Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 146, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STURM-PICONE TYPE THEOREMS FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

AYDIN TIRYAKI

Abstract. The aim of this article is to give Sturm-Picone type theorems for the pair of second-order nonlinear differential equations

$$(p_1(t)|x'|^{\alpha-1}x')' + q_1(t)f_1(x) = 0$$
$$(p_2(t)|y'|^{\alpha-1}y')' + q_2(t)f_2(y) = 0, \quad t_1 < t < t_2$$

in both regular and singular cases. Our results include some earlier results and generalize the well-known comparison theorems given by Sturm [19], Picone [18] and Leighton [15] which play a key role in the qualitative behaviour of the solutions.

1. Introduction

In 1836 the first important comparison theorem was given by Sturm [14, 20], which deals with a pair of linear ordinary differential equations

$$\big(p_1(t)x'\big)' + q_1(t)x = 0, \tag{1.1}$$
$$\big(p_2(t)y'\big)' + q_2(t)y = 0 \tag{1.2}$$

on a bounded interval (t_1, t_2) where p_1, q_1, p_2, q_2 are real-valued continuous functions and $p_1(t) > 0$, $p_2(t) > 0$ on $[t_1, t_2]$. In this celebrated paper, Sturm [19] proved the following remarkable result.

Theorem 1.1 (Sturm's Comparison Theorem). *Suppose $p_1(t) = p_2(t)$ and $q_1(t) > q_2(t)$, $\forall t \in (t_1, t_2)$. If there exists a nontrivial real solution y of* (1.2) *such that $y(t_1) = 0 = y(t_2)$, then every real solution of* (1.1) *has at least one zero in (t_1, t_2).*

In 1909, Picone [18] modified Sturm's theorem as follows.

Theorem 1.2 (Sturm-Picone Theorem). *Suppose that $p_2(t) \geq p_1(t)$ and $q_1(t) \geq q_2(t)$, for all $t \in (t_1, t_2)$. If there exists a nontrivial real solution y of* (1.2) *such that $y(t_1) = 0 = y(t_2)$, then every real solution of* (1.1) *unless a constant multiple of y has at least one zero in (t_1, t_2)*

Note that Theorem 1.2 is a special case of Leighton's theorem [15]. For a detailed study and earlier developments of this subject, we refer the reader to the books

2000 *Mathematics Subject Classification.* 34C10, 34C15.
Key words and phrases. Comparison theorem; Sturm-Picone theorem; half-linear; second order differential equations; singular equation; variational lemma.

Submitted March 5, 2014. Published June 20, 2014.

[14, 20]. Sturm-Picone theorem is extended in several directions, see [2] and [3] for linear systems, [17] for nonselfadjoint differential equations, [22] for implicit differential equations, [7, 11, 16] for half-linear equations, [6] for degenerate elliptic equations, [26] for linear equations on time scales. There is also a good amount of interest in the qualitative theory of partial differential equations to determine whether the given equation is oscillatory or not and Sturm-Picone theorem, also plays an important role in this direction. For earlier developments, we refer to [18, 19, 20], and for recent developments we refer to Yoshida's book [24]. Sturm comparison theorem for the half-linear elliptic equation and Picone type identities have been studied in, for example, [4, 6, 7, 9, 12, 13, 21, 25].

When some or all of p_1, q_1, p_2, q_2 are not continuous at t_1 or t_2 or at t_1 and t_2 both, where the possibility that the interval is unbounded is not excluded, then (1.1), (1.2) are called singular differential equations. Analog of Theorems 1.1, 1.2 and other related theorems for singular differential equations have been obtained earlier (see [20]). Recently, in [1], Sturm's theorem for a pair of singular linear differential equations was proved assuming that the solution of minorant equation is principal at both end points of the interval. Very recently, Tyagi [23] studied a pair of second order nonlinear differential equations

$$(p_1(t)x')' + q_1(t)f_1(x) = 0, \tag{1.3}$$

$$(p_2(t)y')' + q_2(t)f_2(y) = 0, \quad t_1 < t < t_2 \tag{1.4}$$

under suitable sufficient conditions. He gave the generalization of these theorems to (1.3) and (1.4) for regular and singular cases. Tyagi's paper [23] is the first generalization of Sturm-Picone theorem by establishing a nonlinear version of Leighton's variational Lemma. In the linear case, Tyagi's results reduce to the celebrated Sturm-Picone and Leighton theorems. But it is obvious that Tyagi's result does not work for the half-linear case.Our aim is to give an answer for this case. As far as our understanding goes, there is no generalization of Leighton-type theorems for nonlinear differential equations that contain the half-linear equation.

In this paper motivated by the ideas in [23], extending Tyagi's results, we prove a nonlinear analogue for Leighton's theorem and we give a generalization to Sturm-Picone theorem by establishing a suitable nonlinear version of Leighton's variational lemma which contain the half-linear and also the linear equations. Our results also include the singular case.

2. Regular Sturm-Picone theorem for nonlinear equations

Let us consider a pair of second-order nonlinear ordinary differential equations

$$\ell x := \big(p_1(t)|x'|^{\alpha-1}x'\big)' + q_1(t)f_1(x) = 0, \tag{2.1}$$

$$Ly := \big(p_2(t)|y'|^{\alpha-1}y'\big)' + q_2(t)f_2(y) = 0, \quad t_1 < t < t_2 \tag{2.2}$$

where $p_1,\ p_2 \in C^1\big([t_1,t_2],(0,\infty)\big)$, $q_1,\ q_2 \in C\big([t_1,t_2],R\big)$, $f_1,\ f_2 \in C\big(R,R\big)$, α is a real positive constant, l and L are differential operators or mappings whose domains consist of all real-valued functions $x \in C^1[t_1,t_2]$, such that $p_1|x'|^{\alpha-1}x'$ and $p_2|x'|^{\alpha-1}x' \in C^1[t_1,t_2]$, respectively. In what follows, we assume the following hypotheses with respect to functions f_1 and f_2:

- (H1) Let $f_1 \in C^1(R,R)$ and there exist $\alpha_1 > 0$, $\alpha_0 > 0$ such that $\alpha_0|x|^{\alpha-1} \le f_1'(x) \neq 0$ and $\alpha_1|x|^{\alpha-1}x \ge f_1(x) \neq 0$, for all $0 \neq x \in \mathbb{R}$, and $f_1(0) = 0$, $f_1'(0) \ge 0$.

(H2) Let $f_2 \in C(R,R)$ and there exist $\alpha_2,\ \alpha_3 \in (0,\infty)$ such that $\alpha_3|y|^{\alpha+1} \le f_2(y)y \le \alpha_2|y|^{\alpha+1}$, for all $0 \neq y \in \mathbb{R}$.

Remark 2.1. Assumption (H1) motivates us to take the nonlinearities of the form
$$f_1(x) = |x|^{\alpha-1}x\big(1 \mp \text{ a nonlinear part}\big)$$
where nonlinear part is decaying at ∞.

Remark 2.2. Assumption (H2) simply says that $\frac{f_2(y)}{|y|^{\alpha-1}y}$ is bounded, for all $0 \neq y \in \mathbb{R}$.

We begin with a lemma and the definition of some concepts, needed in this article.

Lemma 2.3 ([10, 12])**.** Define $\varphi(u) := |u|^{\alpha-1}u$, $\alpha > 0$. If $x,\ y \in \mathbb{R}$ then
$$x\varphi(x) + \alpha y\varphi(y) - (\alpha+1)x\varphi(y) \ge 0$$
where equality holds if and only if $x = y$.

Let U be the set of all real valued functions $u \in C^1[t_1,t_2]$, such that $u(t_1) = u(t_2) = 0$, where t_1 and t_2 are consecutive zeros of u. Also define the functionals j and $J : U \to R$ by
$$\begin{gathered} j(u) = \int_{t_1}^{t_2} \{p_1(t)|u'(t)|^{\alpha+1} - C_1 q_1(t)|u(t)|^{\alpha+1}\}dt \\ J(u) = \int_{t_1}^{t_2} \{p_2(t)|u'(t)|^{\alpha+1} - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t))|u(t)|^{\alpha+1}\}dt \end{gathered} \tag{2.3}$$
where $C_1 = (\frac{\alpha_0}{\alpha_1\alpha})^\alpha\alpha_1$, $q_2^+ = \max\{q_2, 0\}$ and $q_2^- = \max\{-q_2, 0\}$. The variation $V(u)$ is defined as
$$V(u) = J(u) - j(u). \tag{2.4}$$

Theorem 2.4 (Leighton's variational type lemma)**.** *Suppose that there exists a function $u \in U$, not identically zero in any open subinterval of (t_1,t_2) such that $j(u) \le 0$. If x is a nontrivial solution of* (2.1) *such that* (H1) *holds, then x has a zero in (t_1,t_2) except possibly when $|u|^\alpha = |Kf_1(x)|$ for some nonzero constant K.*

Proof. Assume on the contrary that the statement is false. Let $x(t) \neq 0$ for every $t \in (t_1,t_2)$. We observe that the following equality is valid on (t_1,t_2):
$$\begin{aligned} &(\frac{\alpha u(t)\varphi(u(t))}{f_1(x(t))}p_1(t)\varphi(x'(t)))' \\ &= \frac{\alpha u(t)\varphi(u(t))}{f_1(x(t))}(-q_1(t)f_1(x(t))) + p_1(t)\varphi(x'(t))(\frac{\alpha u(t)\varphi(u(t))}{f_1(x(t))})' \\ &= -\alpha q_1(t)u(t)\varphi(u(t)) + p_1(t)\varphi(x'(t))\Big[\frac{\alpha(\alpha+1)u'(t)\varphi(u(t))}{f_1(x(t))} \\ &\quad - \frac{\alpha u(t)\varphi(u(t))x'(t)f_1'(x(t))}{f_1^2(x(t))}\Big] \\ &= -\alpha q_1(t)u(t)\varphi(u(t)) - p_1(t)\frac{|f_1(x(t))|^{\alpha-1}}{(f_1'(x(t)))^\alpha}\Big\{\alpha^{\alpha+1}u'(t)\varphi(u'(t)) \\ &\quad - \alpha(\alpha+1)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)u'(t) \end{aligned}$$

$$+\alpha\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)-\alpha^{\alpha+1}u'(t)\varphi(u'(t))\Big\}.$$

Using Lemma 2.3 with $x=\alpha u'(t)$ and $y=\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}$ and hypothesis (H1), we obtain

$$\begin{aligned}
&\big(\frac{\alpha u(t)\varphi(u(t))}{f_1(x(t))}p_1(t)\varphi(x'(t))\big)'\\
&\le -\alpha q_1(t)|u(t)|^{\alpha+1}+\alpha^{\alpha+1}p_1(t)\frac{(\alpha_1)^{\alpha-1}}{\alpha_0^{\alpha}}|u'(t)|^{\alpha+1}\\
&\quad -p_1(t)\frac{|f_1(x(t))|^{\alpha-1}}{(f_1'(x(t)))^{\alpha}}\Big[|\alpha u'(t)|^{\alpha+1}+\alpha|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}|^{\alpha+1}\\
&\quad -(\alpha+1)\alpha u'(t)\varphi(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))})\Big].
\end{aligned}$$

This implies

$$\begin{aligned}
&p_1(t)|u'(t)|^{\alpha+1}-C_1q_1(t)|u(t)|^{\alpha+1}\\
&\ge C_1\Big(\frac{u(t)\varphi(u(t))}{f_1(x(t))}p_1(t)\varphi(x'(t))\Big)'\\
&\quad+\frac{C_1}{\alpha}p_1(t)\frac{|f_1(x(t))|^{\alpha-1}}{(f_1'(x(t)))^{\alpha}}\Big\{|\alpha u'(t)|^{\alpha+1}+\alpha|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}|^{\alpha+1}\\
&\quad-\alpha(\alpha+1)u'(t)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)\Big\}.
\end{aligned}\tag{2.5}$$

Integrating over (t_1,t_2), it follows that

$$\begin{aligned}
&\int_{t_1}^{t_2}(p_1(t)|u'(t)|^{\alpha+1}-C_1q_1(t)|u(t)|^{\alpha+1})dt\\
&\ge C_1(\frac{|u(t)|^{\alpha+1}p_1(t)\varphi(x'(t))}{f_1(x(t))})\,\big|_{t_1}^{t_2}+\frac{C_1}{\alpha}\int_{t_1}^{t_2}p_1(t)\frac{|f_1(x(t))|^{\alpha-1}}{(f_1'(x(t)))^{\alpha}}\Big\{|\alpha u'(t)|^{\alpha+1}\\
&\quad+\alpha|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}|^{\alpha+1}-\alpha(\alpha+1)u'(t)\varphi(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))})\Big\}dt.
\end{aligned}\tag{2.6}$$

Now, there are three cases for the behavior of $x(t)$ at t_1 and t_2.

Case 1. If both $x(t_1)\neq 0$ and $x(t_2)\neq 0$, then it follows from (2.6) and $u\in U$ that $j(u)\ge 0$ and from Lemma 2.3

$$\begin{aligned}
&\int_{t_1}^{t_2}p_1(t)\frac{|f_1(x(t))|^{\alpha-1}}{(f_1'(x(t)))^{\alpha}}\{|\alpha u'(t)|^{\alpha+1}+\alpha|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}|^{\alpha+1}\\
&-\alpha(\alpha+1)u'(t)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)\}dt=0
\end{aligned}$$

if and only if

$$\alpha u'(t)-\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\equiv 0.$$

This implies

$$|u(t)|^{\alpha}=|Kf_1(x(t))|,\quad \forall t\in(t_1,t_2)$$

and for some constant K. Since t_1 and t_2 are consecutive zeros of u, this implies that $u(t)\neq 0$ for all $t\in(t_1,t_2)$. So K is a non-zero constant. Using this fact, we obtain $j(u) > 0$, which leads a contradiction. This contradiction shows that x vanishes at least once in (t_1, t_2).

Case 2. If both $x(t_1) = 0$ and $x(t_2) = 0$, then $x'(t_1) \neq 0$ and $x'(t_2) \neq 0$. It follows from the fact that zeros of a nontrivial solution of (2.1) are simple, which can be proved as follows. Indeed we prove only the case $x(t_1) = 0$. Assume on the contrary that $x'(t_1) = 0$. We take $x(t) > 0$ on (t_1, t_2) in the case $x(t) < 0$ on (t_1, t_2) is similar and hence omitted. It follows from (2.1) that

$$x'(t) = \varphi^{-1}\{-\frac{1}{p_1(t)} \int_{t_1}^{t} q_1(s) f_1(x(s)) ds\},$$

where $\varphi^{-1}(s) = |s|^{\frac{1}{\alpha}-1} s$ is the inverse function of φ. Since $x(t_1) = 0$ and $p_1 \in C^1([t_1, t_2], (0, \infty))$,

$$\begin{aligned} x(t) &= \int_{t_1}^{t} \varphi^{-1}\Big(-\frac{1}{p_1(\xi)} \int_{a}^{\xi} q_1(s) f_1(x(s)) ds \Big) d\xi \\ &\leq (t - t_1) \varphi^{-1}\Big(M \int_{t_1}^{t} |q_1(s)||f_1(x(s))| ds \Big) \end{aligned}$$

for $t_1 \leq t \leq t_2$, where

$$M = \max\big\{ \frac{1}{p_1(t)} : t_1 \leq t \leq t_2 \big\}$$

Hence

$$\varphi(x(t)) \leq (t - t_1)^{\alpha} M \int_{t_1}^{t} |q_1(s)| f_1(x(s))| ds \quad \text{for } t_1 \leq t \leq t_2.$$

Using (H1), it follows from the Gronwall inequality that $\varphi(x(t)) = 0$ for each $t \in [t_1, t_2]$. This implies that $x(t) = 0$ on (t_1, t_2), which contradicts the hypothesis $x(t) > 0$ on (t_1, t_2). Then if $x(t_1) = 0$, by L'Hospital's Rule, considering (H1), assuming $x'(t_1) > 0$,

$$\lim_{t \to t_1^+} \varphi\big(\frac{u(t)}{x(t)} \big) = \varphi\big(\lim_{t \to t_1^+} \frac{u'(t)}{x'(t)} \big) < \infty$$

and

$$\begin{aligned} \lim_{t \to t_1^+} \frac{u(t)}{\alpha_1} \varphi\big(\frac{u(t)}{x(t)} \big) p_1(t) \varphi(x'(t)) &\leq \lim_{t \to t_1^+} \frac{u(t) \varphi(u(t)) p_1(t) \varphi(x'(t))}{f_1(x(t))} \\ &\leq \lim_{t \to t_1^+} \frac{\alpha}{\alpha_0} u(t) \varphi\big(\frac{u(t)}{x(t)} \big) p_1(t) \varphi(x'(t)), \end{aligned}$$

we have

$$\lim_{t \to t_1^+} \frac{u(t) \varphi(u(t)) p_1(t) \varphi(x'(t))}{f_1(x(t))} = 0.$$

Similarly,

$$\lim_{t \to t_2^-} \frac{u(t) \varphi(u(t)) p_1(t) \varphi(x'(t))}{f_1(x(t))} = 0,$$

if $x(t_2) = 0$.

Therefore, we obtain from (2.6) that $j(u) \geq 0$ and hence we obtain a contradiction $j(u) > 0$ unless $|f_1(x)|$ is a constant multiple of $|u|^{\alpha}$.

Case 3. If $x(t_1) = 0$ and $x(t_2) \neq 0$ or $x(t_1) \neq 0$, $x(t_2) = 0$, then as in the proof of Case 1, it is obvious that $j(u) > 0$ which leads a contradiction and hence x vanishes at least once in (t_1, t_2). This completes the proof. $\square$

From Theorem 2.4 we have the following result which is an extension of Leighton's Theorem for (2.1) and (2.2).

Theorem 2.5. *Let* (H1), (H2) *hold. If there exists a nontrivial real solution y of $L_y = 0$ in (t_1, t_2) such that $y(t_1) = y(t_2) = 0$ and $V(y) \geq 0$, then every nontrivial solution x of $\ell x = 0$ has one of the following properties:*

(i) *x has a zero in (t_1, t_2) or,*
(ii) *$|f_1(x)|$ is a nonzero constant multiple of $|y|^\alpha$.*

Proof. Since $y(t_1) = 0 = y(t_2)$ and $Ly(t) = 0$, by applying Green's identity, we have

$$y(t)\Big(p_2(t)|y'(t)|^{\alpha-1}y'(t)\Big)' + q_2(t)f_2(y(t))y(t) = 0,$$

$$\begin{aligned}\Big(p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t)\Big)' &= y(t)\Big(p_2(t)|y'(t)|^{\alpha-1}y'(t)\Big)' + |y'(t)|^{\alpha+1}p_2(t) \\ &= -q_2(t)f_2(y(t))y(t) + |y'(t)|^{\alpha+1}p_2(t).\end{aligned}$$

Integrating both side from t_1 and t_2, we obtain

$$\int_{t_1}^{t_2}\Big(q_2(t)f_2(y(t))y(t) - p_2(t)|y'(t)|^{\alpha+1}\Big)dt = 0. \tag{2.7}$$

In view of (H2), one can see that

$$\int_{t_1}^{t_2}\{(q_2(t)f_2(y(t))y(t) - \Big(\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t)\Big)|y(t)|^{\alpha+1}\}dt \leq 0 \tag{2.8}$$

By (2.7) and (2.8), we have $J(y) \leq 0$. Since $V(y) \geq 0$ this implies that

$$j(y) \leq J(y) \leq 0$$

and hence by an application of Theorem 2.4 every nontrivial solution x of $\ell x = 0$ has at least one zero in (t_1, t_2) except possibly when $|f_1(x(t))|$ is a nonzero constant multiple of $|y(t)|^\alpha$. This completes the proof. $\square$

Remark 2.6. If the condition $V(y) \geq 0$ is strengthened to $V(y) > 0$, conclusion (ii) of Theorem 2.5 does not hold.

From Theorem 2.5 we immediately have the following Corollary which is an extension of Sturm-Picone Comparison Theorem for the equations (2.1) and (2.2).

Corollary 2.7. *Let* (H1) *and* (H2) *hold. Suppose there exists a nontrivial solution y of $Ly = 0$ in (t_1, t_2) such that $y(t_1) = 0 = y(t_2)$ if $p_2(t) \geq p_1(t)$ and*

$$C_1 q_1(t) - \Big(\alpha_2 q_2(t) - (\alpha_3 - \alpha_2)q_2^-(t)\Big) \geq 0$$

for every $t \in (t_1, t_2)$, then every nontrivial solution x of $\ell x = 0$ has at least one zero in (t_1, t_2) unless $|f_1(x)|$ is a nonconstant multiple of $|y|^\alpha$.

From Theorem 2.4, Theorem 2.5 and Corollary 2.7 we easily obtain the following results which are straight forward extensions of the variational Lemma, Leighton's theorem and the celebrated Sturm-Picone theorem from [14, 15, 19, 20] valid for linear second order equations to the case of half-linear equations.

Corollary 2.8. *Let $f_1(x) = |x|^{\alpha-1}x$ in (2.1) if*

$$\int_{t_1}^{t_2} \{p_1(t)|u'(t)|^{\alpha+1} - q_1(t)|u(t)|^{\alpha+1}\}dt \le 0,$$

where $u \in U$, not identically zero in any open subinterval of (t_1, t_2), then every nontrivial solution x of (2.1) has a zero in (t_1, t_2) except possibly when $u = Kx$ for some nonzero constant K.

Corollary 2.9. *Let us consider equations (2.1) and (2.2) with $f_1(u) = |u|^{\alpha-1}u = f_2(u)$. Suppose there exists a nontrivial solution y of $Ly = 0$ in (t_1, t_2) such that $y(t_1) = 0 = y(t_2)$. If*

$$\int_{t_1}^{t_2} \{(p_2(t) - p_1(t))|y'(t)|^{\alpha+1} + (q_1(t) - q_2(t))|y(t)|^{\alpha+1}\}dt \ge 0,$$

then every nontrivial solution x of $\ell x = 0$ has at least one zero in (t_1, t_2) except possibly it is a constant multiple of y.

Corollary 2.10. *Consider the equations (2.1) and (2.2) with $f_1(u) = |u|^{\alpha-1}u = f_2(u)$. Let $p_2(t) \ge p_1(t)$ and $q_1(t) \ge q_2(t)$ for every $t \in (t_1, t_2)$. If there exists a nontrivial solution y of $Ly = 0$ in (t_1, t_2) such that $y(t_1) = 0 = y(t_2)$, then any nontrivial solution x of $\ell x = 0$ either has a zero in (t_1, t_2) or it is a nonzero constant multiple of y.*

Note that the Corollaries 2.8–2.10 were also obtained by Jaros and Kusano [11]. But their proofs depend on the Picone-type and Wirtinger-type inequalities. Corollary 2.9 was also obtained by Li and Yeh [16] using different way.

3. Singular Sturm-Picone theorem for nonlinear equations

In this section, we consider the second-order nonlinear singular equations

$$\ell_s x := \Big(p_1(t)|x'|^{\alpha-1}x'\Big)' + q_1(t)f_1(x) = 0 \tag{3.1}$$

$$L_s y := \Big(p_2(t)|y'|^{\alpha-1}y'\Big)' + q_2(t)f_2(y) = 0 \quad t_1 < t < t_2, \tag{3.2}$$

where $p_1, p_2 \in C\big((t_1, t_2), (0, \infty)\big)$, $q_1, q_2 \in C\big((t_1, t_2), R\big)$, some or all of p_1, p_2, q_1, q_2 may not be continuous at t_1 or t_2 or at t_1 and t_2 both, where the possibility that the interval is unbounded is not excluded. Let $f_1, f_2 \in C(R, R)$, ℓ_s and L_s are differential operators or mappings whose domains consists of all real-valued functions $x \in C^1(t_1, t_2)$ such that $p_1|x'|^{\alpha-1}x'$ and $p_2|x'|^{\alpha-1}x' \in C^1(t_1, t_2)$ respectively.

We begin with the following quadratic functionals corresponding to (3.1) and (3.2) respectively. For $t_1 < \xi < \eta < t_2$, let

$$j_{\xi\eta}(u) = \int_\xi^\eta \{p_1(t)|u'(t)|^{\alpha+1} - C_1 q_1(t)|u(t)|^{\alpha+1}\}dt \tag{3.3}$$

$$J_{\xi\eta}(u) = \int_\xi^\eta \{p_2(t)|u'(t)|^{\alpha+1} - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t))|u(t)|^{\alpha+1}\}dt. \tag{3.4}$$

Let us define $j_s(u) = \lim_{\xi\to t_1^+, \eta\to t_2^-} j_{\xi\eta}(u)$ and $J_s(u) = \lim_{\xi\to t_1^+, \eta\to t_2^-} J_{\xi\eta}(u)$ whenever the limit exists.

The domains D_{j_s} of j_s and D_{J_s} of J_s are defined to be the set of all real-valued continuous functions $u \in C^1(t_1,t_2)$ such that $j_s(u)$ and $J_s(u)$ exist. Let us define

$$A_{t_1t_2}[u,x] = \lim_{t\to t_2^-} \frac{\alpha u(t)\varphi(u(t))p_1(t)\varphi(x'(t))}{f_1(x(t))} - \lim_{t\to t_1^+} \frac{\alpha u(t)\varphi(u(t))p_1(t)\varphi(x'(t))}{f_1(x(t))} \tag{3.5}$$

whenever the limits on the right-hand side exist. The variation $V_s(u)$ is defined as

$$V_s(u) = J_s(u) - j_s(u); \tag{3.6}$$

i.e.,

$$V_s(u) = \int_{t_1}^{t_2} \{\Big(p_2(t) - p_2(t)\Big)|u'(t)|^{\alpha+1} + (C_1q_1(t) - (\alpha_2 q_2^+(t) - \alpha_3 q_2^-(t)))|u(t)|^{\alpha+1}\}dt$$

with domain $D := D_{j_s} \cap D_{J_s}$. We begin with the singular version of Leighton's variational type lemma for (3.1).

Theorem 3.1. *Suppose there exists a function $u \in D$, not identically zero in any open interval subinterval of (t_1,t_2) such that $j_s(u) \le 0$. If x is a nontrivial solution of (3.1) such that the hypotheses (H1) holds and $A_{t_1t_2}[u,x] \ge 0$, then x has a zero in (t_1,t_2) unless $|f_1(x)|$ is a nonzero constant multiple of $|u|^\alpha$.*

Proof. Assume for the sake of contradiction that equation (3.1) has a nonzero, nontrivial solution on (t_1,t_2). Along the same lines of proof of Theorem 2.4, we see that the inequality (2.5) holds on (t_1,t_2). An integration of (2.5) over (ξ,η) yields

$$\begin{aligned} j_{\xi\eta}(u) \ge{}& C_1 \frac{u(t)\varphi(u(t))p_1(t)\varphi(x'(t))}{f_1(x(t))}\Big|_\xi^\eta + \frac{C_1}{\alpha}\int_\xi^\eta p_1(t)\frac{|f_1(x(t))|^{\alpha+1}}{(f_1'(x(t)))^\alpha}\Big\{|\alpha u'(t)|^{\alpha+1} \\ &+ \alpha\Big|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big|^{\alpha+1} - \alpha(\alpha+1)u'(t)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)\Big\}dt\,. \end{aligned}$$

Letting $\xi \to t_1^+$, $\eta \to t_2^-$ and using $A_{t_1t_2}[u,x] \ge 0$ we obtain

$$\begin{aligned} j_s(u) \ge{}& \frac{C_1}{\alpha}\int_{t_1}^{t_2} p_1(t)\frac{|f_1(x(t))|^{\alpha+1}}{(f_1'(x(t)))^\alpha}\Big\{|\alpha u'(t)|^{\alpha+1} + \alpha\Big|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big|^{\alpha+1} \\ &- \alpha(\alpha+1)u'(t)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)\Big\}dt \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\int_{t_1}^{t_2} p_1(t)\frac{|f_1(x(t))|^{\alpha+1}}{(f_1'(x(t)))^\alpha}\Big\{|\alpha u'(t)|^{\alpha+1} \\ &+ \alpha\Big|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big|^{\alpha+1} - \alpha(\alpha+1)u'(t)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big)\Big\}dt = 0 \end{aligned}$$

if and only if

$$|\alpha u'(t)|^{\alpha+1} + \alpha\Big|\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big|^{\alpha+1} - \alpha(\alpha+1)u'(t)\varphi\Big(\frac{u(t)x'(t)f_1'(x(t))}{f_1(x(t))}\Big) \equiv 0$$

According to Lemma 2.3, this implies

$$|u(t)|^\alpha = |Kf_1(x(t))| \quad \text{for every } t \in (t_1,t_2)$$

and for some nonzero constant K. Using this fact, we have $j_s(u) > 0$ which leads to a contradiction. This contradiction shows that x vanishes at least once in (t_1,t_2). This completes the proof. ∎

As in Section 2, from Theorem 3.1 plays an important role to establish the following result which is an extension of Leighton's theorem for equations (3.1) and (3.2) for the singular case.

Theorem 3.2. *Suppose that there exists a nontrivial real solution $y \in D$ of $L_s y = 0$ in (t_1, t_2). Let x be any nontrivial solution of $\ell_s x = 0$. Let also (H1) and (H2). If $A_{t_1 t_2}[y, x] \geq 0$,*

$$\lim_{t \to t_1^+} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \geq 0, \quad \lim_{t \to t_2^-} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \leq 0$$

and $V_s(y) > 0$, then x has at least one zero in (t_1, t_2). If the condition $V(y) > 0$ is weakened to $V_s(y) \geq 0$ the same conclusion holds unless $|f_1(x)|$ is a nonzero constant multiple of $|y|^\alpha$.

From Theorem 3.2, we have the following corollary which is the extension of Sturm-Picone comparison theorem for equations (3.1) and (3.2).

Corollary 3.3. *Suppose that there exists a nontrivial real solution $y \in D$ of $L_s y = 0$ in (t_1, t_2). Let x be any nontrivial solution of $\ell_s x = 0$. Let also (H1) and (H2). If $A_{t_1 t_2}[y, x] \geq 0$, $p_2(t) \geq p_1(t)$,*

$$\lim_{t \to t_1^+} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \geq 0, \quad \lim_{t \to t_2^-} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \leq 0,$$

$$C_1 q_1(t) - (\alpha_2 q_2(t) - (\alpha_3 - \alpha_2)q_2^-(t)) \geq 0 \quad \forall t \in (t_1, t_2),$$

then x has at least one zero in (t_1, t_2) unless $|f_1(x)|$ is a nonzero constant multiple $|y|^\alpha$.

Finally the results in Theorems 3.1–3.2 and Corollary 3.3 which are nonlinear extensions of the variational lemma, Leighton's theorem and Sturm-Picone theorem respectively, can also be given for the singular half-linear case as in the following:

Corollary 3.4. *Let $f_1(x) = |x|^{\alpha-1}x$ in (3.1). Suppose that there exists a function $u \in D_{j_s}$, not identically zero in any open subinterval of (t_1, t_2) such that $j_s(u) \leq 0$. If x is a nontrivial solution of (3.1) such that $A_{t_1 t_2}[u, x] \geq 0$, then x has a zero in (t_1, t_2) except possibly when $u = Kx$ for some nonzero constant K.*

Corollary 3.5. *Let us consider equations (3.1) and (3.2) with $f_1(u) = |u|^{\alpha-1}u = f_2(u)$. Suppose that there exists a nontrivial real solution of $y \in D$ of $L_s y = 0$. Let x be any nontrivial solution of $\ell_s x = 0$. If $V_s(y) \geq 0$, $A_{t_1 t_2}[y, x] \geq 0$ and*

$$\lim_{t \to t_1^+} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \geq 0, \quad \lim_{t \to t_2^-} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \leq 0,$$

then x has at least one zero in (t_1, t_2) unless x is a nonzero constant multiple of y.

Corollary 3.6. *Consider the equations (3.1) and (3.2) with $f_1(u) = |u|^{\alpha-1}u = f_2(u)$. Suppose that there exists a nontrivial real solution $y \in D$ of $L_s y = 0$. Let x be any nontrivial solution of $\ell_s(x) = 0$. If $A_{t_1 t_2}[y, x] \geq 0$, $p_2(t) \geq p_1(t)$, $q_1(t) \geq q_2(t)$ for all $t \in (t_1, t_2)$, and*

$$\lim_{t \to t_1^+} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \geq 0, \quad \lim_{t \to t_2^-} p_2(t)y(t)|y'(t)|^{\alpha-1}y'(t) \leq 0,$$

then any nontrivial solution x of $\ell_s x = 0$ either has a zero in (t_1, t_2) or it is a nonzero constant multiple of y.

Acknowledgments. This manuscript is dedicated to the memory of honorable Necdet Doğanata, the founder of Izmir University. The author would like to thank the anonymous referee for his/her useful comments and suggestions.

References

[1] Aharonov, D.; Elias, U.; *Singular Sturm comparison theorems*, J. Math. Anal. Appl. 371(2), (2010), 759-763.

[2] Ahmad, S.; Lazer, A., C.; *A new generalization of the Sturm comparison theorem to self adjoint systems*, Proc. Amer. Math. Soc., 68 (1978), 185-188.

[3] Ahmad, S.; *On Sturmian theory for second order systems*, Proc. Amer. Math. Soc. 87 (1983), 661-665.

[4] Allegretto, W.; *Sturm type theorems for solutions of elliptic nonlinear problems*, Nonlinear Differ. Equ. Appl., 7 (2000), 309-321.

[5] Allegretto, W.; Huang, Y.-X.; *A Picones identitiy for the p-Laplacian and applications*, Nonlinear Analysis T. M. A, 32 (1998), 819-830.

[6] Allegretto, W.; *Sturm theorems for degenerate elliptic equations*, Proc. Amer. Math. Soc. 129 (2001), 3031-3035.

[7] Došllý, O.; *The Picone identity for a class of partial differential equations*, Mathematica Bohemica, 127 (2002), 581-589.

[8] Došlý, O.; Jaroš, J.; *A singular version of Leighton's comparison theorem for forced quasilinear second-order differential equations*, Arch. Math. (BRNO), 39 (2003), 335-345.

[9] Fišnarová, S.; Mařík, R.; *Generalized Picone and Riccati inequalites for half-linear differential operators with arbitrary elliptic matrices*, Electronic J. Diff. Equations, No: 111, (2010), pp. 1-13.

[10] Hardy, G. H., Littlewood, J. E.; Polya, G.; *Inequalities*, Cambridge Univ. Press, 1988.

[11] Jaroš, J.; Kusano T., A; *Picone type identity for second order half-linear differential equations*, Acta Math. Univ. Comenianae, Vol. LXVIII, 1(1999), 137-151.

[12] Jaroš, J.; Kusano, T.; Yoshida, N.; *Forced superlinear oscillations via Picone's identity*, Acta Math. Univ. Comenianae, LXIX (2000), 107-113.

[13] Jaroš, J., Kusano, T.; Yoshida, N.; *Picone type inequalies for half linear elliptic equations and their applications*, Adv. Math. Sci. Appl., 12(2), (2002), 709-724.

[14] Kreith, K.; *Oscillation Theory, Lecture Notes in Mathematics*, no. 324, Springer-Verlag, 1973.

[15] Leighton, W.; *Comparison theorems for linear differential equations of second order*, Proc. Amer. Math. Soc. 13 (1962), 603610.

[16] Li, H. J.; Yeh, C. C.; *Sturmian comparison theorem for half-linear second order differential equations*, Proc. Roy. Soc. Edinburgh 125A (1995), 11931204.

[17] Müller-Pfeiffer, E.; *Sturm Comparison theorems for non-selfadjoint differential equations on non-compact intervals*, Math. Nachr., 159 (1992), 291-298.

[18] Picone, M.; *Sui valori eccezionali di un parametro da cui dipende unequazione differenziale lineare ordinaria del second ordine*, Ann. Scuola Norm. Pisa 11 (1909), 1-141.

[19] Sturm, C.; *Sur les équations difféntielles linéaries du second ordere*, J. Math. Pures. Appl., 1 (1836), 106-186.

[20] Swanson, C. A.; *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

[21] Tadie; *Oscillation criteria for semilinear elliptic equations with a damping term in R^n*, Electronic J. Diff. Equations, No. 51, (2010), pp. 1-5.

[22] Tyagi, T.; Raghavendra, V.; *A note on generalization of Sturm's comparison theorem*, Nonlinear Dyn. Syst. Theory, 8(2) (2008), 213-216.

[23] Tyagi, J.; *Generalization of Sturm-Picone theorem for second order nonlinear differential equations*, Taiwanese Journal of Mathematics, Vol. 17, No. 1 ,(2013) pp. 361-378.

[24] Yoshida, N.; *Oscillation theory of partial differential equations*, World Scientific, 2008.

[25] Yoshida, N.; *A Picone identity for half-linear elliptic equations and its applications to oscillation theory*, Nonlinear Anal., 71 (2009), pp. 4935-4951.

[26] Zhang, C.; Sun, S.; *Sturm-Picone comparison theorem of second-order linear equations on time scales*, Adv. Diff. Eqs., (2009), pp. 12.

Aydin Tiryaki
Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Izmir University, 35350 Uckuyular, Izmir, Turkey
E-mail address: aydin.tiryaki@izmir.edu.tr