Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 154, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OPTIMIZATION OF THE PRINCIPAL EIGENVALUE UNDER MIXED BOUNDARY CONDITIONS

LUCIO CADEDDU, MARIA ANTONIETTA FARINA, GIOVANNI PORRU

Abstract. We investigate minimization and maximization of the principal eigenvalue of the Laplacian under mixed boundary conditions in case the weight has indefinite sign and varies in a class of rearrangements. Biologically, these optimization problems are motivated by the question of determining the most convenient spatial arrangement of favorable and unfavorable resources for a species to survive or to decline. We prove existence and uniqueness results, and present some features of the optimizers. In special cases, we prove results of symmetry and results of symmetry breaking for the minimizer.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain representing a region occupied by a population that diffuses at rate D and grows or declines locally at a rate $g(x)$ (so that $g(x) > 0$ corresponds to local growth and $g(x) < 0$ to local decline). Suppose the boundary $\partial\Omega$ is divided in two parts, Γ and $\partial\Omega \setminus \Gamma$ so that the 1-Lebesgue measure of Γ is positive. Suppose there is an hostile population outside Γ (we have Dirichlet boundary conditions on Γ), and suppose there is not flux of individuals across $\partial\Omega \setminus \Gamma$ (we have Neumann boundary conditions there). If $\phi(x,t)$ is the population density, the behavior of such a population is described by the logistic equation

$$\frac{\partial \phi}{\partial t} = D\Delta\phi + (g(x) - \kappa\phi)\phi \quad \text{in } \Omega \times \mathbb{R}^+,$$
$$\phi = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \quad \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } (\partial\Omega \setminus \Gamma) \times \mathbb{R}^+,$$

where $\Delta\phi$ denotes the spatial Laplacian of $\phi(x,t)$, κ is the carrying capacity and ν is the exterior normal to $\partial\Omega$.

It is known (see [7, 8]) that the logistic equation predicts persistence if and only if $\lambda_g < 1/D$, where λ_g is the (positive) principal eigenvalue in

$$\Delta u + \lambda g(x) u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma.$$

2000 *Mathematics Subject Classification.* 47A75, 35J25, 35Q92, 49J20, 49K20.
Key words and phrases. Principal eigenvalue; rearrangements; minimization; maximization, symmetry breaking; population dynamics.

Submitted November 26, 2013. Published July 5, 2014.

Many results and applications related to such eigenvalue problems are discussed in [2, 9, 20, 22].

In the present paper we consider the following question: *for weights $g(x)$ within the set of rearrangements of a given weight function $g_0(x)$, which, if any, minimizes or maximizes λ_g?*

The corresponding problem with Dirichlet boundary conditions has been investigated by many authors, see [10, 11, 12, 14] and references therein. For the case of the p-Laplacian see [13, 23]. For the case of Neumann boundary conditions see [16]. Eigenvalue problems for nonlinear elliptic equations are discussed in [15]. The problem of competition of more species has been treated in [6, 21].

In what follows, Ω is a bounded smooth domain in $\mathbb{R}^N$. In applications to population dynamics, we have $1 \le N \le 3$, but most of our results hold for general N. If $E \subset \mathbb{R}^N$ is a measurable set we denote with $|E|$ its Lebesgue measure. We say that two measurable functions $f(x)$ and $g(x)$ have the same rearrangement in Ω if

$$|\{x \in \Omega : f(x) \ge \beta\}| = |\{x \in \Omega : g(x) \ge \beta\}| \quad \forall \beta \in \mathbb{R}.$$

If $g_0(x)$ is a bounded function in Ω we denote by $\mathcal{G}$ the class of its rearrangements.

We make use of the following results proved in [3] and [4]. For short, throughout the paper we shall write increasing instead of non-decreasing, and decreasing instead of non-increasing.

Denote with $\overline{\mathcal{G}}$ the weak closure of $\mathcal{G}$ in $L^p(\Omega)$. It is well known that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact (see for example [4, Lemma 2.2]).

Lemma 1.1. *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^\infty(\Omega)$, and let $u \in L^p(\Omega)$, $p \ge 1$. There exists $\hat{g} \in \mathcal{G}$ such that*

$$\int_\Omega g\, u\, dx \le \int_\Omega \hat{g}\, u\, dx \qquad \forall g \in \overline{\mathcal{G}}.$$

The above lemma follows from [4, Lemma 2.4].

Lemma 1.2. *Let $g : \Omega \mapsto \mathbb{R}$ and $w : \Omega \mapsto \mathbb{R}$ be measurable functions, and suppose that every level set of w has measure zero. Then there exists an increasing function ϕ such that $\phi(w)$ is a rearrangement of g. Furthermore, there exists a decreasing function ψ such that $\psi(w)$ is a rearrangement of g.*

The assertions of the above lemma follow from [4, Lemma 2.9].

Lemma 1.3. *Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^p(\Omega)$, $p \ge 1$, and let $w \in L^q(\Omega)$, $q = p/(p-1)$. If there is an increasing function ϕ such that $\phi(w) \in \mathcal{G}$ then*

$$\int_\Omega g\, w\, dx \le \int_\Omega \phi(w)\, w\, dx \quad \forall g \in \overline{\mathcal{G}},$$

and the function $\phi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a decreasing function ψ such that $\psi(w) \in \mathcal{G}$ then

$$\int_\Omega g\, w\, dx \ge \int_\Omega \psi(w)\, w\, dx \qquad \forall g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{G}}$.

The assertions of the above lemma follow from [4, Lemma 2.4]. We recall that the $L^q(\Omega)$ topology on $L^p(\Omega)$ is the weak topology if $1 \le p < \infty$, and the weak* topology if $p = \infty$ [3].

2. Optimization of the principal eigenvalue

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, and let $g_0(x)$ be a bounded measurable function in Ω which takes positive values in a set of positive measure. Suppose Γ is a portion of $\partial\Omega$ with a positive $(N-1)$-Lebesgue measure. Let $\mathcal{G}$ be the class of rearrangements generated by g_0. For $g \in \mathcal{G}$, we consider the eigenvalue problem

$$\Delta u + \lambda g(x)u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma. \tag{2.1}$$

We are interested in the principal eigenvalue, that is, a positive eigenvalue to which corresponds a positive eigenfunction. If

$$W_\Gamma^+ = \Big\{ w \in H^1(\Omega) : \, w = 0 \text{ on } \Gamma, \, \int_\Omega g\, w^2 dx > 0 \Big\},$$

we have

$$\lambda_g = \inf_{w \in W_\Gamma^+} \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\, w^2 dx} = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g u_g^2 dx}, \tag{2.2}$$

where u_g is positive in Ω and unique up to a positive constant. Note that, if u_g is a minimizer, so is $|u_g|$, hence $|u_g|$ satisfies equation (2.1). By Harnack's inequality (see, for example, [24, Theorem 1.1]) we have $|u_g| > 0$ in Ω. By continuity, we have either $u_g > 0$ or $u_g < 0$. We also note that if there are a positive number Λ and a positive function v such that

$$\Delta v + \Lambda g(x)v = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \Gamma, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma,$$

then $\Lambda = \lambda_g$ and $v = cu_g$ for some positive constant c (see [19, Corollary 5.6]). Actually, in [19] the authors consider the case of Dirichlet boundary conditions, however, the same proof works in our situation.

We investigate the problem of finding

$$\inf_{g \in \mathcal{G}} \lambda_g, \quad \sup_{g \in \mathcal{G}} \lambda_g.$$

Let $\overline{\mathcal{G}}$ be the closure of $\mathcal{G}$ with respect to the weak* topology of $L^\infty(\Omega)$. Recall that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact.

Theorem 2.1. *Let λ_g be defined as in* (2.2).
(i) The problem of finding

$$\min_{g \in \mathcal{G}} \lambda_g$$

has (at least) a solution.
(ii) If $\hat{g}$ is a minimizer then $\hat{g} = \phi\big(u_{\hat{g}}\big)$ for some increasing function $\phi(t)$.

Proof. If g_n is a minimizing sequence for $\inf_{g \in \mathcal{G}} \lambda_g$, we have

$$I = \inf_{g \in \mathcal{G}} \lambda_g = \lim_{n \to \infty} \lambda_{g_n} = \lim_{n \to \infty} \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2 dx}. \tag{2.3}$$

We can suppose the sequence λ_{g_n} is decreasing, therefore,

$$\int_\Omega |\nabla u_{g_n}|^2 dx \le C_1 \int_\Omega g_n u_{g_n}^2 dx \le C_2 \int_\Omega u_{g_n}^2 dx, \tag{2.4}$$

for suitable constants C_1, C_2. Let us normalize u_{g_n} so that

$$\int_\Omega u_{g_n}^2 dx = 1. \tag{2.5}$$

By (2.4) and (2.5) we infer that the norm $\|u_{g_n}\|_{H^1(\Omega)}$ is bounded by a constant independent of n. Therefore (see [17]), a sub-sequence of u_{g_n} (denoted again by u_{g_n}) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $z \in H^1(\Omega)$ with $z(x) \geq 0$ in Ω, $z = 0$ on Γ and

$$\int_\Omega z^2 dx = 1.$$

Furthermore, since the sequence g_n is bounded in $L^\infty(\Omega)$, there is a subsequence (denoted again by g_n) which converges to some $\eta \in \overline{\mathcal{G}}$ in the weak* topology of $L^\infty(\Omega)$. We have

$$\int_\Omega g_n u_{g_n}^2 dx - \int_\Omega \eta\, z^2 dx = \int_\Omega (g_n - \eta)\, z^2 dx + \int_\Omega g_n (u_{g_n}^2 - z^2) dx.$$

Since

$$\lim_{n\to\infty} \int_\Omega (g_n - \eta)\, z^2 dx = 0$$

and since

$$\Big| \int_\Omega g_n (u_{g_n}^2 - z^2) dx \Big| \leq C_3 \|u_{g_n} + z\|_{L^2(\Omega)} \|u_{g_n} - z\|_{L^2(\Omega)},$$

we find

$$\lim_{n\to\infty} \int_\Omega g_n u_{g_n}^2 dx = \int_\Omega \eta\, z^2 dx \geq 0. \tag{2.6}$$

Furthermore, since $\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ is a norm equivalent to the usual norm in $H^1(\Omega)$ with $u = 0$ on Γ, we have

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{g_n}|^2 dx \geq \int_\Omega |\nabla z|^2 dx. \tag{2.7}$$

We claim that $\int_\Omega \eta\, z^2 dx > 0$. Indeed, passing to the limit as $n \to \infty$ in

$$\int_\Omega \nabla u_{g_n} \cdot \nabla \psi \, dx = \lambda_{g_n} \int_\Omega g_n \, u_{g_n} \psi \, dx$$

we obtain

$$\int_\Omega \nabla z \cdot \nabla \psi \, dx = I \int_\Omega \eta \, z \, \psi \, dx, \quad \forall \psi \in H^1(\Omega),$$

and

$$\int_\Omega |\nabla z|^2 \, dx = I \int_\Omega \eta \, z^2 dx.$$

If we had $\int_\Omega |\nabla z|^2 \, dx = 0$, we would have $z = 0$, contradicting the condition $\int_\Omega z^2 dx = 1$. The claim follows.

Now, by Lemma 1.1, we find some $\hat{g} \in \mathcal{G}$ such that

$$\int_\Omega \eta \, z^2 dx \leq \int_\Omega \hat{g} \, z^2 dx.$$

Using this estimate and recalling the variational characterization of $\lambda_{\hat{g}}$ we find

$$I = \frac{\int_\Omega |\nabla z|^2 dx}{\int_\Omega \eta \, z^2 dx} \geq \frac{\int_\Omega |\nabla z|^2 dx}{\int_\Omega \hat{g} \, z^2 dx} \geq \lambda_{\hat{g}} \geq I.$$

Therefore,

$$\inf_{g \in \mathcal{G}} \lambda_g = \lambda_{\hat{g}}.$$

Part (i) of the theorem is proved.

Let us prove that $\hat g = \phi(u_{\hat g})$ for some increasing function ϕ. By

$$\frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\, w^2 dx} \geq \frac{\int_\Omega |\nabla u_{\hat g}|^2 dx}{\int_\Omega \hat g\, u_{\hat g}^2 dx} \quad \forall g \in \mathcal{G}, \forall w \in W_\Gamma^+,$$

with $w = u_{\hat g}$ we obtain

$$\int_\Omega g u_{\hat g}^2 dx \leq \int_\Omega \hat g\, u_{\hat g}^2 dx \quad \forall g \in \mathcal{G}. \tag{2.8}$$

The function $u_{\hat g}$ satisfies the equation

$$-\Delta u_{\hat g} = \lambda_{\hat g} \hat g u_{\hat g}. \tag{2.9}$$

Recall that $u_{\hat g} > 0$ in Ω. By equation (2.9), the function $u_{\hat g}$ cannot have flat zones neither in the set

$$F_1 = \{x \in \Omega : \hat g(x) < 0\}$$

nor in the set

$$F_2 = \{x \in \Omega : \hat g(x) > 0\}.$$

By Lemma 1.2, there is an increasing function $\phi_1(t)$ such that $\phi_1(u_{\hat g}^2)$ is a rearrangement of $\hat g(x)$ on $F_1 \cup F_2$. Define

$$\alpha = \inf_{x \in \Omega \setminus F_1} u_{\hat g}^2(x).$$

Using (2.8), one proves that $u_{\hat g}^2(x) \leq \alpha$ in F_1 (see [5, Lemma 2.6] for details). Now define

$$\beta = \sup_{x \in \Omega \setminus F_2} u_{\hat g}^2(x).$$

Using (2.8) again one shows that $u_{\hat g}^2(x) \geq \beta$ in F_2. Since

$$\sup_{F_1} \phi_1(u_{\hat g}^2) = \sup_{F_1} \hat g(x) \leq 0$$

we have $\phi_1(t) \leq 0$ for $t < \alpha$. Similarly, since

$$\inf_{F_2} \phi_1(u_{\hat g}^2) = \inf_{F_2} \hat g(x) \geq 0$$

we have $\phi_1(t) \geq 0$ for $t > \beta$. We put

$$\tilde\phi(t) = \begin{cases} \phi_1(t) & \text{if } 0 \leq t < \alpha \\ 0 & \text{if } \alpha \leq t \leq \beta \\ \phi_1(t) & \text{if } t > \beta. \end{cases}$$

The function $\tilde\phi(t)$ is increasing. Furthermore, $\tilde\phi(u_{\hat g}^2)$ is a rearrangement of $\hat g(x)$ in Ω (the functions $\hat g$ and $\tilde\phi(u_{\hat g}^2)$ have the same rearrangement on $F_1 \cup F_2$, and both vanish on $\Omega \setminus (F_1 \cup F_2)$). By (2.8) and Lemma 1.3 we must have $\hat g = \tilde\phi(u_{\hat g}^2)$. Part (ii) of the theorem follows with $\phi(t) = \tilde\phi(t^2)$. $\square$

Let us prove a continuity result.

Proposition 2.2. *Let λ_g be defined as in* (2.2). *Suppose $g_n \in \mathcal{G}$, $g \in \overline{\mathcal{G}}$ and $g_n \rightharpoonup g$ as $n \to \infty$ with respect to the weak* convergence in $L^\infty(\Omega)$.*
(i) If $g(x) > 0$ in a subset of positive measure then

$$\lim_{n\to\infty} \lambda_{g_n} = \lambda_g.$$

(ii) If $g(x) \le 0$ in Ω then

$$\lim_{n\to\infty} \lambda_{g_n} = +\infty.$$

Proof. To prove Part (i), we follow an argument similar to that used in [13, Lemma 4.2] in the case of Dirichlet boundary conditions and $g(x) \ge 0$. Let u_{g_n} be the eigenfunction corresponding to g_n normalized so that

$$\int_\Omega u_{g_n}^2 dx = 1.$$

We have

$$\lambda_{g_n} = \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2 dx} \le \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g_n u_g^2 dx},$$

where u_g is the principal eigenfunction corresponding to g normalized so that

$$\int_\Omega u_g^2 dx = 1.$$

Since

$$\lambda_g = \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g\, u_g^2 dx},$$

we have

$$\lambda_{g_n} \le \frac{\int_\Omega |\nabla u_g|^2 dx}{\int_\Omega g_n u_g^2 dx} = \lambda_g \frac{\int_\Omega g u_g^2 dx}{\int_\Omega g_n u_g^2 dx}.$$

The assumption $g_n \rightharpoonup g$ with respect to the weak* convergence in $L^\infty(\Omega)$ yields

$$\lim_{n\to\infty} \int_\Omega g_n u_g^2 dx = \int_\Omega g u_g^2 dx.$$

Therefore, for $\epsilon > 0$ we find ν_ϵ such that, for $n > \nu_\epsilon$ we have $\lambda_{g_n} < \lambda_g + \epsilon$. It follows that

$$\limsup_{n\to\infty} \lambda_{g_n} \le \lambda_g.$$

To find the complementary inequality we use the equation

$$-u_{g_n} \Delta u_{g_n} = \lambda_{g_n} g_n u_{g_n}^2.$$

Integrating over Ω, recalling that $u_{g_n} = 0$ on Γ, that the normal derivative of u_{g_n} on $\partial\Omega \setminus \Gamma$ vanishes, and using the inequality $\lambda_{g_n} < \lambda_g + \epsilon$ (for n large), we find a constant C such that

$$\int_\Omega |\nabla u_{g_n}|^2 dx \le (\lambda_g + \epsilon) \int_\Omega g_n u_{g_n}^2 dx \le C,$$

where the boundedness of g_n and the normalization of u_{g_n} have been used. We infer that the norm $\|u_{g_n}\|_{H^1(\Omega)}$ is bounded by a constant independent of n. A subsequence of u_{g_n} (denoted again by u_{g_n}) converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $z \in H^1(\Omega)$ with $z \ge 0$, $z = 0$ on Γ, and

$$\int_\Omega z^2 dx = 1.$$

As a consequence,
$$\liminf_{n\to\infty}\int_\Omega |\nabla u_{g_n}|^2 dx \geq \int_\Omega |\nabla z|^2 dx.$$
Moreover, arguing as in the proof of Theorem 2.1 we obtain
$$\lim_{n\to\infty}\int_\Omega g_n u_{g_n}^2\, dx = \int_\Omega g\, z^2 dx.$$
An argument similar to that used in the proof of Theorem 2.1 shows that we cannot have
$$\int_\Omega |\nabla z|^2\, dx = \int_\Omega g z^2 dx = 0.$$
Therefore, it follows that
$$\liminf_{n\to\infty}\lambda_{g_n} = \liminf_{n\to\infty}\frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2\, dx} \geq \frac{\int_\Omega |\nabla z|^2 dx}{\int_\Omega g\, z^2 dx} \geq \lambda_g.$$
Part (i) of the proposition follows.

To prove Part (ii), we argue by contradiction. Suppose there is a sub-sequence of λ_{g_n}, still denoted λ_{g_n}, and a real number M such that
$$\lambda_{g_n} = \frac{\int_\Omega |\nabla u_{g_n}|^2 dx}{\int_\Omega g_n u_{g_n}^2 dx} \leq M$$
and
$$\int_\Omega u_{g_n}^2 dx = 1.$$
It follows that
$$\int_\Omega |\nabla u_{g_n}|^2 dx \leq M \int_\Omega g_n u_{g_n}^2\, dx \leq \tilde{M}.$$
Therefore, there is a sub-sequence of u_{g_n} (denoted again by u_{g_n}) which converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to some function $z \in H^1(\Omega)$, $z(x) \geq 0$ $z = 0$ on Γ, and such that
$$\int_\Omega z^2 dx = 1.$$
Furthermore, up to a subsequence, we may suppose that
$$\lim_{n\to\infty}\lambda_{g_n} = \tilde{\lambda}.$$
For $n = 1, 2, \dots$ we have
$$\int_\Omega \nabla u_{g_n}\cdot\nabla\psi dx = \lambda_{g_n}\int_\Omega g_n u_{g_n}\psi dx \quad \forall \psi \in H^1(\Omega).$$
Letting $n \to \infty$ we find
$$\int_\Omega \nabla z\cdot\nabla\psi dx = \tilde{\lambda}\int_\Omega g z\psi dx \quad \forall \psi \in H^1(\Omega).$$
By the latter equation we find $z \in C^1(\Omega)$. Furthermore, putting $\psi = z$ we find
$$\int_\Omega |\nabla z|^2 dx = \tilde{\lambda}\int_\Omega g z^2 dx \leq 0,$$
where the assumption $g(x) \leq 0$ has been used. It follows that $|\nabla z| = 0$ in Ω. Therefore, $z = 0$, contradicting the condition $\int_\Omega z^2 dx = 1$. The proof is complete. $\square$

Proposition 2.3. *Let λ_g be defined as in (2.2), and let $J(g) = 1/\lambda_g$. The map $g \mapsto J(g)$ is Gateaux differentiable with derivative*

$$J'(g;h) = \frac{\int_\Omega h\, u_g^2 dx}{\int_\Omega |\nabla u_g|^2 dx}.$$

Furthermore, if g satisfies $\int_\Omega g(x)dx \geq 0$, the map $g \mapsto \lambda_g$ is strictly concave.

In case we have Dirichlet boundary conditions, the proof of this proposition is well known (see, for example, [9, Proposition 1]). The same proof also works under our boundary conditions.

Theorem 2.4. *Let λ_g be defined as in (2.2). The problem of finding*

$$\max_{g \in \overline{\mathcal{G}}} \lambda_g$$

has a solution; if $\int_\Omega g_0(x)dx \geq 0$, the maximizer $\breve{g}$ is unique; if $\int_\Omega g_0(x)dx > 0$, we have $\breve{g} = \psi\big(u_{\breve{g}}\big)$ for some decreasing function $\psi(t)$; finally, if $g_0(x) \geq 0$ then the maximizer $\breve{g}$ belongs to $\mathcal{G}$.

Proof. Since the functional $g \mapsto \lambda_g$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$ (by Proposition 2.2), and since $\overline{\mathcal{G}}$ is weakly compact, a maximizer $\breve{g}$ exists in $\overline{\mathcal{G}}$. Assuming $\int_\Omega g_0(x)dx \geq 0$, the uniqueness of the maximizer follows from the strict concavity of λ_g (see Proposition 2.3). If $\int_\Omega g_0(x)dx > 0$, the maximizer $\breve{g}$ is positive in a subset of positive measure, therefore, $\lambda_{\breve{g}}$ is finite and $u_{\breve{g}}(x) > 0$ a.e. in Ω. If $0 < t < 1$ and if $g_t = \breve{g} + t(g - \breve{g})$, since $J(g)$ is differentiable (see Proposition 2.3), we have

$$J(\breve{g}) \leq J(g_t) = J(\breve{g}) + t\frac{\int_\Omega (g - \breve{g})u_{\breve{g}}^2 dx}{\int_\Omega |\nabla u_{\breve{g}}|^2 dx} + o(t) \quad \text{as} \quad t \to 0.$$

It follows that

$$\int_\Omega (g - \breve{g})u_{\breve{g}}^2 dx \geq 0.$$

Equivalently, we have

$$\int_\Omega g u_{\breve{g}}^2 dx \geq \int_\Omega \breve{g} u_{\breve{g}}^2 dx \quad \forall g \in G. \tag{2.10}$$

The function $u_{\breve{g}}$ satisfies the equation

$$-\Delta u_{\breve{g}} = \lambda_{\breve{g}} \breve{g} u_{\breve{g}}. \tag{2.11}$$

By equation (2.11), the function $u_{\breve{g}}$ cannot have flat zones neither in the set $F_3 = \{x \in \Omega : \breve{g}(x) > 0\}$ nor in the set $F_4 = \{x \in \Omega : \breve{g}(x) < 0\}$. By Lemma 1.2, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u_{\breve{g}}^2)$ is a rearrangement of $\breve{g}(x)$ on $F_3 \cup F_4$. Following the proof of [5, Theorem 2.1], we introduce the class $\mathcal{W}$ of rearrangements of our maximizer $\breve{g}$. Of course, $\mathcal{W} \subset \overline{\mathcal{G}}$. Define

$$\gamma = \inf_{x \in \Omega \setminus F_3} u_{\breve{g}}^2(x).$$

Using (2.10), one proves that $u_{\breve{g}}^2(x) \leq \gamma$ in F_3. Define

$$\delta = \sup_{x \in \Omega \setminus F_4} u_{\breve{g}}^2(x).$$

Using (2.10) again one shows that $u_{\breve g}^2(x) \geq \delta$ in F_4. Now we put

$$\tilde\psi(t) = \begin{cases} \psi_1(t) & \text{if } 0 \leq t < \gamma \\ 0 & \text{if } \gamma \leq t \leq \delta \\ \psi_1(t) & \text{if } t > \delta. \end{cases}$$

The function $\tilde\psi(t)$ is decreasing and $\tilde\psi(u_{\breve g}^2)$ is a rearrangement of $\breve g(x)$ in Ω. Indeed, the functions $\breve g$ and $\tilde\psi(u_{\breve g}^2)$ have the same rearrangement on $F_3 \cup F_4$, and both vanish on $\Omega \setminus (F_3 \cup F_4)$. By (2.10) and Lemma 1.3 we must have $\breve g = \tilde\psi(u_{\breve g}^2) \in \mathcal{W}$.

Note that, in general, the maximizer $\breve g$ does not belong to $\mathcal{G}$ (see next Theorem 2.5). Assuming $g_0(x) \geq 0$, we can prove that $\breve g \in \mathcal{G}$. Indeed, by (2.11), the function $u_{\breve g}$ cannot have flat zones in the set $F = \{x \in \Omega : \breve g(x) > 0\}$. If $|F| < |\Omega|$, since $\breve g \in \overline{\mathcal{G}}$, by [4, Lemma 2.14] we have $|F| \geq |\{x \in \Omega : g_0(x) > 0\}|$. Therefore there is $g_1 \in \mathcal{G}$ such that its support is contained in F. By Lemma 1.3, there is a decreasing function $\psi_1(t)$ such that $\psi_1(u_{\breve g}^2)$ is a rearrangement of $g_1(x)$ on F. Define

$$\gamma = \inf_{x \in \Omega \setminus F} u_{\breve g}^2(x).$$

Using (2.10), one proves that $u_{\breve g}^2(x) \leq \gamma$ in F. By using equation (2.10) once more we find that $u_{\breve g}^2(x) < \gamma$ a.e. in F. Now define

$$\tilde\psi(t) = \begin{cases} \psi_1(t) & \text{if } 0 \leq t < \gamma \\ 0 & \text{if } t \geq \gamma. \end{cases}$$

The function $\tilde\psi(t)$ is decreasing and $\tilde\psi(u_{\breve g}^2)$ is a rearrangement of $g_1 \in \mathcal{G}$ on Ω. Indeed, the functions g_1 and $\tilde\psi(u_{\breve g}^2)$ have the same rearrangement on F, and both vanish on $\Omega \setminus F$. By (2.10) and Lemma 1.3 we must have $\breve g = \tilde\psi(u_{\breve g}^2) \in \mathcal{G}$. Hence, in case of $|F| < |\Omega|$, the conclusion follows with $\psi(t) = \tilde\psi(t^2)$. If $|F| = |\Omega|$, the proof is easier and we do not need the introduction of the function g_1. The statement of the theorem follows. ☐

Theorem 2.5. *Suppose $u \in H^2(\Omega) \cap C^0(\Omega)$ with $u = 0$ on Γ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega \setminus \Gamma$. Here $\Gamma \subset \partial\Omega$ is supposed to be smooth and to have a $(N-1)$-Lebesgue positive measure. Let $u(x) > 0$ in Ω and*

$$-\Delta u = \Lambda \psi(u) u \quad \text{a.e. in } \Omega$$

for some $\Lambda > 0$ and some decreasing bounded function ψ. Then, either $\Delta u \leq 0$ or $\Delta u \geq 0$ a.e. in Ω.

Proof. By contradiction, suppose that the essential range of Δu contains positive and negative values. Since $u > 0$ and $-\Delta u = \Lambda\psi(u)u$, $\psi(t)$ takes positive and negative values for $t > 0$. Let

$$\beta = \sup\{t : \psi(t) \geq 0\}, \quad \Omega_\beta = \{x \in \Omega : u(x) > \beta\}.$$

By our assumptions, the open set Ω_β is not empty. On the other hand, since ψ is decreasing and $u > 0$ we have

$$-\Delta u < 0 \quad \text{in } \Omega_\beta, \quad u = \beta \quad \text{on } \partial\Omega_\beta \setminus \Gamma_\beta \quad \text{and } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_\beta,$$

where Γ_β is a suitable subset of $\partial\Omega \setminus \Gamma$. By second Hopf's boundary Lemma, u cannot have its maximum value on Γ_β. Therefore, the maximum principle for subharmonic functions yields $u(x) \le \beta$ in Ω_β. This contradicts the definition of Ω_β, and the theorem follows □

3. Symmetry

3.1. **The one-dimensional case.** Let $N = 1$ and $\Omega = (0, L)$. Given a measurable function $f : \Omega \to \mathbb{R}$, we denote by f^* the decreasing rearrangement of f (f^* is non increasing on $(0, L)$). Similarly, we denote by f_* the increasing rearrangement of f. The following results are well known.

Lemma 3.1. *Let $N = 1$ and $\Omega = (0, L)$.*
(i) If $f(x)$ and $g(x)$ belong to $L^\infty(\Omega)$ then

$$\int_\Omega f_*(x)g^*(x)dx \le \int_\Omega f(x)g(x)dx \le \int_\Omega f^*(x)g^*(x)dx. \tag{3.1}$$

(ii) If $u \in H^1(\Omega)$, $u(x) \ge 0$ and $u(L) = 0$, then $u^* \in H^1(\Omega)$, $u^*(x) \ge 0$, $u^*(L) = 0$ and

$$\int_\Omega (u')^2 dx \ge \int_\Omega ((u^*)')^2 dx. \tag{3.2}$$

For a proof of the above lemma, see, for example, [1, 18]. Note that, (i) is often proved for non negative functions. However, replacing f by $f + M$ and g by $g + M$ with a suitable constant M, one gets the result for bounded functions.

Theorem 3.2. *Let $\mathcal{G}$ be a class of rearrangements generated by a bounded function g_0. Let $g \in \mathcal{G}$, and let λ_g be defined as in (2.2) with $\Omega = (0, L)$ and $u(L) = 0$. Then we have $\lambda_g \ge \lambda_{g^*}$.*

Proof. If $g \in \mathcal{G}$ and if u_g is a corresponding (positive) principal eigenfunction, we have

$$\lambda_g = \frac{\int_\Omega (u_g')^2 dx}{\int_\Omega g u_g^2 dx}. \tag{3.3}$$

Since $u_g > 0$ we have $(u_g^*)^2 = (u_g^2)^*$, and by (3.1) we find

$$\int_\Omega g u_g^2\, dx \le \int_\Omega g^*\, (u_g^*)^2 dx. \tag{3.4}$$

Note that $u_g^*(L) = 0$. Using (3.1), (3.2), and recalling the variational characterization of λ_{g^*} we find

$$\lambda_g = \frac{\int_\Omega (u_g')^2 dx}{\int_\Omega g u_g^2 dx} \ge \frac{\int_\Omega ((u_g^*)')^2 dx}{\int_\Omega g^*(u_g^*)^2 dx} \ge \frac{\int_\Omega (u_{g^*}')^2 dx}{\int_\Omega g^* u_{g^*}^2 dx} = \lambda_{g^*}.$$

The proof is complete. □

Example 3.3. Let us apply the result of Theorem 3.2 to the following example. For $0 < \alpha \le \beta < L$, let $g(t) = 1$ on a subset E with measure α, $g(t) = -1$ on a subset F with measure $L - \beta$, and $g(t) = 0$ on $(0, L) \setminus (E \cup F)$. By Theorem 3.2, a minimizer is the function

$$g^* = \begin{cases} 1, & 0 < t < \alpha, \\ 0, & \alpha \le t \le \beta, \\ -1, & \beta < t < L. \end{cases}$$

If $\Lambda > 0$ is the corresponding principal eigenvalue and u is a corresponding eigenfunction, we have

$$-u'' = \begin{cases} \Lambda u, & 0 < t < \alpha, \\ 0, & \alpha \le t \le \beta, \\ -\Lambda u, & \beta < t < L, \end{cases}$$

with $u'(0) = u(L) = 0$. This boundary value problem can be solved easily. We find

$$u = \begin{cases} \cos(\sqrt{\Lambda}\, t), & 0 < t < \alpha, \\ At + B & \alpha \le t \le \beta, \\ K \sinh(\sqrt{\Lambda}(L-t)), & \beta < t < L. \end{cases}$$

Since the function u must be continuous and differentiable for $t = \alpha$ and for $t = \beta$, the constants Λ, A, B and K must satisfy the conditions

$$\begin{gathered} \cos(\sqrt{\Lambda}\, \alpha) = A\alpha + B \\ -\sqrt{\Lambda} \sin(\sqrt{\Lambda}\, \alpha) = A, \end{gathered}$$

and

$$\begin{gathered} K \sinh(\sqrt{\Lambda}(L-\beta)) = A\beta + B \\ -K\sqrt{\Lambda} \cosh(\sqrt{\Lambda}(L-\beta)) = A. \end{gathered}$$

Therefore, Λ and K must satisfy

$$\sin(\sqrt{\Lambda}\, \alpha) = K \cosh(\sqrt{\Lambda}(L-\beta))$$

and

$$\cos(\sqrt{\Lambda}\, \alpha) + \alpha\sqrt{\Lambda} \sin(\sqrt{\Lambda}\, \alpha) = K \sinh(\sqrt{\Lambda}(L-\beta)) + K\beta\sqrt{\Lambda} \cosh(\sqrt{\Lambda}(L-\beta)).$$

It follows that

$$\cot(\sqrt{\Lambda}\, \alpha) = \tanh(\sqrt{\Lambda}(L-\beta)) + \sqrt{\Lambda}(\beta-\alpha). \tag{3.5}$$

The function $y(t) = \cot(t\alpha)$, for $0 < t < \pi/(2\alpha)$, satisfies

$$y(0) = +\infty, \quad y'(t) < 0, \quad y\Big(\frac{\pi}{2\alpha}\Big) = 0.$$

Moreover, the function $z(t) = \tanh(t(L-\beta)) + t(\beta-\alpha)$, for $0 < t$, satisfies

$$z(0) = 0, \quad z'(t) > 0, \quad z(t) < 1 + t(\beta - \alpha).$$

It follows that equation (3.5) has a unique solution $\Lambda = \Lambda(\alpha)$ such that

$$\frac{1}{\alpha} \arctan \frac{1}{1 + \sqrt{\Lambda}(\beta-\alpha)} < \sqrt{\Lambda} < \frac{\pi}{2\alpha}.$$

It is clear that $\Lambda \to \infty$ as $\alpha \to 0$.

Theorem 3.4. *Let $\mathcal{G}$ be a class of rearrangements generated by a function g_0 defined in $(0, L)$ such that $\int_0^L g_0(x)dx > 0$. If $g \in \mathcal{G}$, let ρ such that $\int_0^\rho g_*(x)dx = 0$. Define $\breve{g} = 0$ for $0 < x < \rho$, and $\breve{g} = g_*$ for $\rho < x < L$. If λ_g is defined as in* (2.2) *with $\Omega = (0, L)$ and $u(L) = 0$, we have $\lambda_g \le \lambda_{\breve{g}}$.*

Proof. Let u_g be a principal eigenfunction corresponding to $g \in \mathcal{G}$, and let $u_{\breve{g}}$ be a principal eigenfunction corresponding to $\breve{g}$. We have

$$\lambda_g = \frac{\int_\Omega (u_g')^2 dx}{\int_\Omega g u_g^2 dx} \le \frac{\int_\Omega \big(u_{\breve{g}}'\big)^2 dx}{\int_\Omega g u_{\breve{g}}^2 dx}. \tag{3.6}$$

On the other hand, the function $u_{\breve{g}}$ solves the problem

$$-u_{\breve{g}}'' = \lambda_{\breve{g}} \breve{g} u_{\breve{g}}, \ u'(0) = u(L) = 0.$$

Since $u_{\breve{g}}'' = 0$ on $(0, \rho)$ (recall that $\breve{g} = 0$ there) and $u'(0) = 0$, the function $u_{\breve{g}}$ is a positive constant on $(0, \rho)$. Furthermore, since

$$-u_{\breve{g}}'(x) = \lambda_{\breve{g}} \int_0^x \breve{g} u_{\breve{g}} dt \ge 0,$$

the function $u_{\breve{g}}$ is decreasing on $(0, L)$. (Recall that we write decreasing instead of non-increasing). It follows that

$$u_{\breve{g}} = u_{\breve{g}}^*, \quad u_{\breve{g}}^2 = (u_{\breve{g}}^2)^*.$$

Hence, since g_* is increasing, by (3.1) we find

$$\int_0^L g u_{\breve{g}}^2 dx \ge \int_0^L g_* u_{\breve{g}}^2 dx. \tag{3.7}$$

Furthermore, since $u_{\breve{g}}$ is a constant on $(0, \rho)$ and since $\int_0^\rho g_* dx = 0$, we have

$$\int_0^L g_* u_{\breve{g}}^2 dx = c^2 \int_0^\rho g_* dx + \int_\rho^L g_* u_{\breve{g}}^2 dx = \int_0^L \breve{g} u_{\breve{g}}^2 dx.$$

Therefore, by (3.7) we find

$$\int_0^L g u_{\breve{g}}^2 dx \ge \int_0^L \breve{g} u_{\breve{g}}^2 dx.$$

The latter inequality and (3.6) yield

$$\lambda_g \le \frac{\int_\Omega (u_{\breve{g}}')^2 dx}{\int_\Omega \breve{g} u_{\breve{g}}^2 dx} = \lambda_{\breve{g}}.$$

The proof is complete. □

Example 3.5. Let us apply the result of Theorem 3.4 to the following example. For $0 < \alpha < L$, let $g(t) = 1$ on a subset E with measure $L - \alpha$, and $g(t) = 0$ on $(0, L) \setminus E$. By Theorem 3.4, the maximizer is the function

$$g_* = \begin{cases} 0, & 0 < t < \alpha, \\ 1, & \alpha < t < L. \end{cases}$$

If $\Lambda > 0$ is the corresponding principal eigenvalue and u is a corresponding eigenfunction, we have

$$-u'' = \begin{cases} 0, & 0 < t < \alpha, \\ \Lambda u, & \alpha < t < L, \end{cases}$$

with $u'(0) = u(L) = 0$. This boundary value problem can be solved easily. We find

$$u = \begin{cases} 1, & 0 < t < \alpha, \\ \sin(\sqrt{\Lambda}(L - t)), & \alpha < t < L. \end{cases}$$

Since the function u must be continuous and differentiable for $t = \alpha$, $\Lambda > 0$ must satisfy the condition

$$\sqrt{\Lambda} = \frac{\pi}{2(L-\alpha)}.$$

Remark 3.6. The conclusions of Theorems 3.2 and 3.4 continue to hold in the following case. Let $\Omega = (0, L) \times (0, \ell)$, and let

$$\begin{cases} u = 0 & \text{on } \{x_1 = L\}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_2 = \ell\}. \end{cases}$$

Suppose the function g_0 depends on x_1 only, and the class $\mathcal{G}$ is the (restricted) family of all rearrangements of g_0 in Ω depending on x_1 only. In this situation, the principal eigenfunctions depend on x_1 only, and the optimization of the corresponding principal eigenvalue is essentially a one-dimensional problem discussed in Theorems 3.2 and 3.4.

3.2. **α-sector.** For $0 < \alpha \leq \pi$, consider the domain (in polar coordinates (r, θ))

$$D = \{(r, \theta) : \ 0 \leq r < R, \quad 0 < \theta < \alpha\}. \tag{3.8}$$

For a function $f \in L^2(D)$, we consider the radial decreasing rearrangement f^* and the radial increasing rearrangement f_*. We refer to [1] (page 73) for a discussion on this kind of rearrangements. Recall that f^* depends on r only and it is non increasing, f_* depends on r only and it is non decreasing. We have

Lemma 3.7. *If $f, g \in L^2(D)$ we have*

$$\int_D f_* g^* dx \leq \int_D fg \ dx \leq \int_D f^* g^* dx. \tag{3.9}$$

If $u \in H^1(D)$, $u \geq 0$ and $u = 0$ on $r = R$, then, $u^* \in H^1(D)$, $u^* \geq 0$ and $u^* = 0$ on $r = R$. Furthermore,

$$\int_D |\nabla u|^2 dx \geq \int_D |\nabla u^*|^2 dx. \tag{3.10}$$

For a proof of the above lemma, we refer the reader to [1, pages 73-75],

Theorem 3.8. *Let $\mathcal{G}$ be the class of rearrangements generated by a bounded function g_0 defined in the α-sector D introduced in* (3.8)*. For $g \in \mathcal{G}$, let λ_g be defined as in* (2.2) *where $\Omega = D$ and Γ is the portion of ∂D with $r = R$. Then $\lambda_g \geq \lambda_{g^*}$.*

Proof. If λ_g is the corresponding principal eigenvalue, using inequalities (3.9) and (3.10) we find

$$\lambda_g = \frac{\int_D |\nabla u_g|^2 dx}{\int_D g u_g^2 dx} \geq \frac{\int_D |\nabla u_g^*|^2 dx}{\int_D g^* (u_g^*)^2 dx} \geq \frac{\int_D |\nabla u_{g^*}|^2 dx}{\int_D g^* u_{g^*}^2 dx} = \lambda_{g^*}.$$

Note that, since $u_g \geq 0$ in D and it vanishes on Γ, also $u_g^* \geq 0$ in D and vanishes on Γ. The theorem is proved. $\square$

In case the class $\mathcal{G}$ is generated by $g_0 = \chi_E - \chi_F$, where E and F are disjoint subsets of D, we have $g^* = \chi_{\hat{E}} - \chi_{\hat{F}}$, where

$$\hat{E} = \Big\{(r, \theta) \in D : \, r^2 \leq \frac{2|E|}{\alpha}\Big\},$$
$$\hat{F} = \Big\{(r, \theta) \in D : \, r^2 \geq R^2 - \frac{2|F|}{\alpha}\Big\}.$$

Theorem 3.9. *Let D be the α-sector defined in (3.8). Let $\mathcal{G}$ be the class of rearrangements generated by a function g_0 defined in D such that $\int_D g_0(x)dx > 0$. If $g \in \mathcal{G}$, let $D_\rho \subset D$ be the α-sector such that $\int_{D_\rho} g_*(x)dx = 0$. Define $\breve{g} = 0$ for $x \in D_\rho$, and $\breve{g} = g_*$ for $D \setminus D_\rho$. Let λ_g be defined as in (2.2) where $\Omega = D$ and Γ is the portion of ∂D with $r = R$. Then $\lambda_g \leq \lambda_{\breve{g}}$.*

Proof. Let u_g be a principal eigenfunction corresponding to $g \in \mathcal{G}$, and let $u_{\breve{g}}$ be a principal eigenfunction corresponding to $\breve{g}$. We have

$$\lambda_g = \frac{\int_D |\nabla u_g|^2 dx}{\int_D g u_g^2 dx} \leq \frac{\int_D |\nabla u_{\breve{g}}|^2 dx}{\int_D g u_{\breve{g}}^2 dx}. \tag{3.11}$$

On the other hand, the function $u_{\breve{g}}$ satisfies the problem

$$-\Delta u_{\breve{g}} = \lambda_{\breve{g}} \breve{g} u_{\breve{g}} \quad \text{in } D,$$

with $u = 0$ on Γ and $u_\theta = 0$ on the segments $\theta = 0$ and $\theta = \alpha$. The solution $u_{\breve{g}}$ is radial and (since $\breve{g} = 0$ for $x \in D_\rho$) its derivative (with respect to r) is a constant in $(0, \rho)$. Since $u'(0) = 0$, this constant must be zero, and the function $u_{\breve{g}}$ is a positive constant in D_ρ. Furthermore, since

$$-r u_{\breve{g}}'(r) = \lambda_{\breve{g}} \int_0^r t \breve{g} u_{\breve{g}} dt \geq 0,$$

$u_{\breve{g}}(r)$ is decreasing on $(0, R)$. It follows that

$$u_{\breve{g}} = u_{\breve{g}}^*, \quad u_{\breve{g}}^2 = (u_{\breve{g}}^2)^*.$$

Hence, since $g_*(r)$ is increasing, by the left hand side of (3.9) we find

$$\int_D g u_{\breve{g}}^2 dx \geq \int_D g_* u_{\breve{g}}^2 dx. \tag{3.12}$$

Furthermore, since $u_{\breve{g}}$ is a constant in D_ρ and since $\int_{D_\rho} g_* dx = 0$, we have

$$\int_D g_* u_{\breve{g}}^2 dx = c^2 \int_{D_\rho} g_* dx + \int_{D \setminus D_\rho} g_* u_{\breve{g}}^2 dx = \int_D \breve{g} u_{\breve{g}}^2 dx.$$

Therefore, by (3.12) we find

$$\int_D g u_{\breve{g}}^2 dx \geq \int_D \breve{g} u_{\breve{g}}^2 dx.$$

The latter inequality and (3.11) yield

$$\lambda_g \leq \frac{\int_D |\nabla u_{\breve{g}}|^2 dx}{\int_D \breve{g} u_{\breve{g}}^2 dx} = \lambda_{\breve{g}}.$$

The theorem is proved. □

In case the class $\mathcal{G}$ is generated by $g_0 = \chi_E - \chi_F$ with $E \cap F = \emptyset$, $|E| > |F|$, the maximum of λ_g is attained for $\breve{g} = \chi_G$, where G is the set

$$G = \Big\{(r, \theta) \in D : r^2 \geq R^2 - \frac{2(|E| - |F|)}{\alpha}\Big\}.$$

If $|E| \leq |F|$, we have $\sup \lambda_g = +\infty$.

4. Symmetry breaking

Concerning the minimum, the symmetry of the data may not be inherited by the solution.

Theorem 4.1. *Let $N=2$ and $\Omega = B_{a,a+2}$, the annulus of radii a, $a+2$. Suppose $g_0 = \chi_E$, where E is a measurable set contained in Ω and such that $|E| = \pi\rho^2$, $0<\rho<1$. Let $\mathcal{G}$ be the family of rearrangements of g_0. Consider the eigenvalue problem (2.1) in Ω with Γ being the circle with radius $a+2$. If a is large enough then a minimizer of λ_g in $\mathcal{G}$ cannot be radially symmetric with respect to the center of $B_{a,a+2}$.*

Proof. Recall that

$$\lambda_g = \inf\Big\{ \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega g\, w^2 dx}, \quad w\in H^1(\Omega): \quad w=0 \quad \text{on} \quad \Gamma, \quad \int_\Omega g\, w^2 dx > 0\Big\}. \tag{4.1}$$

Let $E = B_\rho$ be a disc with radius ρ and such that its center x_0 lies on $|x| = a+1$. If $g = \chi_{B_\rho}$, the function $z = (\rho^2 - |x-x_0|^2)^+$ vanishes on Γ, hence, if $|x-x_0| = r$,

$$\lambda_g \le \frac{\int_{B_\rho} |\nabla z|^2 dx}{\int_{B_\rho} z^2 dx} = \frac{\int_0^\rho 4r^3 dr}{\int_0^\rho r(\rho^2-r^2)^2 dr} = \frac{6}{\rho^2}. \tag{4.2}$$

Note that this upper bound is independent of a.

Now suppose $g = \chi_E$, with E radially symmetric with respect to the center of $B_{a,a+2}$. With $r = |x|$, put $g(x) = h(r) = \chi_{E_1}$, E_1 being the intersections of E with a ray of B_{a+2}. The corresponding eigenfunction is radially symmetric (by uniqueness), and the inferior in (4.2) can be taken over all $v \in H^1_{rad}$ (the class of radially symmetric functions in $H^1(\Omega)$) with $v(a+2) = 0$. We have

$$\lambda_g = \inf\Big\{ \frac{\int_a^{a+2} r(v')^2 dr}{\int_a^{a+2} rhv^2 dr}, \quad v\in H^1_{rad} : v(a+2) = 0\Big\}.$$

We find

$$\frac{\int_a^{a+2} r(v')^2 dr}{\int_a^{a+2} rhv^2 dr} \ge \frac{\int_a^{a+2} a(v')^2 dr}{\int_a^{a+2} (a+2)hv^2 dr} = \frac{a}{a+2}\, \frac{\int_a^{a+2} (v')^2 dr}{\int_a^{a+2} hv^2 dr}.$$

The 1-measure of E_1 depends on the location of E, however we have

$$|E_1| \le \sqrt{a^2+\rho^2} - a := \ell.$$

Note that $\ell \to 0$ as $a\to\infty$.

Using classical inequalities for decreasing rearrangements we find

$$\frac{\int_a^{a+2} (v')^2 dr}{\int_a^{a+2} h\, v^2 dr} \ge \frac{\int_a^{a+2} ((v^*)')^2 dr}{\int_a^{a+2} h^*\, (v^*)^2 dr} \ge \frac{\int_{-1}^1 (w')^2 dt}{\int_{-1}^1 g^*\, w^2 dt},$$

where $w(t) = v^*(r)$, $t = r-(a+1)$, and

$$g^* = \begin{cases} 1, & -1<t<-1+\ell, \\ 0, & -1+\ell < t < 1. \end{cases}$$

We have

$$\lambda_g \ge \frac{a}{a+2} \inf\Big\{ \frac{\int_{-1}^1 (w')^2 dt}{\int_{-1}^1 g^* w^2 dt}: \quad w\in H^1(-1,1), \quad w(1)=0\Big\} = \frac{a}{a+2}\Lambda_{g^*}. \tag{4.3}$$

To find $\Lambda = \Lambda_{g^*}$, we look for a positive solution of the problem

$$-z'' = \begin{cases} \Lambda z, & -1 < t < -1+\ell, \\ 0, & -1+\ell < t < 1, \end{cases}$$

with $z'(-1) = z(1) = 0$. We have

$$z = \begin{cases} \cos(\sqrt{\Lambda}(t+1)), & -1 < t < -1+\ell, \\ A(1-t) & 1-\ell < t < 1, \end{cases}$$

where A, and Λ satisfy

$$\cos(\sqrt{\Lambda}\ell) = A(2-\ell), \qquad \sqrt{\Lambda}\sin(\sqrt{\Lambda}\ell) = A.$$

It follows that

$$\sqrt{\Lambda}\tan(\sqrt{\Lambda}\ell) = \frac{1}{2-\ell}.$$

Since $\ell \to 0$ as $a \to \infty$, the latter equation shows that we must have $\Lambda \to \infty$ as $a \to \infty$. Then, by (4.3), also $\lambda \to \infty$ as $a \to \infty$. The latter result together with (4.2) show that a minimizer $\hat{g}$ of $g \mapsto \lambda_g$ cannot be symmetric for a large. The proof is complete. □

The situation is different for the maximizer. Indeed, since we have uniqueness of the maximizer (for a class $\mathcal{G}$ generated by $g_0 = \chi_E$), we cannot have symmetry breaking for any annulus.

As already remarked, the solution to the one–dimensional problem treated in Subsection 3.1, also solves the bi-dimensional problem (2.1) in the rectangle $(0, L) \times (0, \ell)$ with Γ being the portion of $\partial\Omega$ with $x_1 = L$, and $\mathcal{G}$ being a class of rearrangements of functions g depending on x_1 only. Indeed, since the eigenfunctions are independent of x_2, the Neumann condition on $x_2 = 0$ and on $x_2 = \ell$ is trivially satisfied. One may ask what happens if $\mathcal{G}$ is the entire family of rearrangements. We prove that for large ℓ we have a sort of symmetry breaking.

Theorem 4.2. *Let $N = 2$ and $\Omega = (0, L) \times (0, \ell)$, $\ell \geq L$. Suppose $g_0 = \chi_E$, where E is a measurable set contained in Ω and such that $|E| = \pi\rho^2$, $0 < \rho < L/2$. Let $\mathcal{G}$ be the family of rearrangements of g_0. Consider the eigenvalue problem* (2.1) *in Ω with Γ being the portion of $\partial\Omega$ with $x_1 = L$. If ℓ is large enough then a minimizer of λ_g in $\mathcal{G}$ cannot be a set of the kind $K \times (0, \ell)$.*

Proof. In case of $E = K \times (0, \ell)$, our problem is essentially one dimensional, which we have treated in Subsection 3.1. The minimum of the eigenvalue (for this kind of sets E) is attained when $E = (0, \tau) \times (0, \ell)$, with $\tau = \frac{\pi\rho^2}{\ell}$. By Example 1 (with $\alpha = \tau$ and $\beta = L$) it is clear that $\lambda_g \to \infty$ as $\ell \to \infty$. On the other hand, if we take $E = B_\rho$, a ball with radius ρ, located in D far from ∂D, the same computation which leads to (4.2) shows that λ_{B_ρ} is bounded independently of ℓ. The statement of the theorem follows. □

References

[1] C. Bandle; *Isoperimetric Inequalities and Applications*, Pitman Monographs and Studies in Mathematics, Vol. **7**, Pitman, Boston, 1980.

[2] F. Belgacem; *Elliptic Boundary Value Problems with Indefinine Weights: Variational Formulations of the Principal Eigenvalue and Applications*, Pitman Research Notes in Mathematics, **368**, Longman, Harlow, Essex, UK, 1997.

[3] G. R. Burton; *Rearrangements of functions, maximization of convex functionals and vortex rings*, Math Ann. **276** (1987), 225-253.

[4] G. R. Burton; *Variational problems on classes of rearrangements and multiple configurations for steady vortices*, Ann. Inst. Henri Poincare. **6** (4) (1989), 295-319.

[5] G. R. Burton, J. B. McLeod; *Maximisation and minimisation on classes of rearrangements*, Proc. Roy. Soc. Edinburgh Sect. A, **119** (3-4) (1991), 287–300.

[6] S. Cano-Casanova, J. López-Gómez; *Permanence under strong aggressions is possible*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **20** no. 6 (2003),(1989), 999-1041.

[7] R. S. Cantrell, C. Cosner; *Diffusive logistic equations with indefinite weights: population models in disrupted environments*, Proc. Roy. Soc. Edinburgh., **112A** (1989), 293–318.

[8] R. S. Cantrell, C. Cosner; *The effects of spatial heterogeneity in population dynamics*, J. Math. Biol., **29** (1991), 315–338.

[9] C. Cosner, F. Cuccu, G. Porru; *Optimization of the first eigenvalue of equations with indefinite weights*, Advanced Nonlinear Studies, **13** (2013), 79–95.

[10] S. Cox; *The two phase drum with the deepest note*, Japan J. Indust. Appl. Math., **8** (1991), 345–355.

[11] S. J. Cox, J. R. McLaughlin; *Extremal eigenvalue problems for composite membranes, I*, Appl. Math. Optim., **22** (1990), 153–167.

[12] S. J. Cox, J. R. McLaughlin; *Extremal eigenvalue problems for composite membranes, II*, Appl. Math. Optim., **22** (1990), 169–187.

[13] F. Cuccu, B. Emamizadeh, G. Porru; *Optimization of the first eigenvalue in problems involving the p-Laplacian*, Proc. Amer. Math. Soc., **137** (2009), 1677-1687.

[14] F. Cuccu, G. Porru; *Optimization in eigenvalue problems*, Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal., **10** (2003), 51–58.

[15] M. E. Filippakis, N. S. Papageorgiu, S. Vasile; *Eigenvalue problems for nonlinear elliptic equations with unilateral constraints*, Nonlinear Analysis, **69** (2008), 85–109.

[16] K. Jha, G. Porru; *Minimization of the principal eigenvalue under Neumann boundary conditions*, Numerical Functional Analysis and Optimization, **32** (2011), 1146-1165.

[17] D. Gilbarg, N.S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, 1977.

[18] B. Kawohl; *Rearrangements and Convexity of Level Sets in PDE*, Lectures Notes in Mathematics, **1150**, Berlin, 1985.

[19] B. Kawohl, M. Lucia, S. Prashanth; *Simplicity of the first eigenvalue for indefinite quasilinear problems*, Adv. Differential Equations, **12** (2007), no. 4, 407–434.

[20] P. Lindqvist; *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$. Proc. Amer. Math. Soc. **109** (1990), 157–164. Addendum ibidem **116**(1992), 583–584.

[21] J. López-Gómez; *Permanence under strong competition*, Dynamical systems and applications, 473–488, World Sci. Ser. Appl. Anal., **4**, World Sci. Publ., River Edge, NJ, 1995.

[22] A. Manes, A.M. Micheletti; *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Boll. Unione Mat. Ital. Ser IV, **7** (1973), 285–301.

[23] G. Porru; *Optimization problems for eigenvalues of p-Laplace equations with indefinite weights*, Boll. Accademia Gioenia di Scienze Naturali - Catania, Vol. 46, n. 376 (2013), 44–56.

[24] N. S. Trudinger; *On Harnack type inequalities and their applications to quasilinear elliptic equations*. Comm. on Pure and Applied Math. Vol. XX (1967), 721–747.

Lucio Cadeddu
Dipartimento di Matematica e Informatica, Univ. di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: cadeddu@unica.it

Maria Antonietta Farina
Dipartimento di Matematica e Informatica, Univ. di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: mafarina@unica.it

Giovanni Porru
Dipartimento di Matematica e Informatica, Univ. di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
E-mail address: porru@unica.it