + \frac{1}{2}|G*1| \\ &\quad + |G*(\rho-1)| + \frac{1}{2}G*(\rho-1)^2 \\ &\le \frac{5(e+1)}{4(e-1)} + \frac{e}{2(e-1)}\|\rho-1\|_{L^2}^2 + (\frac{A^2}{4} + \frac{(e+1)(|\sigma|+2|3-\sigma|)}{4(e-1)})\|u\|_{H^1}^2 \\ &\le \frac{5(e+1)}{4(e-1)} + (\frac{A^2}{4} + \frac{2e+(e+1)(|\sigma|+2|3-\sigma|)}{4(e-1)})\|(u_0,\rho_0-1)\|_{H^1\times L^2}^2. \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18), we obtain

$$|f| \le \frac{5(e+1)}{4(e-1)} + (\frac{A^2}{4} + \frac{2e+(e+1)(|\sigma|+2|3-\sigma|)}{4(e-1)})\|(u_0,\rho_0-1)\|_{H^1\times L^2}^2. \tag{2.19}$$

Since $s \ge 3$, it follows that $u \in C_0^1(S)$ and

$$\inf_{x\in S} u_x(t,x) \le 0, \quad \sup_{x\in S} u_x(t,x) \ge 0, \quad t\in[0,T). \tag{2.20}$$

Hence, we obtain

$$\bar{m}(t) > 0 \quad \text{for } t\in[0,T). \tag{2.21}$$

From the second equation in (2.15), we have

$$\bar{\zeta}(t) = \bar{\zeta}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}, \tag{2.22}$$

$$|\rho(t,q(t,x_1))| = |\bar{\zeta}(t)| \le |\bar{\zeta}(0)| \le \|\rho_0\|_{L^\infty}.$$

For any given $x\in S$, we define

$$P_1(t) = \bar{m}(t) - \|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}.$$

Notice that $P_1(t)$ is a C^1-function in $[0,T)$ and satisfies

$$P_1(0) = \bar{m}(0) - \|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}} \le \bar{m}(0) - \|u_{0x}\|_{L^\infty} \le 0.$$

Next, we claim that

$$P_1(t) \le 0 \quad \text{for } t\in[0,T). \tag{2.23}$$

If not, then suppose that there is a $t_0\in[0,T)$ such that $P_1(t_0) > 0$. Define $t_1 = \max\{t<t_0 : P_1(t)=0\}$, then $P_1(t_1)=0$, $P_1'(t_1)\ge 0$. That is,

$$\bar{m}(t_1) = \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}, \quad \bar{m}'(t_1) = P_1'(t_1) \ge 0.$$

On the other hand, we have

$$\begin{aligned}\bar{m}'(t_1) &= -\frac{\sigma}{2}\bar{m}^2(t_1) - \lambda\bar{m}(t_1) + \frac{1}{2}\bar{\zeta}^2(t_1) + f(t_1, q(t_1, x_1)) \\ &\le -\frac{\sigma}{2}\Big(\|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} + \frac{\|\rho_0\|^2_{L^\infty} + C_1^2}{\sigma}} + \frac{\lambda}{\sigma}\Big)^2 \\ &\quad + \frac{\lambda^2}{2\sigma} + \frac{1}{2}\|\rho_0\|^2_{L^\infty} + \frac{1}{2}C_{_1}^2 < 0.\end{aligned}$$

This yields a contraction. Thus, $P_1(t) \le 0$ for $t \in [0, T)$. Since x is chosen arbitrarily, we obtain (2.6)).

(2) When $\sigma < 0$, we have a finer estimate

$$\begin{aligned}-f &\le -A(\partial_x G * \partial_x u) + G * \frac{3-\sigma}{2}u^2 + \frac{1}{2}(G * 1) + G * (\rho - 1) + \frac{1}{2}G * (\rho - 1)^2 \\ &\le A|\partial_x G * \partial_x u| + |G * \frac{3-\sigma}{2}u^2| + \frac{1}{2}|G * 1| + |G * (\rho - 1)| + \frac{1}{2}|G * (\rho - 1)^2| \\ &\le \frac{5(e+1)}{4(e-1)} + (\frac{A^2}{4} + \frac{(5-\sigma)e + 3 - \sigma}{4(e-1)})\|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2} = \frac{1}{2}C_2^2.\end{aligned} \tag{2.24}$$

We consider the functions $m(t)$ and $\xi(t)$ in Lemma 2.4,

$$m(t) := \inf_{x \in S}[u_x(t, x)], \quad t \in [0, T) \tag{2.25}$$

Then $u_{xx}(t, \xi(t)) = 0$ a.e. on $t \in [0, T)$. Choose $x_2(t) \in S$, such that $q(t, x_2(t)) = \xi(t)$, $t \in [0, T)$. Let $\zeta(t) = \rho(t, q(t, x_2)), t \in [0, T)$. Along the trajectory $q(t, x_2)$, equation (2.10) and the second equation of (2.1) become

$$\begin{gathered}m'(t) = -\frac{\sigma}{2}m^2(t) - \lambda m(t) + \frac{1}{2}\zeta^2(t) + f(t, q(t, x_2)) \\ \zeta'(t) = -\zeta(t)m(t).\end{gathered}$$

Let $P_2(t) = m(t) + \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}$, $\forall x \in \mathbb{R}$. Then $P_2(t)$ is a C^1-function in $[0, T)$ and satisfies

$$P_2(0) = m(0) + \|u_{0x}\|_{L^\infty} + \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} \ge m(0) + \|u_{0x}\|_{L^\infty} \ge 0.$$

Now we claim that

$$P_2(t) \ge 0 \quad \text{for } t \in [0, T). \tag{2.26}$$

Assume that there is a $\bar{t}_0 \in [0, T)$ such that $P_2(\bar{t}_0) < 0$. Define $t_2 = \max\{t < \bar{t}_0 : P_2(t) = 0\}$, then $P_2(t_2) = 0$, $P_2'(t_2) \le 0$. That is,

$$m(t_2) = -\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}}, \quad m'(t_2) = P_2'(t_2) \le 0.$$

In addition, we have

$$\begin{aligned}m'(t_2) &= -\frac{\sigma}{2}m^2(t_2) - \lambda m(t_2) + \frac{1}{2}\zeta^2(t_2) + f(t_2, q(t_2, x_2)) \\ &\ge -\frac{\sigma}{2}(-\|u_{0x}\|_{L^\infty} - \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} + \frac{\lambda}{\sigma})^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}C_2^2 > 0.\end{aligned}$$

This is a contradiction. Then we have $P_2(t) \ge 0$ for $t \in [0, T)$, since x is chosen arbitrarily. □

Now, we present the following estimates for $\|\rho\|_{L^\infty(S)}$, if σu_x is bounded from below.

Lemma 2.9 ([5])**.** *Let $\sigma \neq 0$ and (u,ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and T be the maximal time of the existence. If there is a $M \geq 0$ such that $\inf_{(t,x)\in[0,T)\times S} \sigma u_x \geq -M$, Then we have following two statements.*

(1) If $\sigma > 0$, then $\|\rho(t,\cdot)\|_{L^\infty(S)} \leq \|\rho_0\|_{L^\infty(S)} e^{Mt/\sigma}$.

(2) If $\sigma < 0$, then $\|\rho(t,\cdot)\|_{L^\infty(S)} \leq \|\rho_0\|_{L^\infty(S)} e^{Nt}$,

where $N = \|u_{0x}\|_{L^\infty} + (C_2/\sqrt{-\sigma})$ and C_2 is given in (2.24).

Proof. The proof of Lemma 2.9 is similar to that of [5, Proposition 3.8], so we omit it here. □

From the above results, we can get the necessary and sufficient conditions for the blow-up of solutions.

Theorem 2.10 (Wave-breaking criterion for $\sigma \neq 0$)**.** *Let $\sigma \neq 0$ and (u,ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and T be the maximal time of existence. Then the solution blows up in finite time if and only if*

$$\lim_{t\to T^-} \inf_{x\in S} \sigma u_x(t,x) = -\infty. \tag{2.27}$$

Proof. Assume that $T < \infty$ and (2.27) is not valid, then there is some positive number $M > 0$, such that $\sigma u_x(t,x) \geq -M$, $\forall (t,x) \in [0,T) \times S$. From the above lemmas, we have $|u_x(t,x)| \leq C$, where $C = C(A, M, \sigma, \lambda, \|(u_0, \rho_0 - 1)\|_{H^s\times H^{s-1}})$. Thus, Theorem 2.3 implies that the maximal existence time $T = \infty$, which contradicts the assumption $T < \infty$.

On the other hand, the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ with $s > 1/2$ implies that if (2.27) holds, the corresponding solution blows up in finite time. The proof is complete. □

3. Blow-up scenarios

Theorem 3.1. *Let $\sigma > 0$ and (u,ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and T be the maximal time of existence. Assume that there is some $x_0 \in S$ such that $\rho_0(x_0) = 0, u_{0x}(x_0) = \inf_{x\in S} u_{0x}(x)$ and*

$$\begin{aligned} &\|(u_0, \rho_0 - 1)\|^2_{H^1\times L^2} \\ &< \Big(\frac{8e-10}{18(e-1)} - \frac{\lambda^2}{2\sigma}\Big)\frac{4(e-1)}{(18A^2+19)e - (18A^2+17) + (2|3-\sigma| + \sigma)(e+1)}, \end{aligned} \tag{3.1}$$

then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a T such that

$$\begin{aligned} 0 < T \leq \frac{2}{\sigma - \lambda} &+ \Big(72\sigma(e-1)(1 + |u_{0x}(x_0)|)\Big) \\ &\div \Big(\sigma(32e - 40 - 324e - 324A^2e + 324A^2 + 306) - 36\lambda^2(e-1) \\ &+ (2|3-\sigma| + \sigma)(e-1)\|(u_0,\rho_0-1)\|^2_{H^1\times L^2}\Big) \end{aligned} \tag{3.2}$$

and that $\liminf_{t\to T^-} (\inf_{x\in S} u_x(t,x)) = -\infty$.

Proof. Here we also consider $s \geq 3$. We still consider along the trajectory $q(t, x_2)$ defined as before. In this way, we can write the transport equation of ρ in (2.1) along the trajectory of $q(t, x_2)$ as

$$\frac{d\rho(t,\xi(t))}{dt} = -\rho(t,\xi(t))u_x(t,\xi(t)). \tag{3.3}$$

By the assumption, we have

$$m(0) = u_x(0,\xi(0)) = \inf_{x\in S} u_{0x}(x) = u_{0x}(x_0).$$

Choose $\xi(0) = x_0$ and then $\rho_0(\xi(0)) = \rho_0(x_0) = 0$. Then by (3.3), we derive

$$\rho(t,\xi(t)) = 0, \quad \forall t \in [0,T). \tag{3.4}$$

Evaluating the result at $x = \xi(t)$ and combining (3.4) with $u_{xx}(t,\xi(t)) = 0$, we have

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2}m^2(t) - \lambda m(t) + \frac{3-\sigma}{2}u^2(t,\xi(t)) + A(G_x * u_x)(t,\xi(t)) \\ &\quad - G * (\frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2)(t,\xi(t)) \\ &= -\frac{\sigma}{2}m^2(t) - \lambda m(t) + f(t,q(t,x_2)) \\ &= -\frac{\sigma}{2}(m(t) + \frac{\lambda}{\sigma})^2 + \frac{\lambda^2}{2\sigma} + f(t,q(t,x_2)). \end{aligned} \tag{3.5}$$

We modify the estimates:

$$\begin{aligned} A|G_x * u_x| &\leq A\|G_x\|_{L^2}\|u_x\|_{L^2} \leq \frac{1}{18}\cdot\frac{e+1}{2(e-1)} + \frac{9}{2}A^2\|u_x\|_{L^2}^2, \\ |G*(\rho-1)| &\leq \|G\|_{L^2}\|\rho-1\|_{L^2} \leq \frac{1}{18}\cdot\frac{e+1}{2(e-1)} + \frac{9}{2}\|\rho-1\|_{L^2}^2. \end{aligned}$$

Similarly, we obtain the upper bound of f as

$$\begin{aligned} f &\leq \frac{10-8e}{18(e-1)} + \frac{(18A^2+19)e - (18A^2+17) + (2|3-\sigma| + \sigma)(e+1)}{4(e-1)} \\ &\quad \times \|(u_0, \rho_0 - 1)\|_{H^1\times L^2}^2 := -C_3. \end{aligned}$$

By assumption (3.1), we obtain $\frac{\lambda^2}{2\sigma} - C_3 < 0$ and

$$m'(t) \leq -\frac{\sigma}{2}\big(m(t) + \frac{\lambda}{\sigma}\big)^2 + \frac{\lambda^2}{2\sigma} - C_3 \leq \frac{\lambda^2}{2\sigma} - C_3 < 0, \quad t \in [0,T). \tag{3.6}$$

So $m(t)$ is strictly decreasing in $[0,T)$. If the solution (u,ρ) of (2.1) exists globally in time, that is, $T = \infty$, we will show that it leads to a contradiction.

Let $t_1 = \frac{2\sigma(1+|u_{0x}(x_0)|)}{2\sigma C_3 - \lambda^2}$. Integrating (3.6) over $[0,t_1]$ gives

$$m(t_1) = m(0) + \int_0^{t_1} m'(t)dt \leq |u_{0x}(x_0)| + (\frac{\lambda^2}{2\sigma} - C_3)t_1 = -1. \tag{3.7}$$

For $t \in [t_1, T)$, we have $m(t) \leq m(t_1) \leq -1$. From (3.6), we have

$$m'(t) \leq -\frac{\sigma}{2}\big(m(t) + \frac{\lambda}{\sigma}\big)^2. \tag{3.8}$$

Integrating over $[t_1, T)$, by (3.7), yields

$$-\frac{1}{m(t) + \frac{\lambda}{\sigma}} + \frac{1}{\frac{\lambda}{\sigma} - 1} \leq -\frac{1}{m(t) + \frac{\lambda}{\sigma}} + \frac{1}{m(t_1) + \frac{\lambda}{\sigma}} \leq -\frac{\sigma}{2}(t - t_1), \quad t \in [t_1, T),$$

$$m(t) \le \frac{1}{\frac{\sigma}{2}(t-t_1)+\frac{\sigma}{\lambda-\sigma}} - \frac{\lambda}{\sigma} \to -\infty, \quad \text{as } t \to t_1 + \frac{2}{\sigma-\lambda}.$$

So, $T \le t_1 + \frac{2}{\sigma-\lambda}$, which is a contradiction to $T=\infty$. Consequently, the proofis complete. ☐

Theorem 3.2. *Let $\sigma \neq 0$ and (u,ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and T be the maximal time of the existence.*

(1) When $\sigma > 0$, assume that there is an $x_0 \in S$ such that $\rho_0(x_0) = 0$, $u_{0x}(x_0) = \inf_{x\in S} u_{0x}(x)$ and $u_{0x}(x_0) < -\sqrt{\frac{\lambda^2}{\sigma^2}+\frac{C_1^2}{\sigma}} - \frac{\lambda}{\sigma}$, where C_1 is defined in (2.8). Then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a T_1 such that

$$0 < T_1 \le -\frac{2(\lambda+\sigma u_{0x}(x_0))}{(\lambda+\sigma u_{0x}(x_0))^2 - (\lambda^2+\sigma C_1^2)},$$

and

$$\liminf_{t\to T_1^-}\{\inf_{x\in S} u_x(t,x)\} = -\infty.$$

(2) When $\sigma < 0$, assume that there is some $x_0 \in S$ such that $u_{0x}(x_0) > \sqrt{\frac{\lambda^2}{\sigma^2}-\frac{C_2^2}{\sigma}} - \frac{\lambda}{\sigma}$, where C_2 is defined in (2.9). Then the corresponding solution to system (2.1) blows up in finite time in the following sense: there exists a T_2 such that

$$0 < T_2 \le -\frac{2(\lambda+\sigma u_{0x}(x_0))}{(\lambda+\sigma u_{0x}(x_0))^2 - (\lambda^2-\sigma C_2^2)},$$

and

$$\liminf_{t\to T_2^-}\{\sup_{x\in S} u_x(t,x)\} = \infty.$$

Proof. (1) When $\sigma > 0$, using the upper bound of f in (2.17) and (3.4), we have

$$m'(t) \le -\frac{\sigma}{2}\Big(m(t)+\frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2}C_1^2, \quad t \in [0,T).$$

By the assumption $m(0) = u_{0x}(x_0) < -\sqrt{\frac{\lambda^2}{\sigma^2}+\frac{C_1^2}{\sigma}} - \frac{\lambda}{\sigma}$, we have that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0,T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{\sigma(u_{0x}(x_0)+\frac{\lambda}{\sigma})^2}\Big(\frac{\lambda^2}{2\sigma}+\frac{1}{2}C_1^2\Big) \in (0,\frac{1}{2}).$$

Since $m(t) < m(0) = u_{0x}(x_0) < -\frac{\lambda}{\sigma}$, it holds

$$m'(t) \le -\frac{\sigma}{2}\Big(m(t)+\frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} + \frac{1}{2}C_1^2 \le -\delta\sigma\Big(m(t)+\frac{\lambda}{\sigma}\Big)^2.$$

By a similar argument as in the proof of Theorem 3.1, we obtain

$$m(t) \le \frac{\lambda+\sigma u_{0x}(x_0)}{\sigma+(\delta\sigma^2 u_{0x}(x_0)+\lambda\delta\sigma)t} - \frac{\lambda}{\sigma} \to -\infty \quad \text{as } t \to -\frac{1}{\lambda\delta+\delta\sigma u_{0x}(x_0)}.$$

Thus, we have $0 < T_1 \le -\frac{1}{\lambda\delta+\delta\sigma u_{0x}(x_0)}$.

(2) when $\sigma < 0$, we consider the functions $\bar{m}(t)$ and $\eta(t)$ as defined in (2.11) and take the trajectory $q(t, x_1)$ with x_1 defined in (2.13), then

$$\begin{aligned} \bar{m}'(t) &= -\frac{\sigma}{2}\bar{m}^2(t) - \lambda\bar{m}(t) + \frac{1}{2}\rho^2(t,\eta(t)) + f(t,q(t,x_1)) \\ &\geq -\frac{\sigma}{2}\Big(\bar{m}(t) + \frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} + f(t,q(t,x_1)). \end{aligned} \tag{3.9}$$

From the lower bound of f in (2.24), we obtain

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\Big(\bar{m}(t) + \frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}C_2^2, \quad t \in [0,T).$$

By the assumption $\bar{m}(0) \geq u_{0x}(x_0) > \sqrt{\frac{\lambda^2}{\sigma^2} - \frac{C_2^2}{\sigma}} - \frac{\lambda}{\sigma}$, we have that $\bar{m}'(0) > 0$ and $\bar{m}(t)$ is strictly increasing over $[0,T)$.

Set

$$\theta = \frac{(\sigma u_{0x}(x_0) + \lambda)^2 - (\lambda^2 - \sigma C_2^2)}{2(\sigma u_{0x}(x_0) + \lambda)^2} \in (0, \frac{1}{2}).$$

Since $\bar{m}(t) > \bar{m}(0) \geq u_{0x}(x_0) > -\frac{\lambda}{\sigma}$, we obtain

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\Big(\bar{m}(t) + \frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}C_2^2 \geq -\theta\sigma\Big(\bar{m}(t) + \frac{\lambda}{\sigma}\Big)^2.$$

Similarly, we obtain

$$\bar{m}(t) \geq \frac{\lambda + \sigma u_{0x}(x_0)}{\sigma + (\theta\sigma^2 u_{0x}(x_0) + \lambda\theta\sigma)t} - \frac{\lambda}{\sigma} \to \infty \quad \text{as } t \to -\frac{1}{\lambda\theta + \theta\sigma u_{0x}(x_0)}.$$

Therefore, $0 < T_2 \leq -\frac{1}{\lambda\theta + \theta\sigma u_{0x}(x_0)}$. The proof is complete. □

Remark. If $\sigma = 3$ and $A = 0$, then all solutions of system (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$ with $s > 3/2$ satisfying $u_0 \neq 0$ and $\rho_0(x_0) = 0$ for some $x_0 \in S$, blow up in finite time.

4. Blow-up rate

Theorem 4.1. *Let $\sigma \neq 0$. If $T < \infty$ is the blow-up time of the solution (u, ρ) to (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$ satisfying the assumptions of Theorem 3.2. Then*

$$\lim_{t \to T^-} \{\inf_{x \in S} u_x(t,x)(T-t)\} = -\frac{2}{\sigma}, \quad \sigma > 0, \tag{4.1}$$

$$\lim_{t \to T^-} \{\sup_{x \in S} u_x(t,x)(T-t)\} = -\frac{2}{\sigma}, \quad \sigma < 0. \tag{4.2}$$

Proof. We assume that $s = 3$ to prove the theorem.

(1) when $\sigma > 0$, from (3.5) we have

$$m'(t) = -\frac{\sigma}{2}\Big(m(t) + \frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} + f(t, q(t,x)). \tag{4.3}$$

From (2.19), note that

$$M = \frac{5(e+1)}{4(e-1)} + (\frac{A^2}{4} + \frac{2e + (e+1)(|\sigma| + 2|3-\sigma|)}{4(e-1)})\|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}, \tag{4.4}$$

Then

$$-\frac{\sigma}{2}(m(t) + \frac{\lambda}{\sigma})^2 - \frac{\lambda^2}{2\sigma} - M \leq m'(t) \leq -\frac{\sigma}{2}(m(t) + \frac{\lambda}{\sigma})^2 + \frac{\lambda^2}{2\sigma} + M. \tag{4.5}$$

Choose $\varepsilon \in (0, \frac{\sigma}{2})$, since $\lim_{t\to T^-}\big(m(t)+\frac{\lambda}{\sigma}\big) = -\infty$, there is some $t_0 \in (0,T)$, such that $m(t_0)+\frac{\lambda}{\sigma} < 0$ and $\big(m(t_0)+\frac{\lambda}{\sigma}\big)^2 > \frac{1}{\varepsilon}\big(\frac{\lambda^2}{2\sigma}+M\big)$. Since m is locally Lipschitz, it follows that m is absolutely continuous. We deduce that m is decreasing on $[t_0,T)$ and

$$\Big(m(t)+\frac{\lambda}{\sigma}\Big)^2 > \frac{1}{\varepsilon}\Big(\frac{\lambda^2}{2\sigma}+M\Big), \quad t \in [t_0,T). \tag{4.6}$$

Combining (4.5) with (4.6), we have

$$\frac{\sigma}{2}-\varepsilon \le \frac{d}{dt}\Big(\frac{1}{m(t)+\frac{\lambda}{\sigma}}\Big) \le \frac{\sigma}{2}+\varepsilon, \quad t \in [t_0,T). \tag{4.7}$$

Integrating over (t,T) with $t \in [t_0,T)$ and noticing that $\lim_{t\to T^-}\big(m(t)+\frac{\lambda}{\sigma}\big) = -\infty$, we obtain

$$(\frac{\sigma}{2}-\varepsilon)(T-t) \le -\frac{1}{m(t)+\frac{\lambda}{\sigma}} \le (\frac{\sigma}{2}+\varepsilon)(T-t).$$

Since $\varepsilon \in (0,\frac{\sigma}{2})$ is arbitrary, in view of the definition of $m(t)$, we have

$$\lim_{t\to T^-}\{m(t)(T-t)+\frac{\lambda}{\sigma}(T-t)\} = -\frac{2}{\sigma};$$

that is, $\lim_{t\to T^-}\{inf_{x\in S}\, u_x(t,x)(T-t)\} = -\frac{2}{\sigma}$.

(2) When $\sigma < 0$, we consider the functions $\bar m(t)$ and $\eta(t)$ as defined in (2.11). From (3.9) and (4.4), we have $\bar m'(t) \ge -\frac{\sigma}{2}\Big(\bar m(t)+\frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} - M$.

Because $\bar m(t) \to \infty$ as $t \to T^-$, there is a $t_1 \in (0,T)$, such that $\bar m(t_1) > \sqrt{\frac{\lambda^2}{\sigma^2}-\frac{2M}{\sigma}} - \frac{\lambda}{\sigma} > 0$. Thus, we have that $\bar m'(t) > 0$ and $\bar m(t)$ is strictly increasing on $[t_1,T)$, and

$$\bar m(t) > \bar m(t_1) > 0. \tag{4.8}$$

By the transport equation for ρ, we have

$$\frac{d\rho(t,\eta(t))}{dt} = -\bar m(t)\rho(t,\eta(t)).$$

Then

$$\rho(t,\eta(t)) = \rho(t_1,\eta(t_1))e^{-\int_{t_1}^t \bar m(\tau)d\tau}, \quad t \in [t_1,T). \tag{4.9}$$

Combining (4.8) with (4.9) yields

$$\rho^2(t,\eta(t)) \le \rho^2(t_1,\eta(t_1)), \quad t \in [t_1,T) \tag{4.10}$$

From (3.9) and (4.10), we have

$$\begin{aligned} &-\frac{\sigma}{2}\Big(\bar m+\frac{\lambda}{\sigma}\Big)^2 + \frac{\lambda^2}{2\sigma} - \frac{1}{2}\rho^2(t_1,\eta(t_1)) - M \\ &\le \bar m' \le -\frac{\sigma}{2}\Big(\bar m+\frac{\lambda}{\sigma}\Big)^2 - \frac{\lambda^2}{2\sigma} + \frac{1}{2}\rho^2(t_1,\eta(t_1)) + M. \end{aligned} \tag{4.11}$$

Choose $\varepsilon \in (0,-\frac{\sigma}{2})$, and pick a $t_2 \in [t_1,T)$, such that

$$\Big(\bar m(t_2)+\frac{\lambda}{\sigma}\Big)^2 > \frac{1}{\varepsilon}\Big(\frac{1}{2}\rho^2(t_1,\eta(t_1)) + M - \frac{\lambda^2}{2\sigma}\Big). \tag{4.12}$$

From (4.11) and (4.12), we have

$$\frac{\sigma}{2}-\varepsilon \le \frac{d}{dt}\Big(\frac{1}{\bar m(t)+\frac{\lambda}{\sigma}}\Big) \le \frac{\sigma}{2}+\varepsilon, \quad t \in [t_2,T). \tag{4.13}$$

Integrating (4.13) over $[t,T)$ with $t\in[t_2,T)$ and $\lim_{t\to T^-}\bar{m}(t)=\infty$ gives

$$(\frac{\sigma}{2}-\varepsilon)(T-t)\le -\frac{1}{\bar{m}(t)+\frac{\lambda}{\sigma}}\le(\frac{\sigma}{2}+\varepsilon)(T-t).$$

Since $\varepsilon\in(0,-\frac{\sigma}{2})$ is arbitrary, in view of the definition of $\bar{m}(t)$, we have

$$\lim_{t\to T^-}\{\sup_{x\in S}u_x(t,x)(T-t)\}=-\frac{2}{\sigma}.$$

This completes the proof of Theorem 4.1. □

5. Existence of a global solution

In this section, we provide a sufficient condition for the global solution of system (2.1) in the case when $0<\sigma<2$.

Lemma 5.1. *Let $0<\sigma<2$ and (u,ρ) be the solution of (2.1) with initial data $(u_0,\rho_0-1)\in H^s(S)\times H^{s-1}(S)$, $s>3/2$, and T be the maximal time of existence. Assume that $\inf_{x\in S}\rho_0(x)>0$.*

(1) When $0<\sigma\le 1$, it holds

$$|\inf_{x\in S}u_x(t,x)|\le\frac{1}{\inf_{x\in S}\rho_0(x)}C_4e^{C_3t},$$
$$|\sup_{x\in S}u_x(t,x)|\le\frac{1}{\inf_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)}C_4^{\frac{1}{2-\sigma}}e^{\frac{C_3t}{2-\sigma}}.$$

(2) When $1<\sigma<2$, it holds

$$|\inf_{x\in S}u_x(t,x)|\le\frac{1}{\inf_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)}C_4^{\frac{1}{2-\sigma}}e^{\frac{C_3t}{2-\sigma}},$$
$$|\sup_{x\in S}u_x(t,x)|\le\frac{1}{\inf_{x\in S}\rho_0(x)}C_4e^{C_3t},$$

where constants C_3 and C_4 are defined as follows:

$$C_3=1+\frac{5(e+1)}{4(e-1)}+(\frac{A^2}{4}+\frac{2e+(e+1)(|\sigma|+2|3-\sigma|)}{4(e-1)})\|(u_0,\rho_0-1)\|^2_{H^1\times L^2},$$
$$C_4=1+\|u_{0x}\|^2_{L^\infty}+\|\rho_0\|^2_{L^\infty}.$$

Proof. A density argument indicates that it suffices to prove the desired results for $s\ge 3$. Since $s\ge3$, we have $u\in C_0^1(S)$ and

$$\inf_{x\in S}u_x(t,x)<0,\quad \sup_{x\in S}u_x(t,x)>0,\quad t\in[0,T).$$

(1) First we will derive the estimate for $|\inf_{x\in S}u_x(t,x)|$. Define $m(t)$ and $\xi(t)$ as in (2.25), and consider along the characteristics $q(t,x_2(t))$. Then

$$m(t)\le 0\quad\text{for } t\in[0,T). \tag{5.1}$$

Let $\zeta(t)=\rho(t,\xi(t))$ and evaluating (2.10) and the second equation of system (2.1) at $(t,\xi(t))$, we have

$$\begin{gathered} m'(t)=-\frac{\sigma}{2}m^2(t)-\lambda m(t)+\frac{1}{2}\zeta^2(t)+f(t,q(t,x_2))\\ \zeta'(t)=-\zeta(t)m(t), \end{gathered}\tag{5.2}$$

where f is defined in (2.16). The second equation above implies that $\zeta(t)$ and $\zeta(0)$ are of the same sign.

Next we construct a Lyapunov function for our system as in [13]. Since here we have a free parameter σ, we could not find a uniform Lyapunov function. Instead, we split the case $0 < \sigma \le 1$ and the case $1 < \sigma < 2$. From the assumption of the theorem, we know that $\zeta(0) = \rho(0, \xi(0)) > 0$.

When $0 < \sigma \le 1$, we define the Lyapunov function

$$\omega_1(t) = \zeta(0)\zeta(t) + \frac{\zeta(0)}{\zeta(t)}(1 + m^2(t)),$$

which is always positive for $t \in [0, T)$. Differentiating $\omega_1(t)$ and using (5.2) gives

$$\begin{aligned}
\omega_1'(t) &= \zeta(0)\zeta'(t) - \frac{\zeta(0)}{\zeta^2(t)}(1 + m^2(t))\zeta'(t) + \frac{2\zeta(0)}{\zeta(t)}m(t)m'(t) \\
&= -\zeta(0)\zeta(t)m(t) - \frac{\zeta(0)}{\zeta^2(t)}(1 + m^2(t))(-\zeta(t)m(t)) \\
&\quad + \frac{2\zeta(0)}{\zeta(t)}m(t)(-\frac{\sigma}{2}m^2(t) - \lambda m(t) + \frac{1}{2}\zeta^2(t) + f) \\
&= (1-\sigma)\frac{\zeta(0)}{\zeta(t)}m^3(t) + \frac{\zeta(0)}{\zeta(t)}m(t) - \frac{2\lambda\zeta(0)}{\zeta(t)}m^2(t) + \frac{2\zeta(0)}{\zeta(t)}m(t)f \\
&\le \frac{\zeta(0)}{\zeta(t)}m(t) + \frac{2\zeta(0)}{\zeta(t)}m(t)f \\
&\le \frac{\zeta(0)}{\zeta(t)}(1 + m^2(t))(1 + |f|) \le C_3\omega_1(t),
\end{aligned} \tag{5.3}$$

where

$$C_3 = 1 + \frac{5(e+1)}{4(e-1)} + (\frac{A^2}{4} + \frac{2e + (e+1)(|\sigma| + 2|3-\sigma|)}{4(e-1)})\|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}.$$

This gives

$$\begin{aligned}
\omega_1(t) &\le \omega_1(0)e^{C_3 t} = (\zeta^2(0) + 1 + m^2(0))e^{C_3 t} \\
&\le (1 + \|u_{0x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty})e^{C_3 t} =: C_4 e^{C_3 t},
\end{aligned} \tag{5.4}$$

where $C_4 = 1 + \|u_{0x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty}$.

Recalling that $\zeta(t)$ and $\zeta(0)$ are of the same sign, the definition of $\omega_1(t)$ implies $\zeta(t)\zeta(0) \le \omega_1(t)$ and $|\zeta(0)||m(t)| \le \omega_1(t)$. By (5.4), we obtain

$$|\inf_{x\in S} u_x(t,x)| = |m(t)| \le \frac{\omega_1(t)}{|\zeta(0)|} \le \frac{1}{\inf_{x\in S}\rho_0(x)} C_4 e^{C_3 t}, \quad \text{for } t \in [0, T).$$

When $1 < \sigma < 2$, we define the Lyapunov function

$$\omega_2(t) = \zeta^\sigma(0)\frac{\zeta^2(t) + 1 + m^2(t)}{\zeta^\sigma(t)}. \tag{5.5}$$

Then

$$\begin{aligned}
\omega_2'(t) &= \frac{2\zeta^\sigma(0)}{\zeta^\sigma(t)}m(t)(\frac{\sigma-1}{2}\zeta^2(t) - \lambda m(t) + f + \frac{\sigma}{2}) \\
&\le \frac{\zeta^\sigma(0)}{\zeta^\sigma(t)}(1 + m^2(t))(|f| + \frac{\sigma}{2}) \le \frac{\zeta^\sigma(0)}{\zeta^\sigma(t)}(1 + m^2(t))(|f| + 1) \le C_3\omega_2(t).
\end{aligned} \tag{5.6}$$

Thus, we obtain

$$\begin{aligned}\omega_2(t) &\le \omega_2(0)e^{C_3 t} = (\zeta^2(0) + 1 + m^2(0))e^{C_3 t}\\ &\le (1 + \|u_{0x}\|_{L^\infty}^2 + \|\rho_0\|_{L^\infty}^2)e^{C_3 t} = C_4 e^{C_3 t}.\end{aligned}$$

Applying Young's inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ to (5.5) with $p = \frac{2}{\sigma}$ and $q = \frac{2}{2-\sigma}$ yields

$$\begin{aligned}\frac{\omega_2(t)}{\zeta^\sigma(0)} &= \Big(\zeta^{\frac{\sigma(2-\sigma)}{2}}\Big)^{\frac{2}{\sigma}} + \Big(\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\Big)^{\frac{2}{2-\sigma}}\\ &\ge \frac{\sigma}{2}\Big(\zeta^{\frac{\sigma(2-\sigma)}{2}}\Big)^{\frac{2}{\sigma}} + \frac{2-\sigma}{2}\Big(\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\Big)^{\frac{2}{2-\sigma}}\\ &\ge (1+m^2)^{\frac{2-\sigma}{2}} \ge |m(t)|^{2-\sigma}.\end{aligned}$$

So we have

$$|\inf_{x\in S} u_x(t,x)| \le \Big(\frac{\omega_2(t)}{\zeta^\sigma(0)}\Big)^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3 t}{2-\sigma}}.$$

(2) Now, we estimate $|\sup_{x\in S} u_x(t,x)|$. Consider $\bar m(t), \eta(t), q(t,x_1)$ as in (2.11) and (2.13), and

$$\begin{gathered}\bar m'(t) = -\frac{\sigma}{2}\bar m^2(t) - \lambda\bar m(t) + \frac{1}{2}\bar\zeta^2(t) + f(t, q(t,x_1))\\ \bar\zeta'(t) = -\bar\zeta(t)\bar m(t)\end{gathered} \tag{5.7}$$

for $t \in [0,T)$, where $\bar\zeta(t) = \rho(t,\eta(t))$. We know that

$$\bar m(t) \ge 0 \quad \text{for } t \in [0,T). \tag{5.8}$$

When $0 < \sigma \le 1$, we define the Lyapunov function

$$\bar\omega_1(t) = \bar\zeta^\sigma(0)\frac{\bar\zeta^2(t) + 1 + \bar m^2(t)}{\bar\zeta^\sigma(t)}. \tag{5.9}$$

Then from (5.6) and (5.8), we have $\bar\omega_1'(t) \le C_3\bar\omega_1(t)$, then $\bar\omega_1(t) \le C_4 e^{C_3 t}$. Hence, by a similar argument as before, we obtain

$$\frac{\bar\omega_1(t)}{\bar\zeta^\sigma(0)} \ge |\bar m(t)|^{2-\sigma}.$$

Then

$$|\sup_{x\in S} u_x(t,x)| \le \big(\frac{\bar\omega_1(t)}{\bar\zeta^\sigma(0)}\big)^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in S}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3 t}{2-\sigma}}, \quad t \in [0,T).$$

When $1 < \sigma < 2$, consider the Lyapunov function

$$\bar\omega_2(t) = \bar\zeta(0)\bar\zeta(t) + \frac{\bar\zeta(0)}{\bar\zeta(t)}(1 + \bar m^2(t)). \tag{5.10}$$

From (5.3) and (5.8), we have $\bar\omega_2'(t) \le C_3\bar\omega_2(t)$ and $\bar\omega_2(t) \le C_4 e^{C_3 t}$. Therefore,

$$|\sup_{x\in S} u_x(t,x)| = |\bar m(t)| \le \frac{\bar\omega_2(t)}{\bar\zeta(0)} \le \frac{1}{\inf_{x\in S}\rho_0(x)} C_4 e^{C_3 t}, \quad t \in [0,T).$$

The proof is complete. □

Theorem 5.2. *Let $0 < \sigma < 2$ and (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s(S) \times H^{s-1}(S)$, $s > 3/2$, and T be the maximal time of existence. If $\inf_{x\in S} \rho_0(x) > 0$, then $T = +\infty$ and the solution (u, ρ) is global.*

Proof. Assume on the contrary that $T < +\infty$ and the solution blows up in finite time. It then follows from Theorem 2.3, that

$$\int_0^T \|u_x(t)\|_{L^\infty} dt = \infty. \tag{5.11}$$

However, from the assumptions of the theorem and Lemma 5.1, we have $|u_x(t, x)| < \infty$ for all $(t, x) \in [0, T) \times S$. This is a contradiction to (5.11). So $T = +\infty$, and it means that the solution (u, ρ) is global. □

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