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CONTINUOUS EVOLUTION OF EQUATIONS AND INCLUSIONS INVOLVING SET-VALUED CONTRACTION MAPPINGS WITH APPLICATIONS TO GENERALIZED FRACTAL TRANSFORMS

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Abstract. Let T be a set-valued contraction mapping on a general Banach space $\mathcal{B}$. In the first part of this paper we introduce the evolution inclusion $\dot{x} + x \in Tx$ and study the convergence of solutions to this inclusion toward fixed points of T. Two cases are examined: (i) T has a fixed point $\bar{y} \in \mathcal{B}$ in the usual sense, i.e., $\bar{y} = T\bar{y}$ and (ii) T has a fixed point in the sense of inclusions, i.e., $\bar{y} \in T\bar{y}$. In the second part we extend this analysis to the case of set-valued evolution equations taking the form $\dot{x}+x = Tx$. We also provide some applications to generalized fractal transforms.

1. Introduction

In [2], it was shown that given a a Banach space $\mathcal{B}$ and a contraction mapping $T : \mathcal{B} \to \mathcal{B}$, the initial value problem

$$\dot{y}(t) = Ty(t) - y(t), \quad y(0) = y_0 \in \mathcal{B}, \tag{1.1}$$

admits a unique solution $y : [0, \infty) \to \mathcal{B}$ which converges exponentially rapidly to the unique fixed point $\bar{y} \in \mathcal{B}$ of T. In other words, $\bar{y} = T\bar{y}$ is the unique globally asymptotically stable solution of (1.1). The main purpose in introducing (1.1) was to produce a *continuous evolution* toward $\bar{y}$, as opposed to the usual discrete sequence of iterates $y_n = T^n y_0$ that converges to $\bar{y}$, independent of y_0. The original motivation for such an evolution arose from a desire to perform continuous (in time) nonlocal, fractal-like operations on images, in which $\mathcal{B}$ denotes a Banach space of functions defined on a compact set $X \subset \mathbf{R}^n$.

Nevertheless, the continuous evolution method of (1.1) can also be applied in other settings where discrete iteration has normally been considered, for example, complex analytic dynamics, including (i) the iteration of rational maps and (ii) Newton's method (and its generalizations) in the complex plane.

In this paper, we wish to consider an extension of the evolution equation in (1.1) to the case of inclusions. Set-valued differential inclusions appear to be the most natural way to capture and explain the level of uncertainty, the absence of controls and the variety of available dynamics that arise in many applied disciplines,

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including physics, mechanics and engineering. The importance of inclusions, which has been realized in the control theory literature, motivated our introduction of the notion of an Iterated Multifunction System (IMS) as a natural extension of the classical notion of an Iterated Function System (IFS) [5, 7].

To produce such an extension, first rewrite (1.1) in the form

$$\dot{y}(t) + y(t) = T(y(t)), \quad y(0) = y_0 \in \mathcal{B}. \tag{1.2}$$

Now let $T : \mathcal{B} \rightrightarrows \mathcal{B}$ be a set-valued contraction mapping and consider the following evolution inclusion,

$$\dot{y}(t) + y(t) \in T(y(t)), \quad y(0) = y_0 \in \mathcal{B}. \tag{1.3}$$

The complexity of this inclusion is much greater than (1.2). One might think that the way to proceed is by considering regular contractive selections. In general, however, selections with this property are difficult to establish. It is even difficult to guarantee the existence of Lipschitz selections. As such, we consider here only cases in which the existence of continuous selections is guaranteed.

Later in the paper we extend this approach to the case of set-valued equations

$$\dot{y}(t) + y(t) = T(y(t)), \quad y(0) = y_0 \in \mathcal{B}, \tag{1.4}$$

in which $y : [0, \infty) \rightrightarrows \mathcal{B}$ is a set-valued solution and the derivative of y w.r.t time is constructed by means of the Minkowski sum.

We end this section with a few remarks regarding (1.1) that will be helpful in our analysis. In [2], the solution of (1.1) was easily accomplished by applying classical techniques to (1.2), namely, Duhamel's formula which leads to the equation

$$y(t) = y_0 e^{-t} + e^{-t} \int_0^t e^s (Ty)(s)\, ds. \tag{1.5}$$

In the special case $y_0 = \bar{y}$, we have $y(t) = \bar{y} = T\bar{y}$ so that (1.5) leads to the trivial equation,

$$\bar{y} = \bar{y} e^{-t} + e^{-t} \int_0^t e^s (T\bar{y})(s)\, ds. \tag{1.6}$$

Subtraction, followed by Minkowski's integral inequality, etc., leads to the desired result,

$$\|y(t) - \bar{y}\| \le \|y_0 - \bar{y}\| e^{(c_T - 1)t}, \tag{1.7}$$

where $c_T \in [0, 1)$ is the contractivity factor of T. One may also examine this problem in terms of semigroups, first by writing (1.1) as

$$\dot{x}(t) = Tx(t) - x(t) = (T - I)x(t)\,. \tag{1.8}$$

The existence of a unique classical solution to (1.8) can be established using the following result.

Theorem 1.1 ([2]). *Let $\mathcal{B}$ be a real Banach space and $T : \mathcal{B} \to \mathcal{B}$ a contraction map on $\mathcal{B}$ with fixed point function $\bar{x}$. Let us suppose that $T - I$ is a closed operator and that the resolvent set of $T - I$ is nonempty. Then for any initial value $x_0 \in \mathcal{B}$, the unique solution $x(t)$ to (1.8) converges exponentially rapidly to $\bar{x}$ as $t \to +\infty$.*

The paper is organized as follows. In Section 2 we provide a brief overview of the method of Iterated Function Systems with Mappings which will be useful in the sequel. In Section 3 we present an extension to set-valued inclusions and we provide some results related to the convergence of fixed points. In Section 4 a different approach, involving set-valued equations in Banach spaces, is considered.

2. Iterated Function Systems with Mappings and their continuous evolution

In general, the action of a generalized fractal transform (GFT) $T : S \to S$ on an element u of the complete metric space (S, d) can be summarized in the following steps. It first produces a set of N spatially-contracted copies of u and then modifies the values of these copies by means of a suitable range-mapping. Finally, it recombines these modified copies by means of an appropriate operator in order to define the element $v \in S$, $v = Tu$. Under appropriate conditions, the fractal transform T is a contraction on (S, d) which, from Banach's Fixed Point Theorem, guarantees the existence of a unique fixed point $\bar{u} = T\bar{u}$.

A special case of GFTs which operate on functions are Iterated Function Systems with Mappings (IFSM), as formulated in [4]. We consider the case in which $u : [0, 1] \to \mathbb{R}$ and thus the IFSM acts on the Banach space

$$\mathcal{B} = \{u : [0, 1] \to \mathbb{R}, u \in L^2[0, 1]\}. \tag{2.1}$$

The ingredients of an N-map IFSM on $\mathcal{B}$ are

(1) A set of N contractive mappings $w = \{w_1, w_2, \dots, w_N\}$, $w_i(x) : [0, 1] \to [0, 1]$, such that $[0, 1] = \cup_{i=1}^N w_i([0, 1])$. In most practical situations, the w_i are assumed to be affine, i.e.,

$$w_i(x) = s_i x + a_i, \quad 0 \le s_i < 1,\ 0 \le a_i \le 1,\ 0 \le s_i + a_i \le 1,\ i = 1, 2, \dots, N; \tag{2.2}$$

(2) A set of associated functions—the so-called *greyscale maps*— $\phi = \{\phi_1, \phi_2, \dots, \phi_N\}$, $\phi_i : \mathbb{R} \to \mathbb{R}$, assumed to be Lipschitz. Once again, in most practical situations, affine maps are employed, i.e.,

$$\phi_i(t) = \alpha_i t + \beta_i, \tag{2.3}$$

Associated with the N-map IFSM (w, ϕ) is the GFT operator T, the action of which on a function $u \in \mathcal{B}$ is given by

$$(Tu)(x) = {\sum_{i=1}^N}' \phi_i(u(w_i^{-1}(x))), \tag{2.4}$$

where the prime denotes that the sum operates on all those terms for which $w_i^{-1}(x)$ is defined.

Theorem 2.1. [4] *$T : \mathcal{B} \to \mathcal{B}$ and for any $u, v \in \mathcal{B}$ we have*

$$d_2(Tu, Tv) \le C d_2(u, v)\,, \tag{2.5}$$

where

$$C = \sum_{i=1}^N s_i^{1/2} \alpha_i. \tag{2.6}$$

When $C < 1$, then T is contractive on X, implying the existence of a unique fixed point $\bar{u} \in \mathcal{B}$ such that $\bar{u} = T\bar{u}$. Also, from Banach's Fixed Point Theorem, the sequence

$$u_{n+1} = Tu_n \tag{2.7}$$

converges to $\bar{u}$ for any initial value u_0.

Example: The 3-map IFSM on $[0,1]$ defined as follows,

$$\begin{gathered} w_1(x) = 0.5x\,, \quad \phi_1(t) = 0.6t + 0.2\,, \\ w_2(x) = 0.4x + 0.3\,, \quad \phi_2(t) = 0.25t + 0.25\,, \\ w_3(x) = 0.6x + 0.4\,, \quad \phi_3(t) = 0.4t + 0.6\,. \end{gathered} \tag{2.8}$$

A quick calculation shows that the Lipschitz constant in (2.6) is $C \approx 0.8922 < 1$, which implies that the fractal transform T associated with this IFSM is contractive. In Figure 1 are plotted the functions $u_1(x), u_2(x)$ and $u_3(x)$ obtained from the iteration process in (2.7) with seed function $u_0(x) = 0$. An approximation to the attractor function $\bar{u}(x)$ is also plotted. These plots were computed over a grid of 2001 equipartition points over $[0,1]$. The approximation to $\bar{u}$ required 43 iterations of T to achieve convergence to one part in 10^{-5}. (The computation time was only 0.003 seconds.)

Note from this figure that the iterate $u_1(x)$ is a piecewise constant function. This is a consequence of the use of the seed function $u_0(x) = 0$ in the iteration process involving an IFSM with affine greyscale maps. In general, from (2.3) and (2.4), if $u_0(x) = 0$, then

$$u_1(x) = \sum_{i=1}^{N} \beta_i \, I_{X_i}(x), \quad x \in [0,1], \tag{2.9}$$

where $X_i = w_i([0,1])$ and $I_S(x)$ denotes the indicator function of a set $S \subseteq [0,1]$. In this example,

$$u_1(x) = 0.2\, I_{[0,0.5]}(x) + 0.25\, I_{[0.3,0.7]}(x) + 0.6\, I_{[0.4,1.0]}(x)\,. \tag{2.10}$$

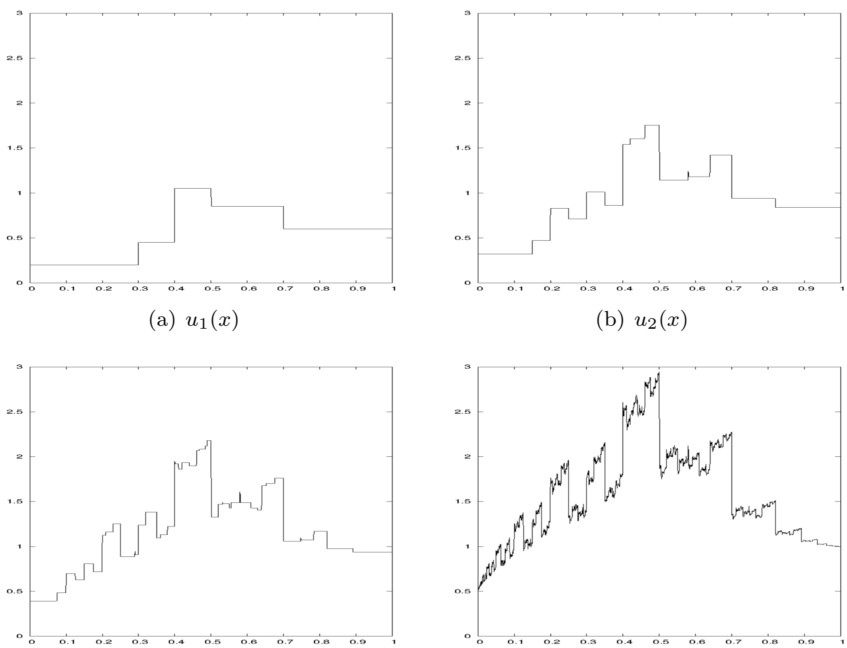
We now consider the continuous evolution associated with this fractal transform T, as defined by (1.1). It is convenient to rewrite the evolution equation in the form

$$\frac{\partial u}{\partial t} = Tu - u, \quad u(x,0) = u_0(x)\,, \tag{2.11}$$

where the solution $u(x,t)$ is now expressed as a function of the spatial variable $x \in [0,1]$ and the time variable $t \geq 0$. From the discussion in the Introduction, all solutions $u(x,t)$ approach the attractor function $\bar{u}(x)$ of T as $t \to \infty$.

In Figure 2 are plotted approximations to the solution $u(x,t)$ to (2.11), with initial condition $u_0(x) = 0$, at times $t = 0.1, 0.5, 1.0$ and 2.0. The solutions were computed over the 2001 equally-spaced gridpoints used in the previous example by means of a simple forward Euler time-difference scheme with step size $h = 0.01$. (Discretization of the time derivative of (1.1) is discussed in [2] as well as in Section 3.1 of this paper.) The vertical scale of Figure 2 has been expanded somewhat from that of Figure 1 in order to accentuate the differences between the graphs. Note that the solution $u(x,1)$ at time $t = 1$ is more "fractal-like" and not identical to the piecewise constant solution $u_1(x)$ obtained from the discrete iteration process of (2.7) and shown in Figure 1. In other words, solutions to the continuous evolution equation do not necessarily interpolate the iterates of the discrete evolution process. This feature was discussed in [2].

The solutions $u(x,1)$ and $u(x,2)$ in Figure 2 demonstrate an evolution of $u(x,t)$ toward the attractor function $\bar{u}(x)$ of T plotted in Figure 1. Further evidence of numerical convergence is shown in Figure 3, where the difference functions



(a) $u_1(x)$ (b) $u_2(x)$

(c) $u_3(x)$ (d) $\bar{u}(x)$

Figure 1. Iterates u_1, u_2 and u_3 produced by the iteration scheme $u_{n+1} = Tu_n$ for the IFSM in the Example in the text, starting with the seed function $u_0(x) = 0$. The attractor function $\bar{u}(x)$ of the IFSM is also shown.

$\Delta(x,t) = \bar{u}(x) - u(x,t)$ are plotted for $t = 5, 10$ and 15. At $t = 20$, the differences between $u(x, 20)$ and $\bar{u}(x)$ are within 2 parts in 10^{-3} over all 2001 gridpoints. (The computation time is only 0.002 seconds.)

3. Set-valued inclusions involving set-valued contraction mappings

We now consider the following evolution inclusion,

$$\dot{y}(t) + y(t) \in T(y(t)), \quad y(0) = y_0 \in \mathcal{B}, \tag{3.1}$$

where $T : \mathcal{B} \to \mathcal{B}$ is a set-valued contraction mapping, as defined below.

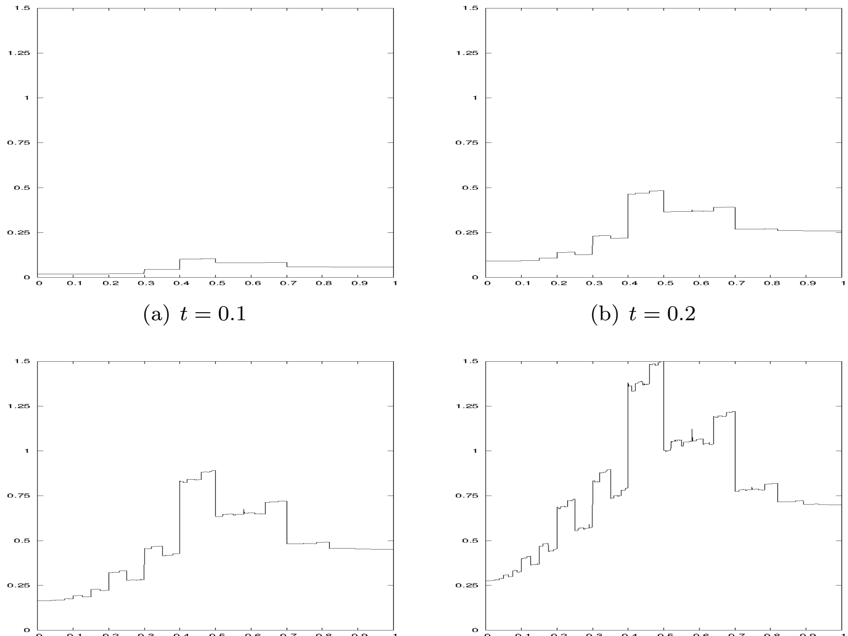
Definition 3.1. Let (Z, d) be a metric space and $T : Z \rightrightarrows Z$ be a set-valued mapping. We say that T is a contraction if there exists a $c_T \in [0, 1)$ such that the following property holds

$$d_H(T(z_1), T(z_2)) \le c_T d(z_1, z_2) \tag{3.2}$$

for all $z_1, z_2 \in Z$, where

$$d_H(T(x_1), T(x_2)) = \max\Big\{ \sup_{a_1 \in Tx_1} \inf_{a_2 \in Tx_2} \|a_1 - a_2\|, \sup_{a_2 \in Tx_2} \inf_{a_1 \in Tx_1} \|a_1 - a_2\| \Big\}$$

is the standard Hausdorff distance between sets.



(a) $t = 0.1$ (b) $t = 0.2$

(c) $t = 1$ (d) $t = 2$

FIGURE 2. Some solutions $u(x,t)$ to the continuous evolution equation (2.11) where T is the fractal transform associated with the 3-map IFSM in (2.8). Initial condition: $u(x,0) = 0$.

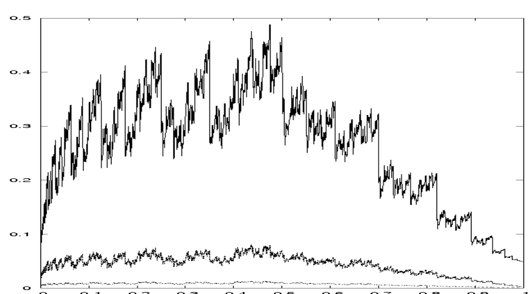


FIGURE 3. A plot of the difference functions $\Delta(x,t) = \bar{u}(x) - u(x,t)$ for $t = 5$ (top), $t = 10$ (middle) and $t = 15$ (bottom) for $0 \le x \le 1$, demonstrating the numerical convergence of $u(x,t)$ to $\bar{u}(x)$.

For compact and convex sets $A, B \subset \mathcal{B}$, it is easy to see that

$$d_H(A,B) = \sup_{p \in \mathcal{B}^*, \|p\|=1} |\operatorname{supp}(p,A) - \operatorname{supp}(p,B)|,$$

where

$$\operatorname{supp}(\cdot, A) : \mathcal{B}^* \to \mathbb{R} \text{ given by } p \mapsto \sup_{a \in A} p(a)$$

is the *support function* of the convex set A (see [11, Ch. 13] for details on support functions).

Theorem 3.2 ([3])**.** *Let $\mathcal{B}$ be a Banach space and $T : \mathcal{B} \rightrightarrows \mathcal{B}$ be a set-valued contraction mapping taking compact and convex values. Then there exists at least one solution $\bar{z} \in \mathcal{B}$ to the fixed point inclusion $z \in T(z)$.*

As mentioned earlier, the complexity of (3.1) is much greater than that of its counterpart in (1.2). The existence of contractive selections is, in general, difficult to establish. It is even difficult to guarantee the existence of Lipschitz selections. For a finite dimensional Banach space the following result holds.

Theorem 3.3. [1] *Consider a Lipschitz set-valued mapping F from a metric space to nonempty closed convex subsets of $\mathbb{R}^n$. Then F has a Lipschitz selection f.*

The extension of this result to infinite dimensional Banach spaces is quite complex and it holds if extra conditions are satisfied. The existence of a continuous selection is guaranteed by the Michael selection theorem (see [9]).

Theorem 3.4 ([9])**.** *A multivalued mapping $T : \mathcal{B}_1 \rightrightarrows \mathcal{B}_2$ admits a continuous single-valued selection, provided that the following conditions are satisfied:*

- *$\mathcal{B}_1$ is a paracompact space,*
- *$\mathcal{B}_2$ is a Banach space,*
- *T is a lower semicontinuous mapping,*
- *For every $x \in X$, $T(x)$ is a nonempty convex subset of $\mathcal{B}_2$, and*
- *For every $x \in X$, $T(x)$ is a closed subset of $\mathcal{B}_2$.*

Since all metric (and thus Banach) spaces are paracompact, we do not need to consider paracompactness in what follows.

We now list some results which guarantee the convergence of trajectories towards fixed points. We assume that the existence of a continuous selection can be guaranteed, for example by assuming that $\mathcal{B}$ is a Banach space and T takes nonempty closed and convex values. We consider two possible cases for the contractive set-valued mapping $T : \mathcal{B} \rightrightarrows \mathcal{B}$.

Case 1: A fixed point $\bar{y} \in \mathcal{B}$ satisfying the equation $\bar{y} = T\bar{y}$ exists. Note that a contractive set-valued mapping $T : \mathcal{B} \rightrightarrows \mathcal{B}$ cannot have two distinct fixed points $\bar{y}_1$ and $\bar{y}_2$ since this would imply that

$$\|\bar{y}_1 - \bar{y}_2\| = d_H(T(\bar{y}_1), T(\bar{y}_2)) \leq c_T \|\bar{y}_1 - \bar{y}_2\| . \tag{3.3}$$

This, in turn, implies that $c_T \geq 1$, contradicting the hypothesis of the contractivity of T.

Proposition 3.5. *Let $\mathcal{B}$ be a real Banach space, $T : \mathcal{B} \rightrightarrows \mathcal{B}$ be a set-valued contraction mapping taking nonempty closed and convex values, and $v : \mathcal{B} \to \mathcal{B}$, $v(x) \in T(x)$ for all $x \in \mathcal{B}$, be a continuous selection. Let $y(t) : [0, +\infty) \to \mathcal{B}$ be a continuous solution to the evolution equation*

$$\dot{y}(t) + y(t) = v(y(t)), \quad y(0) = y_0 \in \mathcal{B} \tag{3.4}$$

Suppose that there exists a $\bar{y} \in \mathcal{B}$ such that $\bar{y} = T\bar{y}$. Then $\lim_{t \to +\infty} \|y(t) - \bar{y}\| = 0$.

Proof. We proceed in a manner somewhat parallel to the approach employed in [2]. Let $y : [0, +\infty) \to \mathcal{B}$ be a solution to the equation

$$\dot{y}(t) + y(t) = v(y(t)) . \tag{3.5}$$

Multiplying both sides by e^t, we obtain

$$e^t\dot{y}(t) + e^t y(t) = e^t v(y(t)); \tag{3.6}$$

that is,

$$(e^t y(t))' = e^t v(y(t)). \tag{3.7}$$

This implies that

$$y(t) = y_0 e^{-t} + \int_0^t e^{s-t} v(y(s))\, ds. \tag{3.8}$$

In the case $y_0 = \bar{y}$, the above equation becomes

$$\bar{y} = \bar{y}e^{-t} + \int_0^t e^{s-t}\, \bar{y}\, ds\,. \tag{3.9}$$

Subtracting (3.9) from (3.8) yields

$$y(t) - \bar{y} = e^{-t}(y_0 - \bar{y}) + \int_0^t e^{s-t}(v(y(s)) - \bar{y})\, ds\,, \tag{3.10}$$

from which it follows that

$$\|y(t) - \bar{y}\| \le e^{-t}\|y_0 - \bar{y}\| + \int_0^t e^{s-t}\|v(y(s)) - \bar{y}\|\, ds\,. \tag{3.11}$$

For each fixed $s \in [0, t]$, we take the the supremum with respect to $v \in T(y(s))$ to obtain

$$\begin{aligned} \|y(t) - \bar{y}\| &\le e^{-t}\|y_0 - \bar{v}\| + \int_0^t e^{s-t} \sup_{v\in T(y(s))} \|v - \bar{y}\|\, ds \\ &\le e^{-t}\|y_0 - \bar{y}\| + \int_0^t e^{s-t}\, d_H(T(y(s)), T(\bar{y}))\, ds \\ &\le e^{-t}\|y_0 - \bar{y}\| + c_T \int_0^t e^{s-t}\|y(s) - \bar{y}\|\, ds\,. \end{aligned} \tag{3.12}$$

Gronwall's lemma then implies that for $t \ge 0$,

$$\|y(t) - \bar{y}\| \le \|y_0 - \bar{y}\| e^{(c_T - 1)t}\,. \tag{3.13}$$

Since $c_T \in [0, 1)$, it follows that $\|y(t) - \bar{y}\| \to 0$ as $t \to \infty$, once again exponentially rapidly. □

Case 2: No points $\bar{y} \in \mathcal{B}$ satisfying the fixed point equation $\bar{y} = T\bar{y}$ exist.

Proposition 3.6. *Let $\mathcal{B}$ be a real Banach space, $T : \mathcal{B} \rightrightarrows \mathcal{B}$ be a set-valued contraction mapping taking nonempty closed and convex values, and $v : \mathcal{B} \to \mathcal{B}$, $v(x) \in T(x)$ for all $x \in \mathcal{B}$, be a continuous selection. Let $\bar{y} \in T\bar{y}$ be one of its fixed points – in the sense of inclusions – and suppose that*

$$\sup_{s\in T(x)} \sup_{l\in T(\bar{y})} \|s - l\| \le c\,\|x - \bar{y}\| \tag{3.14}$$

for all $x \in \mathcal{B}$ and for some $c \in [0, 1)$. Furthermore, let $y(t) : [0, +\infty) \to \mathcal{B}$ be a solution to the evolution equation

$$\dot{y}(t) + y(t) = v(y(t)), \quad y(0) = y_0 \in \mathcal{B}. \tag{3.15}$$

Then $\lim_{t\to+\infty} \|y(t) - \bar{y}\| = 0$.

Proof. The first part of the proof is identical to that of Proposition 3.5 and is therefore omitted. The only difference lies in the following development,

$$\begin{aligned}\|y(t)-\bar{y}\| &\le e^{-t}\|y_0-\bar{y}\| + \int_0^t e^{s-t}\|v(y(s))-\bar{y}\|\,ds \\ &\le e^{-t}\|y_0-\bar{y}\| + \int_0^t e^{s-t} \sup_{v\in T(y(s))}\sup_{l\in T\bar{y}}\|v-l\|\,ds \\ &\le e^{-t}\|y_0-\bar{y}\| + c\int_0^t e^{s-t}\|y(s)-\bar{y}\|\,ds.\end{aligned} \tag{3.16}$$

Once again, Gronwall's lemma can be used to establish the desired result. □

Lemma 3.7. *Let $\mathcal{B}$ be a real Banach space, $T:\mathcal{B}\rightrightarrows\mathcal{B}$ be a set-valued contraction mapping taking nonempty closed and convex values and let $\bar{y}\in\mathcal{B}$ one of its fixed points. Construct the single-valued map*

$$\psi(x) = \Pi_{T(x)}\bar{y}\,, \tag{3.17}$$

where $\Pi_{T(x)}\bar{y}$ is the projection of $\bar{y}$ onto $T(x)$. Then ψ is locally contractive at $\bar{y}$.

Proof. By computing, we have:

$$\|\psi(x)-\psi(\bar{y})\| = \|\psi(x)-\bar{y}\| = d(T(x),\bar{y}) \le d_H(T(x),T(\bar{y})) \le c\|x-\bar{y}\|. \tag{3.18}$$

□

Proposition 3.8. *Let $\mathcal{B}$ be a real Banach space and $T:\mathcal{B}\rightrightarrows\mathcal{B}$ be a set-valued contraction mapping taking nonempty closed and convex values. Let $\bar{y}\in T\bar{y}$ be one of its fixed points – in the sense of inclusions. Now consider the selection $\psi:\mathcal{B}\to\mathcal{B}$ defined as follows,*

$$\psi(x) = \Pi_{T(x)}\bar{y}\,, \tag{3.19}$$

where $\Pi_{T(x)}\bar{y}$ denotes the projection of the element $\bar{y}$ onto the set $T(x)$. Suppose that the differential equation,

$$\dot{y}(t) + y(t) = \psi(y(t)), \quad y(0) = y_0 \in \mathcal{B}, \tag{3.20}$$

admits a solution, $\forall t\ge 0$. Then $\lim_{t\to+\infty}\|y(t)-\bar{y}\| = 0$.

Proof. Once again, the first part of the proof is identical to that of Proposition 3.5 and is omitted. We proceed with the following development,

$$\begin{aligned}\|y(t)-\bar{y}\| &\le e^{-t}\|y_0-\bar{y}\| + \int_0^t e^{s-t}\|\psi(y(s))-\bar{y}\|\,ds \\ &\le e^{-t}\|y_0-\bar{y}\| + \int_0^t e^{s-t}\inf_{v\in T(y(s))}\|v-\bar{y}\|\,ds \\ &\le e^{-t}\|y_0-\bar{y}\| + \int_0^t e^{s-t}\sup_{l\in T(\bar{y})}\inf_{v\in T(y(s))}\|v-l\|\,ds \\ &\le e^{-t}\|y_0-\bar{y}\| + \int_0^t e^{s-t}\, d_H(T(y(s)),T(\bar{y}))\,ds \\ &\le e^{-t}\|y_0-\bar{y}\| + c_T\int_0^t e^{s-t}\,\|y(s)-\bar{y}\|\,ds.\end{aligned} \tag{3.21}$$

The proof then follows from Gronwall's lemma. □

Since a set-valued contraction mapping has, in general, more than one fixed point it looks reasonable to use a particular $\bar{y}$ in the construction of (3.20). The following result states a global convergence result.

Proposition 3.9. *Let $\mathcal{B}$ be a real Banach space, $T : \mathcal{B} \rightrightarrows \mathcal{B}$ be a set-valued contraction mapping taking nonempty closed and convex values. Let $\bar{y} \in T\bar{y}$ be one of its fixed points – in the sense of inclusions – and suppose that*

$$d_H(Tx, \bar{y}) \le c\|x - \bar{y}\| \tag{3.22}$$

for all $x \in \mathcal{B}$ and for some $c \in [0, 1)$. Let $y(t) : [0, +\infty) \to \mathcal{B}$ be a continuous solution to the evolution inclusion

$$\dot{y}(t) + y(t) \in T(y(t)), \quad y(0) = y_0 \in \mathcal{B}. \tag{3.23}$$

Then $\lim_{t\to+\infty} \|y(t) - \bar{y}\| = 0$.

Proof. By taking any p in the dual of $\mathcal{B}$, we obtain

$$p(y(t))' + p(y(t)) \le \operatorname{supp}(p, T(y(t))). \tag{3.24}$$

Standard arguments imply that the function $t \to \operatorname{supp}(p, T(y(t)))$ is measurable, and so we have

$$p(y(t)) \le e^{-t} \int_0^t e^s \operatorname{supp}(p, T(y(s)))ds + e^{-t} p(y(0)). \tag{3.25}$$

Then

$$\begin{aligned} p(y(t)) - p(\bar{y}) &\le e^{-t} \int_0^t e^s (\operatorname{supp}(p, T(y(s))) - p(\bar{y}))ds + e^{-t}(p(y(0)) - p(\bar{y})) \\ &\le e^{-t} \int_0^t e^s (\operatorname{supp}(p, T(y(s))) - \operatorname{supp}(p, \bar{y}))ds + e^{-t}(p(y(0)) - p(\bar{y})) \\ &\le e^{-t} \int_0^t e^s d_H(T(y(s)), \bar{y})ds + e^{-t}(p(y(0)) - p(\bar{y})) \\ &\le e^{-t} \int_0^t e^s \|y(s) - \bar{y}\|ds + e^{-t}(p(y(0)) - p(\bar{y})). \end{aligned} \tag{3.26}$$

Gronwall's lemma can once again be used to establish the desired result. □

3.1. **Discretization.** As studied in [2], employing the simple forward Euler scheme with time step $h > 0$ for the derivative in (1.1) leads to the discrete dynamical system,

$$y_{n+1} = y_n + (Ty_n - y_n)h, \quad n = 0, 1, 2, \cdots . \tag{3.27}$$

Here, y_0 is the initial condition. In the special case $h = 1$, (3.27) becomes the usual iteration procedure,

$$y_{n+1} = Ty_n, \tag{3.28}$$

which necessarily converges to the fixed point $\bar{y} = T\bar{y}$.

The continuous evolution inclusion,

$$\dot{y}(t) + y(t) \in T(y(t)), \tag{3.29}$$

can also be discretized according to classical numerical schemes, including the above-mentioned Euler scheme with time step $h > 0$, yielding the discrete dynamical inclusion,

$$y_{n+1} - y_n \in (T(y_n) - y_n)h, \tag{3.30}$$

where, once again, $y_0 \in \mathcal{B}$ denotes the initial condition. When $h = 1$, (3.30) becomes

$$y_{n+1} \in T(y_n). \tag{3.31}$$

Let us suppose that $\bar{y}$ is a fixed point of T and construct the map $\psi(x) = \Pi_{T(x)}\bar{y}$. In the special case that for each $n \geq 1$, we choose the particular point,

$$y_{n+1} = \Pi_{T(y_n)}\bar{y} \in T(y_n), \tag{3.32}$$

that is, the projection Π of $\bar{y}$ onto $T(y_n)$, then we have a converging discrete iteration process (see also [5]).

3.2. **An application to fractal transforms.** The idea of using set-valued analysis in fractal theory is recent (see [6] and references therein). In this paragraph we introduce a set-valued extension of the classical IFSM operator which satisfies the hypotheses of the main results presented in the previous section. Let $X = [0,1]$ and $\mathcal{B} = L^p([0,1])$ be the Banach space of all p-integrable functions and consider the set-valued map $T : L^p([0,1]) \rightrightarrows L^p([0,1])$ defined as

$$\begin{aligned}(Tf)(x) = \Big\{ g \in L^p([0,1]) : g(x) = \sum_{i=1}^{n} \alpha_i f(w_i^{-1}(x)) + \beta_i, \alpha_i \in [\gamma_1, \gamma_2], \\ \beta_i \in [\theta_1, \theta_2] \Big\} \subset L^p([0,1])\end{aligned} \tag{3.33}$$

where $w_i : [0,1] \to [0,1]$ is a set of N contractive affine maps, $w_i(x) = s_i x + a_i$, $i = 1 \dots N$. The following results prove that T takes convex and closed values and that it is a contraction.

Proposition 3.10. *$T : L^p([0,1]) \rightrightarrows L^p([0,1])$ takes compact and convex values.*

Proof. It is easy to see that for all $f \in L^p([0,1])$ we have that Tf is a compact subset of $L^p([0,1])$, this follows from the fact that Tf is homeomorphic to the compact set $[\gamma_1, \gamma_2] \times [\theta_1, \theta_2]$. To prove Tf is convex, let $\xi_1, \xi_2 \in Tf$ and $\lambda \in [0,1]$. We have

$$\xi_1(x) = \sum_{i=1}^{N} \alpha_i^* f(w_i^{-1}(x)) + \beta_i^*, \tag{3.34}$$

$$\xi_2(x) = \sum_{i=1}^{N} \alpha_i^{**} f(w_i^{-1}(x)) + \beta_i^{**} \tag{3.35}$$

and then

$$\lambda\xi_1(x) + (1-\lambda)\xi_2(x) = \sum_{i=1}^{N} (\lambda\alpha_i^* + (1-\lambda)\alpha_i^{**}) f(w_i^{-1}(x)) + \lambda\beta_i^* + (1-\lambda)\beta_i^{**} \tag{3.36}$$

which proves the thesis. □

Proposition 3.11. *$T : L^p([0,1]) \rightrightarrows L^p([0,1])$ is a contractive multifunction if $\sum_{i=1}^{N} s_i^{1/p} \max\{|\gamma_1|, |\gamma_2|\} < 1$.*

Proof. By means of some straightforward calculations,

$$\max_{\alpha_i^{**} \in [\gamma_1,\gamma_2],\ \beta_i^{**} \in [\theta_1,\theta_2]} \ \min_{\alpha_i^{*} \in [\gamma_1,\gamma_2],\ \beta_i^{*} \in [\theta_1,\theta_2]} \Big\| \sum_{i=1}^{N} \alpha_i^* f_1(w_i^{-1}(x)) + \beta_i^*$$

$$
\begin{aligned}
&- \sum_{i=1}^{N} \alpha_i^{**} f_2(w_i^{-1}(x)) - \beta_i^{**} \Big\|_p \\
&\le \max_{\alpha_i^{**}\in[\gamma_1,\gamma_2],\ \beta_i^{**}\in[\theta_1,\theta_2]} \Big\| \sum_{i=1}^{N} \alpha_i^{**} f_1(w_i^{-1}(x)) - \sum_{i=1}^{N} \alpha_i^{**} f_2(w_i^{-1}(x)) \Big\|_p \\
&= \max_{\alpha_i^{**}\in[\gamma_1,\gamma_2]} \Big\| \sum_{i=1}^{N} \alpha_i^{**} f_1(w_i^{-1}(x)) - \sum_{i=1}^{N} \alpha_i^{**} f_2(w_i^{-1}(x)) \Big\|_p \\
&\le \Big(\sum_{i=1}^{N} s_i^{1/p} \max\{|\gamma_1|, |\gamma_2|\} \Big) \|f_1 - f_2\|_p \,,
\end{aligned}
$$

which implies that T is contractive multifunction. □

The results presented in the previous section can be applied to this set-valued mapping.

4. Alternative approach: Set-valued ODEs in Banach spaces

A completely different alternative approach to generalizing the results of [2] is to define a set-valued evolution equation. This can be done in two equivalent ways, either directly as a set-valued ODE or as an ODE in a Banach space via the natural embedding of the space of compact and convex sets as the positive cone in a Banach space [10]. In this section we discuss this approach.

Let $\mathcal{B}$ be a Banach space and $\mathcal{K}_{cc} = \{A \subset \mathcal{B} : \emptyset \neq A \text{ is compact and convex}\}$. Recall that $\mathcal{K}_{cc}$ is complete under the Hausdorff metric. Let $\mathcal{S}^* = \{p \in \mathcal{B}^* : \|p\| = 1\}$ be the collection of all continuous linear functionals on $\mathcal{B}$ of unit norm. We assume that $T : \mathcal{B} \rightrightarrows \mathcal{B}$ is a contractive set-valued function with $T(x) \in \mathcal{K}_{cc}$ for all x. We also assume that T satisfies the convexity condition

$$
\lambda T(x) + (1-\lambda)T(y) \subseteq T(\lambda x + (1-\lambda)y). \tag{4.1}
$$

These conditions on T imply that if $A \in \mathcal{K}_{cc}$ then $T(A) = \{T(a) : a \in A\} \in \mathcal{K}_{cc}$ as well. In particular, this means that T induces a mapping $T : \mathcal{K}_{cc} \to \mathcal{K}_{cc}$ with the same contractivity factor as T and so there is some $\bar{y} \in \mathcal{K}_{cc}$ with $T(\bar{y}) = \bar{y}$.

Definition 4.1. For a set-valued mapping $T : \mathbb{R} \to \mathcal{K}_{cc}$, we say that $A \in \mathcal{K}_{cc}$ is the *derivative of T at t_0* if

$$
\lim_{h\to 0^+} \frac{d_H(T(t_0 - h) + Ah, T(t_0))}{h} = 0, \quad \lim_{h\to 0^+} \frac{d_H(T(t_0 + h), T(t_0) + Ah)}{h} = 0. \tag{4.2}
$$

In this case, we write $A = T'(t_0)$.

This is a natural extension of the notion of derivative for a function $f : \mathbb{R}^p \to \mathbb{R}^q$ but where we now use the Hausdorff metric to measure distance. This definition of derivative is also very naturally related to the embedding from [10].

The idea behind this embedding is that $\mathcal{K}_{cc}$ has a natural addition (Minkowski addition) and a natural scalar multiplication (at least for $\lambda \geq 0$). In addition, $\lambda(A + B) = \lambda A + \lambda B$ for all λ and $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$ as long as λ_1 and λ_2 have the same sign. Furthermore, the Hausdorff distance on $\mathcal{K}_{cc}$ defines a natural "norm" by $\|A\| = d_H(A, \{0\})$, since $\{0\}$ is the additive identity for this semigroup. The semigroup structure, along with the norm, is used to define the Banach space

$\mathbb{X}$ in which $\mathcal{K}_{cc}$ is embedded as a closed cone. The space $\mathbb{X}$ can be chosen so that the closed cone has nonempty interior and we do so. Since the norm on $\mathbb{X}$ is given by an extension of the Hausdorff distance between elements of $\mathcal{K}_{cc}$, the definition of derivative in (4.2) agrees with the derivative as defined in $\mathbb{X}$, at least for elements of the cone associated with $\mathcal{K}_{cc}$.

Using definition 4.1, we interpret (1.2) directly as a set-valued evolution equation with initial condition $y(0) = y_0 \in \mathcal{K}_{cc}$.

By using support functions we can convert the set-valued ODE (1.2) into a collection of scalar ODEs. That is, for each $p \in \mathcal{S}^*$ (each linear functional p of norm one), we have

$$\begin{gathered}\operatorname{supp}(p, y'(t)) + \operatorname{supp}(p, y(t)) = \operatorname{supp}(p, T(y(t)), \\ \operatorname{supp}(p, y(0)) = \operatorname{supp}(p, y_0) \quad \text{for } y_0 \in \mathcal{K}_{cc}.\end{gathered} \tag{4.3}$$

Now for $A, B \in \mathcal{K}_{cc}$, we have $d_H(A, B) = \sup\{|\operatorname{supp}(p, A) - \operatorname{supp}(p, B)| : p \in \mathcal{S}^*\}$ and thus if $A = T'(t_0)$, then we have for all $p \in \mathcal{B}^*$,

$$\operatorname{supp}(p, A) = \operatorname{supp}(p, T'(t_0)) = \frac{d}{dt}\left(\operatorname{supp}(p, T(t))\right|_{t=t_0}. \tag{4.4}$$

Thus, if we let $y_p(t) = \operatorname{supp}(p, y(t))$, we have the collection of ODEs (one for each $p \in \mathcal{S}^*$):

$$y_p'(t) + y_p(t) = \operatorname{supp}(p, T(y(t))), \quad y_p(0) = \operatorname{supp}(p, y_0). \tag{4.5}$$

Notice that since T is contractive, $x \mapsto \operatorname{supp}(p, T(x))$ is Lipschitz for all $p \in \mathcal{S}^*$ and thus we have existence and uniqueness of solutions for (4.5) for all $p \in \mathcal{S}^*$. However, this by itself is not enough to have existence and uniqueness of solutions to the set-valued ODE. In order for $y_p(t)$ (as p ranges over $\mathcal{S}^*$) to be a collection of support functions for some compact and convex set $y(t)$, these functions must satisfy some additional conditions. These conditions are not obviously true just from being solutions to (4.5).

However, looking at (1.2) instead as an ODE in the embedded Banach space $\mathbb{X}$ we do have existence and uniqueness of solutions in $\mathbb{X}$, and as long as these solutions remain in the cone associated with $\mathcal{K}_{cc}$ we can interpret $y(t)$ as an element of $\mathcal{K}_{cc}$ and thus $y_p(t)$ is a collection of support functions for $y(t) \in \mathcal{K}_{cc}$. Our next result shows that the cone $\mathcal{K}_{cc}$ in $\mathbb{X}$ is positively invariant and thus $y(t) \in \mathcal{K}_{cc}$ for all $t \geq 0$.

Lemma 4.2. *Let $\mathbb{X}$ be a Banach space, $K \subset \mathbb{X}$ be closed and convex and $T : \mathbb{X} \to \mathbb{X}$ be Lipschitz and satisfy $T(K) \subseteq K$. Then the solution $y(t)$ to*

$$y' + y = T(y), \quad y(0) = y_0 \in K \tag{4.6}$$

satisfies $y(t) \in K$ for all $t \geq 0$.

Proof. We know that the solution to (4.6) can be written as

$$y(t) = y_0 e^{-t} + e^{-t} \int_0^t e^{-s} T(y(s))\, ds. \tag{4.7}$$

Choose $p \in \mathcal{S}^*$ and let $u = \operatorname{supp}(p, K)$. Then

$$\begin{aligned} p(y(t)) &= e^{-t} p(y_0) + e^{-t} \int_0^t e^{-s} p(T(y(s)))\, ds \\ &\leq e^{-t} u + e^{-t} \int_0^t e^{-s} u\, ds \end{aligned}$$

$$= e^{-t}u + u(1 - e^{-t}) = u.$$

That is, $p(y(t)) \le u := \operatorname{supp}(p, K)$ for all $t \ge 0$. Since this is true for all $p \in \mathcal{S}^*$, this means that $y(t) \in K$ for all $t \ge 0$. □

In summary, we can either view (1.2) as a set-valued evolution equation or alternatively as a family (4.5) of scalar ODEs, one for each "direction" $p \in \mathcal{S}^*$.

With this framework in place, we have the following result. All the hard work is in setting the framework.

Proposition 4.3. *Let $T : \mathcal{B} \to \mathcal{K}_{cc}$ be contractive and satisfy the convexity condition* (4.1)*. Then the solution $y : [0, \infty) \to \mathcal{K}_{cc}$ of the set-valued evolution equation*

$$y'(t) + y(t) = T(y(t)), \quad y(0) = y_0 \in \mathcal{K}_{cc}, \tag{4.8}$$

satisfies $d_H(y(t), \bar{y}) \to 0$ as $t \to \infty$.

The idea of this proof is the same as the proof of [2, Theorem 1], and is omitted.

4.1. **Weak* compact and convex sets.** For some applications (in particular to spaces of probability measures) it is important to be able to use weak* compact sets rather than norm compact sets. Since the embedding from [10] also works in this situation, the same framework will work. To this end, in the case where $\mathcal{B}$ is separable and the dual of a Banach space, we also define $\mathcal{K}_{wcc} = \{A \subset \mathcal{B} : \emptyset \ne A$ is bounded, weak* compact and convex$\}$.

Since $\mathcal{B}$ is separable, the weak* topology is metrizable but only on norm bounded subsets; this is the reason for the boundedness restriction in the definition of $\mathcal{K}_{wcc}$. We need T to preserve boundedness, so we assume that

$$\text{there is some } \kappa > 0 \text{ so that whenever } \|A\| \le \kappa \text{ then } \|T(A)\| \le \kappa \text{ as well.} \tag{4.9}$$

With this, we use as our "base" space $Y = \{x \in \mathcal{B} : \|x\| \le \kappa\}$ and then $\Omega = \{A \subseteq Y : \emptyset \ne A$ is weak* compact and convex$\}$ replaces $\mathcal{K}_{wcc}$. Since $Y \subset \mathcal{B}$ is bounded, the weak* topology on Y is metrizable and this in turn induces the Hausdorff metric on Ω. Further, Ω is complete because Y is complete. Note that we also have $\Omega = \{A \subset \mathcal{B} : A \in \mathcal{K}_{wcc}, \|A\| \le \kappa\}$. As a subspace of $\mathbb{X}$, Ω is the cone $\mathcal{K}_{wcc}$ intersected with the ball of radius κ centered at 0. In particular, Ω is weak* compact and convex.

Given all of this setup, we obtain the same result as before.

Proposition 4.4. *Let $T : \mathcal{B} \to \mathcal{K}_{wcc}$ be contractive and satisfy the convexity condition* (4.1) *and the boundedness condition* (4.9)*. Let $y_0 \in \mathcal{K}_{wcc}$ be such that $\|y_0\| \le \kappa$. Then the solution $y : [0, \infty) \to \mathcal{K}_{cc}$ of the set-valued evolution equation*

$$y'(t) + y(t) = T(y(t)), \quad y(0) = y_0, \tag{4.10}$$

satisfies $d_H(y(t), \bar{y}) \to 0$ as $t \to \infty$.

The collection of ODEs in (4.5), one for each $p \in \mathcal{S}$, gives a practical method for obtaining a finite dimensional polyhedral approximation to the set-valued solution of the evolution equation (4.10). This is especially simple in the case that $\mathcal{B}$ is a Hilbert space.

The strategy is to choose finitely many $p_i \in \mathcal{S}, i = 1, 2, \dots, M$ and solve the corresponding scalar ODEs (4.5) for each p_i; call this solution $y_i(t)$. Then define the polyhedron $P(t)$ to be the convex hull, $\overline{co}(y_i(t)p_i)$, of the points $y_i(t)p_i \in \mathcal{B}$. Since y_0 is compact, it is certainly possible to choose p_i so that $\overline{co}(p_i)$ closely approximates y_0

in the Hausdorff distance. In principle because $y(t)$ is a differentiable function, it is also possible to choose the p_i in such a way that $\overline{co}(y_i(t)p_i)$ is a close approximation to $y(t)$ for all t in some finite time interval $[0,\tau]$. However, in practice it is difficult to know *a priori* which p_i to choose since $y(t)$ is changing. In the example below (using (4.11) and (4.13)), the first few elements of a Haar basis for $L^2([0,1])$ adapted to the IFS $\{w_i\}$ would work well.

4.2. **An application to fractal transforms.** Let $X=[0,1]$ and $\mathcal{B}=L^p([0,1])$ be the Banach space of all p-integrable functions and $\mathcal{K}_{cc}$ be the space of all nonempty, compact and convex subsets of $\mathcal{B}$. Let $w_i : X \to X$, $i=1,2,\dots,N$, be affine with the contraction factor of w_i being s_i and let $\alpha_i \in \mathbb{R}$. Then for any $f \in L^p([0,1])$ the function $\sum_i \alpha_i f \circ w^{-1}$ is in $L^p([0,1])$ as well. Next choose a fixed $\beta \in \mathcal{K}_{cc}$ (so that $\beta \subset L^p([0,1])$ is compact and convex) and consider the set-valued map $T : L^p([0,1]) \rightrightarrows L^p([0,1])$ defined as

$$(Tf) = \beta + \sum_i \alpha_i f \circ w_i^{-1}. \tag{4.11}$$

Notice that by definition $T(f) \in \mathcal{K}_{cc}$ for each $f \in \mathcal{B}$, since $T(f)$ is a translate of β.

For $\lambda \in [0,1]$ we have

$$\begin{aligned}
(\lambda T(f) + (1-\lambda)T(g)) &= \lambda\Big(\beta + \sum_i \alpha_i f \circ w_i^{-1}\Big) + (1-\lambda)\Big(\beta + \sum_i \alpha_i g \circ w_i^{-1}\Big)\\
&= \lambda\beta + (1-\lambda)\beta + \sum_i \alpha_i(\lambda f + (1-\lambda)g) \circ w_i^{-1}\\
&= \beta + \sum_i \alpha_i(\lambda f + (1-\lambda)g) \circ w_i^{-1}\\
&= T(\lambda f + (1-\lambda)g).
\end{aligned}$$

Note that $\lambda\beta + (1-\lambda)\beta = \beta$ since $\beta \in \mathcal{K}_{cc}$. Thus T satisfies the convexity condition (4.1) and so T induces a mapping $T : \mathcal{K}_{cc} \to \mathcal{K}_{cc}$.

Proposition 4.5. *$T : L^p([0,1]) \rightrightarrows L^p([0,1])$ as defined in* (4.11) *is a contractive multifunction if $\sum_{i=1}^n s_i^{1/p}|\alpha_i| < 1$.*

Proof. This proof is very similar to the proof of proposition 3.11. With s_i the contraction factor of w_i and γ_1, γ_2 selections of β, we compute

$$\begin{aligned}
&\max_{\gamma_1 \in \beta} \min_{\gamma_2 \in \beta} \|\gamma_1 + \sum_i \alpha_i f \circ w_i^{-1} - \gamma_2 - \sum_i \alpha_i g \circ w_i^{-1}\|_p\\
&\le \|\sum_i \alpha_i (f \circ w_i^{-1} - g \circ w_i^{-1})\|_p\\
&\le \sum_i |\alpha_i| \|f \circ w_i^{-1} - g \circ w_i^{-1}\|_p\\
&\le \Big(\sum_i s_i^{1/p}|\alpha_i|\Big)\|f-g\|_p.
\end{aligned}$$

Note that the inequality in the first line comes from choosing $\gamma_2 = \gamma_1$. Thus we have our result. □

In particular, the induced mapping $T : \mathcal{K}_{cc} \to \mathcal{K}_{cc}$ has the same Lipschitz constant of $\sum_i s_i^{1/p}|\alpha_i|$ as T. Thus we have a unique fixed set $A \in \mathcal{K}_{cc}$ and the solution $y(t)$ to the set-valued evolution equation (1.2) will converge to A. As in section 5 of [8], we can write the fixed set A of T in $\mathcal{K}_{cc}$ as

$$A = \beta + \sum_{n=1}^{\infty} \sum_{\sigma \in \{1,2,\dots,N\}^n} \alpha_{\sigma_1}\alpha_{\sigma_2}\cdots\alpha_{\sigma_n}\beta \circ (w_{\sigma_1}^{-1} \circ w_{\sigma_2}^{-1} \circ \dots \circ w_{\sigma_n}^{-1}), \tag{4.12}$$

where we use the notation $\beta \circ \phi = \{g \circ \phi : g \in \beta\}$. Each set in the sum (4.12) is in $\mathcal{K}_{cc}$ and the limit for the infinite sum is taken in the Hausdorff metric on $\mathcal{K}_{cc}$.

For a more definite example, we can take

$$\beta = \cup_{c \in [\beta_1, \beta_2]} \{g(x) = c \text{ for all } x \in [0,1]\} \subset L^p([0,1]). \tag{4.13}$$

Then the operator (4.11) is related in a simple way to that in (3.33) in that we vary the β values but not the αs. Using this choice of β, for each $g \in \beta$, the IFSM (see [4])

$$T_g(f) = g + \sum_i \alpha_i f \circ w_i^{-1} \tag{4.14}$$

has a unique fixed point f_g and the fixed set A of T (given by (4.12)) contains all of these fixed points. Each of these operators T_g could be thought of as a "selection" of the operator T. In a sense, the set-valued evolution will converge to a set which contains all the fixed points f_g for all the "selections" T_g of T. Of course, A contains much more than just the collection $\{f_g : g \in \beta\}$, just like the attractor of a standard IFS contains much more than the collection of the fixed points of the individual w_i. Even though this β is one-dimensional, the limiting set A is infinite dimensional. To see this, we just notice that $T(\beta)$ consists of functions which are piecewise constant on the N sets $w_i([0,1])$, $T^2(\beta)$ consists of functions which are piecewise constant on the N^2 sets $w_i \circ w_j([0,1])$, etc..

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