NODAL SOLUTIONS FOR SINGULAR SECOND-ORDER
BOUNDARY-VALUE PROBLEMS

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Abstract. We use a global bifurcation theorem to prove the existence of nodal solutions to the singular second-order two-point boundary-value problem

\[-(pu')'(t) = f(t, u(t)), \quad t \in (\xi, \eta),\]

\[au(\xi) - b \lim_{t \to \xi^+} p(t)u'(t) = 0,\]

\[cu(\eta) + d \lim_{t \to \eta^-} p(t)u'(t) = 0,\]

where \(\xi, \eta, a, b, c, d\) are real numbers with \(\xi < \eta\), \(a, b, c, d \geq 0\), \(p : (\xi, \eta) \to [0, +\infty)\) is a measurable function with \(\int_\xi^\eta 1/p(s) \, ds < \infty\) and \(f : [\xi, \eta] \times [0, +\infty) \to [0, +\infty)\) is a Carathéodory function.

1. Introduction

Many articles concerning the existence of nodal solutions for second-order differential equations subject to various boundary conditions, have appeared during the previous five decades; see for example [4, 5, 8, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31] and references therein.

Ma and Thompson [19, 20, 21] considered the boundary-value problem (bvp for short),

\[-u'' = a(t)f(u), \quad t \in (0, 1),\]

\[u(0) = u(1) = 0\]  

(1.1)

where \(a : [0, 1] \to [0, +\infty)\) is continuous and does not vanish identically, and \(f : \mathbb{R} \to \mathbb{R}\) is continuous with \(f(s)s > 0\) for \(s \neq 0\). They proved also, that bvp (1.1) admits \(2k\) nodal solutions when the interval whose extremities are \(\lim_{u \to 0} f(u)/u\) and \(\lim_{|u| \to +\infty} f(u)/u\) contains \(k\) eigenvalues of the linear bvp associated with (1.1),

\[-u'' = \lambda a(t)u, \quad t \in (0, 1),\]

\[u(0) = u(1) = 0.\]
Articles [17] and [29] were devoted to the multipoint bvp,

\[-u'' = f(u), \quad t \in (0,1),\]
\[u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),\]  \hspace{1cm} (1.2)

where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is \( C^1 \) with \( f(0) = 0 \), \( m \geq 3 \), \( \eta_i \in (0,1) \) and \( \alpha_i > 0 \) for \( i = 1, \ldots, m-2 \) with \( \sum_{i=1}^{m-2} \alpha_i < 1 \), by which Rynne [29] extended the result and filled some gaps in [17].

Roughly speaking, Rynne proved that bvp (1.2) admits \( 2^k \) nodal solutions when the interval whose extremities are \( \lim_{u \to 0} f(u)/u \) and \( \lim_{|u| \to +\infty} f(u)/u \) contains \( k \) eigenvalues of the linear bvp associated with (1.2),

\[-u'' = \lambda u, \quad t \in (0,1),\]
\[u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i).\]

This result was extended by Genoud and Rynne in [12] to the case with variable coefficients.

Existence and multiplicity of positive solutions for second order bvp's having singular dependence on the independent variable, have been considered in many papers; see, for example, [1, 3, 7, 9, 13, 14, 15, 32, 33, 34] and references therein. In particular, it is proved in [9, 15, 32] that, if the function \( a \) in bvp (1.1) is just continuous on \((0,1)\) and satisfies

\[\int_0^1 t(1-t)a(t)dt < \infty,\]  \hspace{1cm} (1.3)

then (1.1) admits one or more positive solutions under some additional conditions on the behavior of the ratio \( f(u)/u \) at 0 and \(+\infty\). A natural question becomes,

Is it possible to obtain existence results for nodal solutions to bvp (1.1) under Hypothesis (1.3)?

So, the main goal of this paper is to give an answer to this question.

In fact, we will give an answer for a more general bvp having a nonlinearity more general than (1.1), under a hypothesis looking like (1.3). This answer will be based on the knowledge of the spectrum of the linear problem associated with the nonlinear bvp. This was the case also for all the works in [4, 11, 12, 16, 17, 18, 19, 20, 21, 27, 29, 30, 31].

We need also in this work to introduce the concept of half-eigenvalue which generalizes the notion of eigenvalue. The definition of half-eigenvalue here is not the same given by Berysticki (see Remark 3.8), and for the role that will be played by this notion, we refer the reader to [4, 6, 11, 28, 29, 30].

A typical example of a weight function satisfying Hypothesis (1.3) is \( a(t) = t^{-3/2}(1-t)^{-3/2} \). Note that such a weight \( a \) is not integrable near 0 and 1. We have a similar situation in this work and this causes many difficulties in proving existence of half-eigenvalues as well as in proving the main results of this paper. The existence of half-eigenvalues will be obtained by sequential arguments. We will use in this work, the global bifurcation theorem of Rabinowitz to obtain our main results.
2. Main results

This article concerns the existence of nodal solutions for the bvp,

\[-(pu')(t) = f(t, u(t)), \text{ a.e. } t \in (\xi, \eta)\]

\[au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,\]

\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0,\]  \(2.1\)

where \(\xi, \eta \in \mathbb{R}\) with \(\xi < \eta\), \(a, b, c, d \in \mathbb{R}^+ = [0, +\infty)\), \(p : (\xi, \eta) \to \mathbb{R}^+\) is a measurable function and \(f : (\xi, \eta) \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function \((f(\cdot, u)\) is measurable for \(u\) fixed and \(f(t, \cdot)\) is continuous for \(t \in (\xi, \eta)\) a.e.).

Throughout this article, we assume that

\[\int_{\xi}^\eta \frac{d\tau}{p(\tau)} < \infty,\]  \(2.2\)

\[\Delta = ad + ac \int_{\xi}^\eta \frac{d\tau}{p(\tau)} + bc > 0.\]  \(2.3\)

Let

\[L^1_G[\xi, \eta] = \{q : (\xi, \eta) \to \mathbb{R} \text{ measurable, } \int_{\xi}^\eta G(t, t)|q(t)|dt < \infty\}\]

and let \(K_G\) be the cone of all functions \(q \in L^1_G[\xi, \eta]\) such that \(q(t) \geq 0\) a.e. \(t \in [\xi, \eta]\) and \(q > 0\) in a subset of a positive measure of \([\xi, \eta]\) where

\[G(t, s) = \frac{1}{\Delta} \begin{cases} \Phi_{ab}(s)\Psi_{cd}(t), & \xi \leq s \leq t \leq \eta, \\ \Phi_{ab}(t)\Psi_{cd}(s), & \xi \leq t \leq s \leq \eta. \end{cases} \]

is the Green’s function associated with the bvp

\[-(pu')(t) = 0, \text{ a.e. } t \in (\xi, \eta),\]

\[au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,\]

\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0,\]

and the functions \(\Phi_{ab}(t) = b + a \int_{t}^{1} 1/p(\tau) d\tau\) and \(\Psi_{cd}(t) = d + c \int_{t}^{s} 1/p(\tau) d\tau\) are well defined on \([\xi, \eta]\).

Note that the space \(L^1_G[\xi, \eta]\) depends on the parameters \(b, d, a\). In fact, we have that \(L^1_G[\xi, \eta] = L^1[\xi, \eta]\) if \(bd \neq 0\) and \(L^1_G[\xi, \eta]\) is nonempty if \(bd = 0\).

More precisely, we have that \(q \in L^1_G[\xi, \eta]\) is not integrable at \(\xi\) if and only if \(b = 0\) and \(q\) is not integrable at \(\eta\) if and only if \(d = 0\). For example, if \(p = 1\) and \(b = d = 0\) the function \(q(s) = (s(1-s))^{-3/2} \in L^1_G[\xi, \eta]\) \(L^1_G[\xi, \eta]\). Moreover, we have that \(L^1_G[\xi, \eta] \subset L^1_{loc}(\xi, \eta, G)\).

The main result of this article (Theorem 2.9) will be obtained under the following additional conditions on the nonlinearity \(f\):

There exist functions \(\alpha_{\infty}, \beta_{\infty}, \gamma_{\infty}, \delta_{\infty}\) and \(q_0\) in \(K_G\) such that the set

\[\{t \in (\xi, \eta) : \alpha_{\infty}(t)\beta_{\infty}(t) > 0\}\]
is of a positive measure,
\[
\lim_{u \to 0} \frac{f(t, u)}{u} = q_0(t) \quad \text{for } t \in [\xi, \eta] \text{ a.e.,}
\]
\[
\lim_{u \to -\infty} \frac{f(t, u)}{u} = \beta_\infty(t) \quad \text{for } t \in [\xi, \eta] \text{ a.e.,}
\]
\[
\lim_{u \to +\infty} \frac{f(t, u)}{u} = \alpha_\infty(t) \quad \text{for } t \in [\xi, \eta] \text{ a.e.}
\]
and
\[
\delta_\infty(t) \leq \frac{f(t, u)}{u} \leq \gamma_\infty(t) \quad \text{for all } u \in \mathbb{R} \text{ and } t \in [\xi, \eta] \text{ a.e.}
\] (2.5)

From all the above hypotheses, we understand that a solution to bvp (2.1) is a function \(u \in C[\xi, \eta] \cap C^1(\xi, \eta)\) with \((pu')' \in L^1_{\mathcal{C}}[\xi, \eta]\), satisfying all equations in (2.1).

**Remark 2.1.** Note that Hypothesis (2.5) implies that the nonlinearity \(f\) satisfies the following sign condition:
\[
f(t, u)u \geq 0 \quad \text{for all } u \in \mathbb{R} \text{ and } t \in [\xi, \eta] \text{ a.e.}
\]

**Example 2.2.** A typical example of a nonlinearity satisfying Hypotheses (2.4)-(2.5), when \(p = 1\) and \(b = d = 0\), is
\[
f(t, u) = At^{-3/2}(1-t)^{-5/4}u + Bt^{-7/6}(1-t)^{-7/4}\frac{u^3}{1 + u^2 + e^{-u}}
\]
\[
+ Ct^{-11/7}(1-t)^{-13/10}\frac{u^3}{1 + u^2 + e^{u}},
\]
where \(A, B, C\) are positive constants.

Throughout this article, we denote by \(E\) the Banach space of all continuous functions defined on \([\xi, \eta]\), equipped with the sup-norm denoted \(\| \cdot \|_\infty\) and by \(Y\) the Banach space defined as
\[
Y = \left\{ v \in AC[\xi, \eta] : pv' \in C[\xi, \eta] \text{ and } av(\xi) - b \lim_{t \to \xi} p(t)v'(t) = cv(\eta) + d \lim_{t \to \eta} p(t)v'(t) = 0 \right\}
\]
equipped with the norm \(\|v\|_Y = \|v\|_\infty + \|pv'\|_\infty\) for \(v \in Y\). In all this paper, \(\mathcal{L}\) is the differential operator given by
\[
\mathcal{L}u(x) = -(pv')'(x)
\]
with domain
\[
D(\mathcal{L}) = \left\{ v \in AC[\xi, \eta] : pv' \in C(\xi, \eta) \text{ and } (pv')' \in L^1_{\mathcal{C}}[\xi, \eta] \right\}.
\]
Set
\[
Y_\# = \left\{ v \in D(\mathcal{L}) : av(\xi) - b \lim_{t \to \xi} p(t)v'(t) = cv(\eta) + d \lim_{t \to \eta} p(t)v'(t) = 0 \right\}.
\]
We have that \(\mathcal{L} : Y_\# \to L^1_{\mathcal{C}}[\xi, \eta]\) is one to one, with
\[
\mathcal{L}^{-1}v(t) = \int_\xi^t G(t, s)v(s)ds \text{ for all } v \in L^1_{\mathcal{C}}[\xi, \eta].
\]
For \(u \in AC[\xi, \eta]\), \(u^{[1]}\) is the quasiderivative of \(u\), for \(t \in [\xi, \eta]\); that is, \(u^{[1]}(t) = \lim_{\tau \to t} p(\tau)u'(\tau)\) when it exists.
For $k \geq 1$, let $S_k^+$ denote the set of all functions $v \in AC[\xi, \eta]$ with $pv' \in C(\xi, \eta)$, having exactly $(k - 1)$ simple zeros in $(\xi, \eta)$ (if $v(\tau) = 0$ then $v^{[1]}(\tau) \neq 0$) and $v$ is positive in a right neighborhood of $\xi$, and denote $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. Let

$$\rho_0 = \left( \int_{\xi}^{\eta} \frac{ds}{p(s)} \right)^{-1} (\eta - \xi),$$

$$C_{\#}^1[\xi, \eta] = \{ v \in C^1[\xi, \eta] : av(\xi) - b p_0 v'(\xi) = 0 \text{ and } cv(\eta) + d \rho_0 v'(\eta) = 0 \}$$

equipped with the $C^1$-norm and, for all $k \geq 1$ let $\Theta_k^\pm$ be the set of all functions $v \in C_{\#}^1[\xi, \eta]$ having exactly $(k - 1)$ simple zeros in $(\xi, \eta)$ and $v$ is positive in a right neighborhood of $\xi$, $\Theta_k^- = -\Theta_k^+$ and $\Theta_k^\pm = \Theta_k^+ \cup \Theta_k^-$. It is well known that $\Theta_k^\pm$, $\Theta_k^\pm$ and $\Theta_k^\pm$ are open sets in $C_{\#}^1[\xi, \eta]$. Since for all $k \geq 1$ and $\nu = +$ or $-$, $\Phi(S_k^\nu \cap Y) = \Theta_k^\nu$ where $\Phi: Y \to C_{\#}^1[\xi, \eta]$ is the homeomorphism between Banach spaces defined by

$$\Phi(u) = u \circ \varphi^{-1} \text{ with } \varphi(t) = \xi + \rho_0 \int_{\xi}^{t} \frac{ds}{p(s)},$$

we have that $S_k^\nu \cap Y$ is an open set in $Y$. Moreover, since if $u \in \partial \Theta_k^\nu$ then there exists $\tau \in [\xi, \eta]$ such that $u(\tau) = u'(\tau) = 0$, we have that for all $v \in \partial(S_k^\nu \cap Y)$ there exists $\tau \in [\xi, \eta]$ such that $u(\tau) = u^{[1]}(\tau) = 0$.

For $\nu = +$ or $-$, let $I^\nu : E \to E$ be defined by $I^\nu u(x) = \max(\nu u(x), 0)$, for $u \in E$. For all $u \in E$, we have

$$u = I^+ u - I^- u \quad \text{and} \quad |u| = I^+ u + I^- u.$$  

This implies that, for all $u, v \in E$,

$$|I^+ u - I^+ v| \leq \frac{|u - v|}{2} + \frac{||u| - |v||}{2} \leq |u - v|,$$

$$|I^- u - I^- v| \leq \frac{|u - v|}{2} + \frac{||u| - |v||}{2} \leq |u - v|,$$

and the operators $I^+$, $I^-$ are continuous.

For sake of simplicity, throughout this paper, we will use $u^+$ and $u^-$ instead of $I^+ u$ and $I^- u$, respectively. Now we focus our attention on the eigenvalue bvp

$$-(pu')(t) = \lambda (\alpha(t) u^+(t) - \beta(t) u^-(t)), \quad \text{a.e. } t \in (\xi, \eta),$$

$$au(\xi) - \beta \lim_{t \to \xi} p(t) u'(t) = 0,$$

$$cu(\eta) + d \lim_{t \to \eta} p(t) u'(t) = 0,$$

where $\alpha$ and $\beta$ are functions in $K_G$ such that the set $\{ t \in (\xi, \eta) : \alpha(t)\beta(t) > 0 \}$ is of a positive measure, and $\lambda$ is a real parameter.

**Definition 2.3.** We say that $\lambda$ is a half-eigenvalue of (2.7) if there exists a non-trivial solution $(\lambda, u_\lambda)$ of (2.7). In this situation, $\{(\lambda, t u_\lambda), \ t > 0 \}$ is a half-line of nontrivial solutions of (2.7) and $\lambda$ is said to be simple if all solutions $(\lambda, v)$ of (2.7) with $v$ and $u_\lambda$ having the same sign on a right neighborhood of $\xi$ are on this half-line. There may exist another half-line of solutions $\{(\lambda, tv_\lambda), \ t > 0 \}$, but then we say that $\lambda$ is simple if $u_\lambda$ and $v_\lambda$ have different signs on a right neighborhood of $\xi$ and all solutions $(\lambda, v)$ of (2.7) lie on these two half lines.
Theorem 2.4. Assume that (2.2) and (2.3) hold, and $\alpha, \beta \in K_G \cap L^1[\xi, \eta]$. Then the set of half-eigenvalues to bvp (2.7) consists of two increasing sequences $(\lambda^+_k)_{k \geq 1}$ and $(\lambda^-_k)_{k \geq 1}$ such that for all $k \geq 1$ and $\nu = +$ or $-$,

1. $\lambda^+_k$ is simple and is the unique half-eigenvalue having a half-line of solutions in $\{ \lambda^+_k \} \times S^+_k$.
2. $\lambda^-_k$ is a nondecreasing function with the respect of each of the weights $\alpha$ and $\beta$ lying in $L^1[\xi, \eta]$.

Theorem 2.5. Assume that (2.2) and (2.3) hold. Then the set of half-eigenvalues to bvp (2.7) consists of two nondecreasing sequences $(\lambda^+_k)_{k \geq 1}$ and $(\lambda^-_k)_{k \geq 1}$ such that for all $k \geq 1$ and $\nu = +$ or $-$, $\lambda^+_k$ is the unique half eigenvalue having a half-line of solutions in $\{ \lambda^+_k \} \times S^+_k$. Moreover, for all $k \geq 1$ and $\nu = +$ or $-$, $\lambda^+_k$ is a nondecreasing function with the respect of each of the weights $\alpha$ and $\beta$ lying in $L^1[\xi, \eta]$.

Remark 2.6. It is clear that for all $k \geq 1$ and $\nu = +$ or $-$, $\lambda^+_k$ depends on the weights $p, \alpha, \beta$ and on $(\xi, \eta, a, b, c, d)$. When there is no confusion, we just denote $\lambda^+_k$, and when we need to be more precise, we write $\lambda^+_k(\alpha, \beta)$.

Consider the bvp,

$$Lu(t) = \mu q(t)u(t), \quad \text{a.e. } t \in (\xi, \eta),$$
$$au(\xi) - b \lim_{t \to \xi^-} p(t)u'(t) = 0,$$
$$cu(\eta) + d \lim_{t \to \eta^+} p(t)u'(t) = 0,$$

(2.8)

where $\mu$ is a real parameter, and $q \in K_G$.

It is clear that if $\mu$ is an eigenvalue for (2.8) then $\mu$ depends on the weights $p, q$ and on $(\xi, \eta, a, b, c, d)$. When there is no confusion, we just denote $\mu(q)$, and when we need to be more precise, we write $\mu(q, [\xi, \eta])$.

Theorem 2.7. Assume that (2.2) and (2.3) hold. Then bvp (2.8) admits a sequence of eigenvalues $(\mu_k(q))_{k \geq 1}$ such that:

1. For all $k \geq 3$, $\mu_k(q)$ is simple and the associated eigenfunction $\phi_k \in S_k$.
2. For all $k \geq 3$, $\mu_k(q) < \mu_{k+1}(q)$.
3. If $bd \neq 0$, or $bd = 0$ and $q \in L^1[\xi, \eta]$, then $\mu_1(q, [\xi, \eta]) < \mu_2(q, [\xi, \eta])$ and $\mu_1(q) = \mu_2(q)$ are simple having eigenvectors respectively in $S_1$ and $S_2$. If $bd = 0$ and $q \notin L^1[\xi, \eta]$, then $\mu_1(q) = \mu_2(q)$ and $\mu_1(q) = \mu_2(q)$ is double having two eigenvectors $\phi_{1,1} \in S_1$ and $\phi_{1,2} \in S_2$.
4. For all $k \geq 1$ and $\theta > 0$, $\mu_k(q \theta) = \frac{\mu_k(q)}{\theta}$.
5. Let $q_1 \in K_G$. We have $\mu_k(q_1) \geq \mu_k(q)$ for all $k \geq 1$ whenever $q_1 \leq q$.
6. If $[\xi_1, \eta_2] \subset [\xi, \eta]$ then $\mu_k(q, [\xi, \eta]) \leq \mu_k(q, [\xi_1, \eta_2])$.
7. For all $k \geq 1$, $\mu_k([\xi, \eta]) : K_G \to \mathbb{R}$ is continuous.

Remark 2.8. Since the weight $q$ in Theorem 2.7 is not necessarily in $L^1[\xi, \eta]$, Theorem 2.7 is not covered by [34] Theorems 4.3.1, 4.3.2, 4.3.3, 4.3.4.

For the statement of the main results of this paper we introduce the following notation:

$$\theta_\infty(t) = \max(\alpha_\infty(t), \beta_\infty(t)), \quad \vartheta_\infty(t) = \min(\alpha_\infty(t), \beta_\infty(t)),$$

and let $(\mu_k(\theta_\infty))_{k \geq 1}$, $(\mu_k(\vartheta_\infty))_{k \geq 1}$ and $(\mu_k(q_0))_{k \geq 1}$ be respectively the sequences of eigenvalues given by Theorem 2.7 for $q = \theta_\infty$, $q = \vartheta_\infty$ and $q = q_0$. 
Theorem 2.9. Assume that (2.4)–(2.5) hold and that there exist two integers \( k, l \) with \( 2 < k < l \) such that one of the following situations holds:
\[
\mu_l(q_0) < 1 < \mu_k(\theta_\infty)
\]
or
\[
\mu_l(\theta_\infty) < 1 < \mu_k(q_0).
\]
Then for all \( i \in \{k, \ldots, l\} \) and \( \nu = + \) or \( - \), (2.1) has a solution in \( S^\nu_i \).

Consider the separated variables case
\[
Lu(t) = q_{sv}(t)h(u(t)), \quad \text{a.e. } t \in (\xi, \eta)
\]
\[
au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0
\]
\[
cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0
\]
where \( q_{sv} \) is a nonnegative function in \( L^1_{\mathcal{V}}[\xi, \eta] \) which does not vanish identically in
\([\xi, \eta]\) and \( h : \mathbb{R} \to \mathbb{R} \) is a continuous function such that
\[
h(x)x > 0 \quad \text{for all } x \neq 0
\]
\[
\lim_{x \to -\infty} \frac{h(x)}{x} = h_0, \quad \lim_{x \to +\infty} \frac{h(x)}{x} = h_{+\infty},
\]
\[
\lim_{x \to +\infty} h(x) = h_{-\infty} \quad \text{with } h_0, h_{+\infty}, h_{-\infty} \in (0, +\infty).
\]

The following corollary provides an answer to a more general situation than those
studied in [20] and [21] and also covers [22, Theorems 2 and 3].

Corollary 2.10. Assume that (2.10) holds and there exist two integers \( k, l \) with
\( 2 < k < l \), such that one of the following situations holds:
\[
h_{+\infty}, h_{-\infty} < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_0
\]
or
\[
h_0 < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_{+\infty}, h_{-\infty}.
\]
Then for all \( i \in \{k, \ldots, l\} \) and \( \nu = + \) or \( - \), (2.4) has a solution in \( S^\nu_i \).

Proof. Set \( f(t, u) = q_{sv}(t)h(u) \). It is easy to check that \( f \) satisfies Hypotheses
(2.4)–(2.5), with
\[
\alpha_{\infty}(t) = h_{+\infty}q_{sv}(t), \quad \beta_{\infty}(t) = h_{-\infty}q_{sv}(t), \quad q_0(t) = h_0q_{sv}(t),
\]
\[
\gamma_{\infty} = h_{\sup}q_{sv}(t), \quad \delta_{\infty} = h_{\inf}q_{sv}(t),
\]
\[
h_{\inf} = \inf\{h(u)/u : u \neq 0\},
\]
\[
h_{\sup} = \sup\{h(u)/u : u \neq 0\}.
\]
Also, we have
\[
\theta_{\infty}(t) = \max(h_{+\infty}, h_{-\infty})q_{sv}(t), \quad \vartheta_{\infty}(t) = \min(h_{+\infty}, h_{-\infty})q_{sv}(t).
\]
Since Property 2 of Theorem 2.7 implies that for all \( n \geq 1 \),
\[
\mu_n(\theta_\infty) = \frac{\mu_n(q_{sv})}{\max(h_{+\infty}, h_{-\infty})}, \quad \mu_n(\vartheta_\infty) = \frac{\mu_n(q_{sv})}{\min(h_{+\infty}, h_{-\infty})}, \quad \mu_n(q_0) = \frac{\mu_n(q_{sv})}{h_0},
\]
we obtain that \( \mu_1(q_0) < 1 < \mu_k(\theta_\infty) \) if \( h_{+\infty}, h_{-\infty} < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_0 \) and \( \mu_l(\vartheta_\infty) < 1 < \mu_2(q_0) \) if \( h_0 < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_{+\infty}, h_{-\infty} \). Thus, the conclusion of Corollary 2.10 follows from Theorem 2.9.\( \square \)
3. Background

3.1. A comparison result.

**Theorem 3.1.** Let u and v be two functions in $S_k^v$. Then, there exist two intervals $[\xi_i, \eta_i]$ and $[\xi_2, \eta_2]$ such that $uv \geq 0$ in $[\xi_i, \eta_i]$ if $i = 1, 2$. Moreover if $pu', pv' \in \text{AC} [\xi_i, \eta_i]$ then

$$
\int_{\xi_1}^{\eta_1} vLu - uLv \geq 0, \quad \int_{\xi_2}^{\eta_2} vLu - uLv \leq 0.
$$

The proof of this theorem is based on the following lemma.

**Lemma 3.2**. Let j and k be two integers such that $j \geq k \geq 2$. Suppose that there exist two families of real numbers

$$
\xi_0 = \xi < \xi_1 < \xi_2 < \ldots < \xi_{k-1} < \xi_k = \eta
$$

and

$$
\eta_0 = \xi < \eta_1 < \eta_2 < \ldots < \eta_{j-1} < \eta_j = \eta.
$$

Then, if $\xi_1 \leq \eta_1$ there exist two intervals $j_0$ and $k_0$ having the same parity, $1 \leq j_0 \leq j - 1$, $1 \leq k_0 \leq k - 1$ such that

$$
\xi_{j_0} \leq \eta_{j_0} \leq \eta_{j_0+1} \leq \xi_{k_0+1}.
$$

**Proof of Theorem 3.1.** The case $k = 1$ is obvious. Let

$$
x_0 = \xi < x_1 < x_2 < \ldots < x_{k-1} < x_k = \eta,
$$

$$
z_0 = \xi < z_1 < z_2 < \ldots < z_{k-1} < z_k = \eta
$$

be the sequences of zeros of u and v, respectively.

Suppose $x_1 \leq z_1$. Then we deduce from Lemma 3.2 the existence of integers $k_0$ and $j_0$ having the same parity such that $x_{j_0} \leq z_{j_0} \leq x_{j_0+1} \leq x_{k_0+1}.

Therefore, we choose $\xi_1 = x_0$, $\eta_1 = x_1$, $\xi_2 = z_{j_0}$ and $\eta_2 = z_{j_0+1}$, and since $k_0$ and $j_0$ have the same parity, u and v have the same sign in both the intervals $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$.

Now if $u, v \in S_k^v \cap Y$, then for $i = 1, 2$, $u^{[1]}(\xi_i)$, $v^{[1]}(\xi_i)$, $u^{[1]}(\eta_i)$ and $v^{[1]}(\eta_i)$ exist and are finite. Moreover if $u \geq 0$ and $v \geq 0$ in $[\xi_i, \eta_i]$ for $i = 1, 2$ (the other cases can be checked similarly) then we have since $u$ and $v$ satisfy the boundary condition at $\xi_1$,

$$
v(\xi_1)u^{[1]}(\xi_1) - u(\xi_1)v^{[1]}(\xi_1) = 0, \quad u^{[1]}(\eta_1) \leq 0,
$$

and

$$
v^{[1]}(\xi_2) \geq 0, \quad v^{[1]}(\eta_2) \leq 0,
$$

$$
-v(\eta_2)u^{[1]}(\eta_2) + u(\eta_2)v^{[1]}(\eta_2) = \begin{cases} 0, & \text{if } \eta_2 = \eta, \\ u(\eta_2)v^{[1]}(\eta_2) \leq 0, & \text{if } \eta_2 < \eta. \end{cases}
$$

Thus, we obtain

$$
\int_{\xi_1}^{\eta_1} vLu - uLv = -v(\eta_1)u^{[1]}(\eta_1) \geq 0, \quad \int_{\xi_2}^{\eta_2} vLu - uLv = -v(\eta_2)u^{[1]}(\eta_2) + u(\eta_2)v^{[1]}(\eta_2) - u(\xi_2)v^{[1]}(\xi_2) \leq 0.
$$

This completes the proof. \qed
3.2. **Spectral radius of a positive operator.** Let $Z$ be a real Banach space and $L(Z)$ the Banach space of linear continuous operators from $Z$ into $Z$. For $L \in L(Z)$, $r(L) = \lim \|L^n\|^\frac{1}{n}$ denotes the spectral radius of $L$.

A nonempty closed convex subset $K$ of $Z$ is said to be an ordered cone if $(tK) \subseteq K$ for all $t \geq 0$ and $K \cap (-K) = \{0\}$. Moreover if $Z = K - K$, the cone $K$ is said to be total. It well known that an ordered cone $K$ induces a partial order on the Banach space $Z$ ($x \leq y$ if and only if $y - x \in K$ for all $x, y \in Z$).

Let $K$ be an ordered cone of $Z$ and $L \in L(Z)$. $L$ is said to be positive if $L(K) \subseteq K$ and $\mu \in \mathbb{R}$ is said to be a positive eigenvalue of $L$ if there exists $u \in K \setminus \{0\}$ such that $Lu = \mu u$.

Let $L_1, L_2 \in L(Z)$ be two positive operators. We write $L_1 \leq L_2$ if $L_1 u \leq L_2 u$ for all $u \in K$.

We will use in this work the following result known as the Krein-Rutman Theorem.

**Theorem 3.3** ([35 Proposition 7.26]). Assume that the cone $K$ is total and $L \in L(Z)$ is compact and positive with $r(L) > 0$. Then $r(L)$ is a positive eigenvalue of $L$.

We will use also the following lemma.

**Lemma 3.4** ([35 Corollary 7.28]). Assume that the cone $K$ is total and let $L_1, L_2$ in $L(Z)$ be two compact and positive operators. If $L_1 \leq L_2$, then $r(L_1) \leq r(L_2)$.

Next we recall a fundamental result proved by Nussbaum in [23] and used in [3].

**Lemma 3.5.** Let $(L_n)$ be a sequence of compact linear operators on a Banach space $Z$ and suppose that $L_n \rightarrow L$ in operator norm as $n \rightarrow \infty$. Then $r(L_n) \rightarrow r(L)$.

3.3. The linear eigenvalue bvp in the integrable case.

**Theorem 3.6.** Assume that Hypotheses (2.2) and (2.3) hold and $q \in K_G \cap L^1[\xi, \eta]$. Then the set of eigenvalues to bvp (2.8) consists of an increasing sequence of simple eigenvalues $(\mu_k(q))_{k \geq 1}$ tending to $+\infty$, such that for all $k \geq 1$,

1. The eigenfunction $\phi_k$ associated with $\mu_k(q)$ belongs to $S_k$.
2. If $\theta > 0$ then $\mu_k(\theta q) = \frac{\mu_k(q)}{\theta}$.
3. Let $q_1$ be a nonnegative function in $L^1[\xi, \eta]$ which does not vanish identically in $[\xi, \eta]$. We have $\mu_k(q_1) \geq \mu_k(q)$ for all $k \geq 1$ whenever $q_1 \leq q$. Moreover, if $q_1 < q$ in a subset of a positive measure then $\mu_k(q_1) > \mu_k(q)$.
4. If $[\xi_1, \eta_1] \subseteq [\xi, \eta]$ then $\mu_k(q, [\xi, \eta]) < \mu_k(q, [\xi_1, \eta_1])$.
5. $\mu_k$ is a continuous function with respect to the variable $q$ lying in $L^1[\xi, \eta]$.

**Proof.** From Theorem 4.3.2 in [30], the bvp (2.8) has only real and simple eigenvalues and they are ordered to satisfy

$$-\infty < \mu_1 < \mu_2 < \cdots < \lim_{k \rightarrow \infty} \mu_k = +\infty.$$ 

Moreover if $\phi_k$ is an eigenfunction of $\mu_k$, and $n_k$ denotes the number of zeros of $\phi_k$ in $(\xi, \eta)$, then $\phi_k \in AC[\xi, \eta], p\phi'_k \in AC[\xi, \eta]$ and $n_{k+1} = n_k + 1$. Now, we have

$$0 < -\phi_k(\eta)\phi_k(\eta) + \phi_k(\xi)\phi_k(\xi) + \int_\xi^\eta p(\phi'_k)^2 = \int_\xi^\eta \phi_k L \phi_k = \mu_k \int_\xi^\eta q\phi_k^2$$

leading to $\mu_k > 0$ for all $k \geq 1$. 
Let \( L_q : E \to E \) be defined by
\[
L_q u(t) = \int_{\xi}^{t} G(t, s) q(s) u(s) ds = \mathcal{L}^{-1}(q u)(t).
\]

It is easy to see that \( L_q \) is a positive operator with respect to the total cone of nonnegative functions in \( E \) and \( \lambda \) is an eigenvalue of \( L_q \) if and only if \( \lambda^{-1} \) is an eigenvalue of \( \text{bvp (2.8)} \). Also, the presence of eigenvalues implies that \( r(L_q) > 0 \). Thus, we deduce from Theorem 3.3 that \( r(L_q) \) is the largest and positive eigenvalue of \( L_q \), and so, we have \( \mu_1 = 1/r(L_q) \) and \( n_1 = 0 \) and for all \( k \geq 2 \), \( n_k = k - 1 \). That is \( \phi_k \in S_k \) and Assertion 1, is proved.

Assertion 2 is obvious and since \( \mu_k > 0 \) for all \( k \geq 1 \). Assertion 3 follows directly from [36] Theorem 4.9.1.

To prove Property 4, let \([\xi', \eta'] \subset [\xi, \eta]\), and \( \phi \) and \( \psi \) be such that
\[
\mathcal{L} \phi = \mu_k(q) q \phi, \quad \text{a.e. } t \in (\xi, \eta),
\]
\[
a u(\xi) - b \lim_{t \to \xi} p(t) u'(t) = 0,
\]
\[
c u(\eta) + d \lim_{t \to \eta} p(t) u'(t) = 0,
\]
and
\[
\mathcal{L} \psi = \mu_k(q, [\xi_1, \eta_1]) q \psi, \quad \text{a.e. } t \in (\xi, \eta),
\]
\[
a u(\xi_1) - b \lim_{t \to \xi_1} p(t) u'(t) = 0,
\]
\[
c u(\eta_1) + d \lim_{t \to \eta_1} p(t) u'(t) = 0.
\]

Denote by \((x_i)_{1 \leq i \leq k}\) and \((y_j)_{1 \leq j \leq k}\), respectively, the two sequences of zeros of \( \phi \) and \( \psi \). There exist two integers \( 1 \leq i_0, j_0 \leq k \) such that one of the following two situations holds:
\[
\xi \leq x_{i_0-1} \leq y_{j_0-1} < y_{j_0} < x_{i_0} \leq \eta;
\]
\[
\xi \leq x_{i_0-1} < y_{j_0-1} < y_{j_0} \leq x_{i_0} \leq \eta.
\]

Without loss of generality, suppose \( \phi \) and \( \psi \) are positive, respectively, in \((x_{i_0-1}, x_{i_0})\) and \((y_{j_0-1}, y_{j_0})\) and \([3.3] \) holds. Then we have
\[
\psi(y_{j_0}) = 0, \quad \phi(y_{j_0}) > 0, \quad \psi^{[1]}(y_{j_0}) < 0, \quad \psi^{[1]}(y_{j_0-1}) \geq 0, \quad \phi(y_{j_0-1}) \geq 0,
\]
and
\[
\psi^{[1]}(y_{j_0-1}) \phi(y_{j_0-1}) + \phi^{[1]}(y_{j_0-1}) \psi(y_{j_0-1})
\]
\[
= 0 \quad \text{if } y_{j_0-1} = \xi
\]
\[
= -\psi^{[1]}(y_{j_0-1}) \phi(y_{j_0-1}) \leq 0 \quad \text{if } y_{j_0-1} > \xi.
\]

From which we obtain
\[
\int_{y_{j_0}}^{y_{j_0}} q \psi \phi
\]
\[
= \int_{y_{j_0}}^{y_{j_0}} \psi \mathcal{L} \phi - \phi \mathcal{L} \psi
\]
\[
= \psi^{[1]}(y_{j_0}) \phi(y_{j_0}) - \psi^{[1]}(y_{j_0-1}) \phi(y_{j_0-1}) + \phi^{[1]}(y_{j_0-1}) \psi(y_{j_0-1}) < 0
\]
leading to
\[ \mu_k(q, \xi, \eta) < \mu_k(q, [\xi_1, \eta_1]). \]
Finally, Property 5 is obtained from \[35\] Theorem 3.5.2. \[ \square \]

3.4. **Berestycki’s half-eigenvalue bvp.** Let \( m, \alpha \) and \( \beta \) be three continuous functions on \( \xi, \eta \) with \( m > 0 \) in \( \xi, \eta \) and consider the bvp,
\[
\mathcal{L}u = \lambda mu + \alpha u^+ - \beta u^- \quad \text{in} \ (\xi, \eta), \\
au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0 \\
cu(\eta) + d \lim_{t \to \eta} p(t)u'(\eta) = 0, \\
\]
where \( \lambda \) is a real parameter.

Bvp (3.1) is called half-linear since it is linear and positively homogeneous in the cones \( u \geq 0 \) and \( u \leq 0 \).

**Definition 3.7.** We say that \( \lambda \) is a half-eigenvalue of (3.1) if there exists a nontrivial solution \((\lambda, u_\lambda)\) of (3.1). In this situation, \((\lambda, tu_\lambda), t > 0\) is a half-line of nontrivial solutions of (3.1) and \( \lambda \) is said to be simple if all solutions \((\lambda, v)\) of (3.1) with \( v \) and \( u \) having the same sign on a deleted neighborhood of \( \xi \) are on this half-line. There may exist another half-line of solutions \((\lambda, tv_\lambda), t > 0\), but then we say that \( \lambda \) is simple if \( u_\lambda \) and \( v_\lambda \) have different signs on a deleted neighborhood of \( \xi \) and all solutions \((\lambda, v)\) of (3.1) lie on these two half lines.

**Remark 3.8.** Note that the position of the real parameter in the differential equation in (3.1) is not same as in Problem (2.7). Moreover, we have Problem (3.1) coincides with the linear eigenvalue problem when \( \alpha = \beta = 0 \), even though Problem (2.7) coincides with the linear eigenvalue problem when \( \alpha = \beta \).

Berestycki proved in [4] the following theorem.

**Theorem 3.9.** Assume that \( p \in C^1[\xi, \eta] \) and \( p > 0 \) in \([\xi, \eta]\). Then the set of half eigenvalues of bvp (3.1) consists of two increasing sequences of simple half-eigenvalues for bvp (3.1) \((\lambda_k)_{k \geq 1}\) and \((\lambda_k')_{k \geq 1}\), such that for all \( k \geq 1 \) and \( \nu = + \) or \(-\), the corresponding half-lines of solutions are in \((\lambda'_k) \times S'_k\).

**Proposition 3.10.** Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in C([\xi, \eta]). \) We have
- If \( \alpha_1 \leq \alpha_2 \), then \( \lambda_k'(\alpha_2) \geq \lambda_k'(\alpha_1) \), for all \( k \geq 1 \) and \( \nu = + \) or \(-\).
- If \( \beta_1 \leq \beta_2 \), then \( \lambda_k(\beta_2) \geq \lambda_k(\beta_1) \), for all \( k \geq 1 \) and \( \nu = + \) or \(-\).

**Proof.** We present the proof of the first assertion. The second one can be proved in a similar way. Let \( \phi_1, \phi_2 \) be such that
\[
\mathcal{L}\phi_1 = \lambda_k'(\alpha_1)m\phi_1 + \alpha_1\phi_1^+ - \beta\phi_1^- \quad \text{in} \ (\xi, \eta), \\
ap(\xi) - b \lim_{t \to \xi} p(t)\phi_1'(t) = 0, \\
c\phi_1(\eta) + d \lim_{t \to \eta} p(t)\phi_1'(\eta) = 0, \\
\]
and
\[
\mathcal{L}\phi_2 = \lambda_k'(\alpha_2)m\phi_2 + \alpha_1\phi_2^+ - \beta\phi_2^- \quad \text{in} \ (\xi, \eta), \\
ap(\xi) - b \lim_{t \to \xi} p(t)\phi_2'(t) = 0, \\
c\phi_2(\eta) + d \lim_{t \to \eta} p(t)\phi_2'(\eta) = 0. \\
\]
Note that \( \phi_1, \phi_2 \in S_k \cap C^2[\xi, \eta] \) and let \([\xi_1, \eta_1]\) be the interval given by Theorem 3.1 for the functions \( \phi_1 \) and \( \phi_2 \). Since \( \phi_1 \) and \( \phi_2 \) have the same sign in \((\xi_1, \eta_1)\), after simple computations we obtain
\[
(\lambda^+_k(\alpha_1) - \lambda^+_k(\alpha_2)) \int_{\xi_1}^{\eta_1} m \phi_1 \phi_2 = - \int_{\xi_1}^{\eta_1} (\alpha_1 - \alpha_2) \phi_1 \phi_2 + \int_{\xi_1}^{\eta_1} \phi_2 L \phi_1 - \phi_1 L \phi_2 \geq 0
\]
leading to
\[
\lambda^+_k(\alpha_1) \geq \lambda^+_k(\alpha_2).
\]
This completes the proof. \(\blacksquare\)

**Remark 3.11.** Naturally one can ask, is it possible to extend Berestycki’s theorem to the case where the weight \( m \), as well as \( \alpha \) and \( \beta \) all belong to \( L^1[\xi, \eta] \)?

This is technically difficult since a half-eigenvalue of (3.1) is decreasing with respect to the weight \( m \) only if it is positive.

### 3.5. Fučík spectrum

Consider now the bvp,
\[
-u''(t) = \alpha u^+(t) - \beta u^-(t), \quad t \in (\xi, \eta),
\]
\[
au(\xi) - bu'(\xi) = 0,
\]
\[
au(\eta) + du'(\eta) = 0,
\]
where \( \alpha, \beta \) are positive real parameters and \( a, b, c, d \in \mathbb{R}^+ \) with \( ac + ad + bc > 0 \).

The statement of the next result requires introducing the functions \( \Lambda_{a,b,c,d}, \Lambda_{a,b} : (0, +\infty) \to (0, +\infty) \) defined, for \( \sigma > 0 \), by
\[
\Lambda_{a,b,c,d}(\sigma) = \frac{1}{\sqrt{\sigma}} \left( \pi - \arcsin \left( \sqrt{\frac{b^2 \sigma}{a^2 + b^2 \sigma}} \right) - \arcsin \left( \sqrt{\frac{d^2 \sigma}{c^2 + d^2 \sigma}} \right) \right),
\]
and
\[
\Lambda_{a,b}(\sigma) = \frac{1}{\sqrt{\sigma}} \left( \pi - \arcsin \left( \sqrt{\frac{b^2 \sigma}{a^2 + b^2 \sigma}} \right) \right).
\]

Note that \( \Lambda_{a,b} = \Lambda_{a,b,1,0} = \Lambda_{1,0,a,b} \). The sets \( S^+_k \), \( S^-_k \) and \( S_k \) are those introduced in Section 2 for \( p = 1 \). The main goal of this subsection is to describe the set
\[
F_s = \{ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : (3.2) \text{ has a solution } \}
\]
known as the Fučík spectrum.

**Theorem 3.12.** Let \( S \) be the set of solutions to bvp (3.2). Then \( S \subset \bigcup_{k \geq 1} S_k \).

Moreover bvp (3.2) admits a solution
1. in \( S^+_1 \) if and only if \( \Lambda_{a,b,c,d}(\alpha) = \eta - \xi \),
2. in \( S^-_1 \) if and only if \( \Lambda_{a,b,c,d}(\beta) = \eta - \xi \),
3. in \( S^+_{2l} \) with \( l \geq 1 \) if and only if
\[
\Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\beta) + \pi(l - 1) \left( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} \right) = \eta - \xi,
\]
4. in \( S^-_{2l} \) with \( l \geq 1 \) if and only if
\[
\Lambda_{a,b}(\beta) + \Lambda_{c,d}(\alpha) + \pi(l - 1) \left( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} \right) = \eta - \xi,
\]
5. in \( S^+_{2l+1} \) with \( l \geq 1 \) if and only if
\[
\Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\alpha) + \frac{\pi(l - 1)}{\sqrt{\alpha}} + \frac{\pi l}{\sqrt{\beta}} = \eta - \xi,
\]
(6) in $S_{2l}^{\nu}$ with $l \geq 1$ if and only if

$$\Lambda_{a,b}(\beta) + \Lambda_{c,d}(\beta) + \frac{\pi(l-1)}{\sqrt{\beta}} + \frac{\pi l}{\sqrt{\alpha}} = \eta - \xi.$$  

**Proof.** First, note that $u$ is a solution to (3.2) if and only if $v(t) = u((\eta - \xi) t + \xi)$ is a solution to the bvp

$$-v''(t) = (\eta - \xi)^2 \alpha v^+(t) - (\eta - \xi)^2 \beta v^-(t), \quad t \in (0, 1),$$

$$av(0) - \frac{b}{(\eta - \xi)} v'(0) = 0,$$

$$cv(1) + \frac{d}{(\eta - \xi)} v'(1) = 0.$$  

Then, Assertions 1 and 2 of Theorem 3.12 follow from Proposition 3.1 in [2].

Now, for the sake of brevity, we prove only Assertion 3 (the others can be proved similarly). Note that $u \in S_{2l}^{\nu}$ is a solution to (3.2) if and only if there exists a finite sequence $(x_i)_{i=0}^{2l}$ such that

$$\xi = x_0 < x_1 < \cdots < x_{2l-1} < x_{2l} = \eta$$

and

$$u > 0 \quad \text{in} \quad (x_{2i}, x_{2i+1}) \quad \text{for} \quad i = 0, \ldots, (l - 1),$$

$$u < 0 \quad \text{in} \quad (x_{2i-1}, x_{2i}) \quad \text{for} \quad i = 1, \ldots, l.$$  

Moreover, $u$ satisfies

$$-u''(t) = \alpha u(t), \quad t \in (\xi, x_1),$$

$$au(\xi) - bu'(\xi) = u(x_1) = 0,$$

and for $i = 1, \ldots, (l - 1)$:

$$-u''(t) = \alpha u(t), \quad t \in (x_{2i}, x_{2i+1}), \quad u(x_{2i+1}) = 0,$$

$$-u''(t) = \beta u(t), \quad t \in (x_{2i-1}, x_{2i}),$$

$$u(x_{2i-1}) = u(x_{2i}) = 0,$$

and

$$-u''(t) = \alpha u^+(t) - \beta u^-(t), \quad t \in (x_{2l-1}, \eta),$$

$$u(x_{2l-1}) = cu(\eta) + du'(\eta) = 0.$$  

Hence, from Assertions 1 and 2, we obtain

$$\frac{1}{\sqrt{\alpha}} (\pi - \arcsin \left( \sqrt{\frac{b^2 \alpha}{a^2 + b^2 \alpha}} \right)) = x_1 - \xi,$$

$$\frac{\pi}{\sqrt{\alpha}} = (x_{2i+1} - x_{2i}) \quad \text{for} \quad i = 1, \ldots, (l - 1),$$

$$\frac{\pi}{\sqrt{\beta}} = (x_{2i} - x_{2i-1}) \quad \text{for} \quad i = 1, \ldots, (l - 1),$$

$$\frac{1}{\sqrt{\beta}} (\pi - \arcsin \left( \sqrt{\frac{d^2 \beta}{c^2 + d^2 \beta}} \right)) = (\eta - x_{2l-1}).$$
Summing the above equalities, we obtain
\[ \Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\beta) + \pi(t - 1) \left( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} \right) = \eta - \xi. \]
Conversely, let \( \alpha, \beta > 0 \) be such that
\[ \Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\beta) + \pi(t - 1) \left( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} \right) = \eta - \xi, \tag{3.3} \]
and let \( (x_i)_{i=0}^{2l} \) be the sequence defined by
\[
\begin{align*}
x_0 &= \xi, & x_1 &= \xi + \frac{1}{\sqrt{\alpha}} \left( \pi - \arcsin \left( \sqrt{\frac{b^2 \alpha}{a^2 + b^2 \alpha}} \right) \right), \\
x_{2i} &= x_{2i-1} + \pi \frac{\sqrt{\beta}}{\sqrt{\alpha}} \quad \text{for } i = 1, \ldots, (l - 1), \\
x_{2i+1} &= x_{2i} + \pi \frac{\sqrt{\alpha}}{\sqrt{\beta}} \quad \text{for } i = 1, \ldots, (l - 1), \quad x_{2l} = \eta. 
\end{align*}
\tag{3.4}
\]
Observe that from (3.4) and (3.3) we have
\[ \Lambda_{a,b,1,0}(\alpha) = x_1 - \xi \quad \text{and} \quad \Lambda_{1,0,c,d}(\beta) = \eta - x_{2l-1}, \]
that is, 1 is the smallest eigenvalue of each of the bvps
\[
\begin{align*}
&-u'' = \alpha u \quad \text{in } (\xi, x_1), \\
&au(\xi) - bu'(\xi) = u(x_1) = 0, \tag{3.5}
\end{align*}
\]
and
\[
\begin{align*}
&-u'' = \beta u \quad \text{in } (x_{2l-1}, \eta), \\
&u(x_{2l-1}) = cu(\eta) + du'(\eta) = 0. \tag{3.6}
\end{align*}
\]
Thus, we consider the function
\[
\phi(t) = \begin{cases} 
\phi_1(t) & \text{for } t \in [\xi, x_1], \\
\phi_{2i}(t) & \text{for } t \in [x_{2i-1}, x_{2i}], \quad i = 1, \ldots, (l - 1), \\
\phi_{2i+1}(t) & \text{for } t \in [x_{2i}, x_{2i+1}], \quad i = 1, \ldots, (l - 1), \\
\phi_{2l}(t) & \text{for } t \in [x_{2l-1}, \eta],
\end{cases}
\]
where \( \phi_1 \) is the positive eigenfunction associated with the eigenvalue 1 of (3.5) satisfying \( \phi'_1(x_1) = -1 \),
\[
\phi_{2i}(t) = -\frac{1}{\sqrt{\beta}} \sin \left( \sqrt{\beta}(t - x_{2i-1}) \right) \quad \text{for } i = 1, \ldots, (l - 1), \\
\phi_{2i+1}(t) = \frac{1}{\sqrt{\alpha}} \sin \left( \sqrt{\alpha}(t - x_{2i}) \right) \quad \text{for } i = 1, \ldots, (l - 1),
\]
and \( \phi_{2l} \) is the negative eigenfunction associated with the eigenvalue 1 of (3.6) satisfying \( \phi''_{2l}(x_{2l-1}) = -1 \).

Thus, by simple computations we find that
\[
\begin{align*}
\phi'_{2i-1}(x_{2i-1}) &= \phi'_1(x_{2i-1}) = -1 \quad \text{for } i = 1, \ldots, l, \\
\phi''_{2i-1}(x_{2i-1}) &= \phi''_1(x_{2i-1}) = 1 \quad \text{for } i = 1, \ldots, (l - 1), \\
\phi''_{2i}(x_{2i}) &= \phi''_{2i+1}(x_{2i}) = 0 \quad \text{for } i = 1, \ldots, (l - 1), \\
\phi''_{2i+1}(x_{2i+1}) &= \phi''_{2i+1}(x_{2i+1}) = 0 \quad \text{for } i = 1, \ldots, (l - 1), \\
\phi'_1(x_1) &= \phi''_{2l}(x_{2l-1}) = 0.
\end{align*}
\]
All the above equalities make \( \phi \) a function in \( S^+_2 \cap C^2[\xi, \eta] \) satisfying bvp (3.2). This completes the proof.

4. Proofs of main results

4.1. Auxiliary results. Let \( q \in L^1_0[\xi, \eta] \). For \( \varkappa \in (\xi, \eta) \) we define the operators
\[ L_{q, \varepsilon, \xi} : C[\varkappa, \xi] \to C[\varkappa, \xi] \]
and
\[ L_{q, \varepsilon, \eta} : C[\varkappa, \eta] \to C[\varkappa, \eta] \]
by
\[
L_{q, \varepsilon, \xi} u(t) = \int_{\xi}^{t} G_{\varepsilon, \xi}(t, s) q(s) u(s) ds,
\]
\[
L_{q, \varepsilon, \eta} u(t) = \int_{\varepsilon}^{\eta} G_{\varepsilon, \eta}(t, s) q(s) u(s) ds,
\]
where
\[
G_{\varepsilon, \xi}(t, s) = \left( b + a \int_{\xi}^{t} \frac{d\tau}{p(\tau)} \right)^{-1}
\left( b + a \int_{\xi}^{t} \frac{d\tau}{p(\tau)} \right)^{-1} \int_{\xi}^{t} q(s) u(s) ds,
\]
and
\[
G_{\varepsilon, \eta}(t, s) = \left( d + c \int_{\varepsilon}^{\eta} \frac{d\tau}{p(\tau)} \right)^{-1}
\left( d + c \int_{\varepsilon}^{\eta} \frac{d\tau}{p(\tau)} \right)^{-1} \int_{\varepsilon}^{\eta} q(s) u(s) ds,
\]

Lemma 4.1. Assume that (2.2) and (2.3) hold. Then, for every function \( q \in K_G \),
\( \lim_{\varepsilon \to \xi} r(L_{q, \varepsilon, \xi}) = 0 \) and \( \lim_{\varepsilon \to \eta} r(L_{q, \varepsilon, \eta}) = 0 \).

Proof. We will prove that \( \lim_{\varepsilon \to \xi} r(L_{q, \varepsilon, \xi}) = 0 \). The other limit can be obtained similarly. We distinguish two cases:

- \( b \neq 0 \): In this case \( q \in L^1[\xi, \xi + \eta/2] \) and we have
\[
r(L_{q, \varepsilon, \xi}) \leq \int_{\xi}^{\varepsilon} G_{\varepsilon, \xi}(s, s) q(s) ds
\]
\[
\leq \left( b + a \int_{\xi}^{\varepsilon} \frac{d\tau}{p(\tau)} \right)^{-1} \int_{\xi}^{\varepsilon} \left( b + a \int_{\xi}^{t} \frac{d\tau}{p(\tau)} \right) q(s) ds
\]
\[
\leq \int_{\xi}^{\varepsilon} \frac{d\sigma}{p(\sigma)} \int_{\xi}^{\varepsilon} q(s) ds,
\]
from which we obtain that \( \lim_{\varepsilon \to \xi} r(L_{q, \varepsilon, \xi}) = 0 \).

- \( b = 0 \): In this case \( a \neq 0 \) and \( q(s) \int_{\xi}^{s} 1/p(c) d\varepsilon \in L^1[\xi, (\xi + \eta)/2] \) and we have
\[
r(L_{q, \varepsilon, \xi}) \leq \int_{\xi}^{\varepsilon} G_{\varepsilon, \xi}(s, s) q(s) ds
\]
\[
\leq \int_{\xi}^{\varepsilon} \left( \int_{\xi}^{s} \frac{d\sigma}{p(\sigma)} \right) q(s) ds - \left( \int_{\xi}^{\varepsilon} \frac{d\sigma}{p(\sigma)} \right)^{-1} \int_{\xi}^{\varepsilon} \left( \int_{\xi}^{s} \frac{d\sigma}{p(\sigma)} \right) q(s) ds
\]
\[
\leq 2 \int_{\xi}^{\varepsilon} (q(s) \int_{\xi}^{s} \frac{d\sigma}{p(\sigma)} ds)
\]
leading to \( \lim_{\varepsilon \to \xi} r(L_{q, \varepsilon, \xi}) = 0 \). This completes the proof.

Lemma 4.2. Assume that (2.2) and (2.3) hold and let \( \alpha, \beta \) be two functions in \( K_G \). If \( u \) is a nontrivial solution of
\[
-(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)),
\]
\( u(t_0) = 0 \)
with $t_0 \in \{\xi, \eta\}$, then $t_0$ is an isolated zero of $u$ (i.e. there exists a neighborhood $V_0$ of $t_0$ such that $u(t) \neq 0$ for all $t \in V_0$). Moreover, we have that $\lim_{t \to t_0} p(t)u'(t)$ exists.

**Proof.** We present the proof for $t_0 = \xi$ the other case is similar. Let $t_\ast > \xi$ be such that $u$ does not vanish identically in $(\xi, t_\ast)$ and suppose that $\alpha + \beta > 0$ a.e. in $(\xi, t_\ast)$ (the case $\alpha + \beta = 0$ a.e. in $(\xi, t_\ast)$ is obvious). For the purpose of contradiction, suppose that there is a sequence $(\tau_n) \subset (\xi, t_\ast)$ such that $u(\tau_n) = 0$ for all $n \in \mathbb{N}$ and $\lim \tau_n = \xi$. In this case, $u$ satisfies for all $n \in \mathbb{N}$,

$$-(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)), \quad \text{ a. e. } t \in (\xi, \tau_{n+1}),$$

$$u(\xi) = u(\tau_{n+1}) = 0. \tag{4.1}$$

Without loss of generality, assume that $u$ is positive in $(\tau_n, \tau_{n+1})$ and let $\mu_{n,1}(\alpha)$ be the first eigenvalue given by Theorem 3.6 associated with a positive eigenvector $\psi_{n,1}$ of

$$-(p\psi')'(t) = \mu\alpha(t)\psi(t), \quad t \in (\tau_n, \tau_{n+1}),$$

$$\psi(\tau_n) = \psi(\tau_{n+1}) = 0.$$

Multiplying the differential equation in (4.1) by $\psi_{n,1}$, we obtain after two integrations

$$0 \leq u^{[1]}(\tau_n)\psi_{n,1}(\tau_n) = \int_{\tau_n}^{\tau_{n+1}} (\lambda - \mu_{n,1}(\alpha))\alpha\psi_{n,1}u$$

leading to

$$\lambda \geq \mu_{n,1}(\alpha). \tag{4.2}$$

Now, let $\mu_{n,1}^* = 1/\min\{L_{\alpha,\tau_{n+1},l}\}$ and let $\psi_{n,1}^*$ be the associated positive eigenvector. $\mu_{n,1}^*$ and $\psi_{n,1}^*$ satisfy

$$-(p\psi_{n,1}^*)'(t) = \mu_{n,1}^*\alpha(t)\psi_{n,1}^*(t), \quad t \in (\xi, \tau_{n+1})$$

$$\psi_{n,1}^*(\xi) = \psi_{n,1}^*(\tau_{n+1}) = 0.$$

Again, multiplying the differential equation in (4.1) by $\psi_{n,1}^*$, we obtain after two integrations

$$0 \geq -\psi_{n,1}^{[1]}(\tau_n)\psi_{n,1}^*(\tau_n) = \int_{\tau_n}^{\tau_{n+1}} (\mu_{n,1}^* - \mu_{n,1}(\alpha))\alpha\psi_{n,1}^*\psi_{n,1}$$

leading to

$$\mu_{n,1}(\alpha) \geq \mu_{n,1}^*. \tag{4.3}$$

Thus, from (4.2), (4.3) and Lemma 4.1 we obtain the contradiction

$$\lambda \geq \lim \mu_{n,1}(\alpha) \geq \lim \mu_{n,1}^* = 1/\lim \min\{L_{\alpha,\tau_{n+1},l}\} = +\infty.$$

Now, suppose that $u > 0$ on $(\xi, t_\#)$ for some $t_\# > \xi$. We have by simple integration over $(t, t_\#) \subset (\xi, t_\#)$

$$p(t)u'(t) - p(t_\#)u'(t_\#) = \lambda \int_{t}^{t_\#} \alpha(s)u(s)ds$$

leading to

$$\lim_{t \to \xi} p(t)u'(t) = p(t_\#)u'(t_\#) + \lim_{t \to \xi} \int_{t}^{t_\#} \alpha(s)u(s)ds.$$

This completes the proof. \qed
Let \( \alpha, \beta \) be two functions in \( L^1_{\text{loc}}(\xi, \eta) \) such that \( \alpha(t) \geq 0 \), \( \beta(t) \geq 0 \) for \( t \in [\xi, \eta] \) a.e. and each of \( \alpha \) and \( \beta \) is positive in a subset of a positive measure; and consider the initial-value problem

\[
-(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)),
\]

\[
u(t_0) = \lim_{t \to t_0} p(t)u'(t) = 0.
\] (4.4)

By a solution to (4.4) we mean a function \( u \in C(I) \cap C^1(I) \) with \( (pu')' \in L^1(I) \) where \( I \subset (\xi, \eta) \) is an open interval such that \( t_0 \in I \) and \( u \) satisfies all equations in (4.4).

**Lemma 4.3.** Assume that (2.2) holds and let \( \alpha, \beta \) be two functions in \( K_G \). Then, for all \( t_0 \in [\xi, \eta] \), \( u \equiv 0 \) is the unique solution of the initial value problem (4.4).

**Proof.** The case \( \lambda = 0 \) is obvious. Let \( \lambda \neq 0 \) and \( u \) be a solution of (4.4) defined on some interval \( [t_0, t_*] \) with \( t_* \in (t_0, \eta) \) (the case \( u \) defined on \( [t_0, t_*] \) with \( t_* \in (\xi, t_0) \) can be checked similarly). Since \( L^1_G(\xi, \eta) \subset L^1_{\text{loc}}(\xi, \eta) \) and \( u \) is continuous on \( [t_0, t_*] \), \( (pu')' \in L^1_{\text{loc}}(t_0, t_*) \). We distinguish two cases:

- **Case 1:** \( t_0 \in (\xi, \eta). \) Let \( (z_i)_{i=0}^{n} \) be such that \( t_0 = z_0 < z_1 < \cdots < z_n = t_* \),

  \[
  k_i = |\lambda| \int_{z_i}^{z_{i+1}} \frac{dr}{p(\tau)} \int_{z_i}^{z_{i+1}} (\alpha(\tau) + \beta(\tau)) d\tau < 1.
  \]

Set for \( i \in \{0, 1, \ldots, (n - 1)\} \), \( J_i = [z_i, z_{i+1}] \), \( X_i = C(J_i) \) equipped with the sup-norm \( \| \cdot \|_{i, \infty} \) and \( T_i : X_i \to X_i \) with

\[
T_i v(t) = -\int_{z_i}^{t} \left( \frac{\lambda}{p(s)} \int_{z_i}^{s} (\alpha(\tau)v^+(\tau) - \beta(\tau)v^-(\tau)) d\tau \right) ds.
\]

Let \( v, w \in X_i \), we have

\[
|T_i v(t) - T_i w(t)| \leq \int_{z_i}^{t} \left( \frac{|\lambda|}{p(s)} \int_{z_i}^{s} |\alpha(\tau)||v^+(\tau) - w^+(\tau)| d\tau \right) ds
+ \int_{z_i}^{t} \left( \frac{|\lambda|}{p(s)} \int_{z_i}^{s} |\beta(\tau)||v^-(\tau) - w^-(\tau)| d\tau \right) ds
\]

then from (2.6)

\[
\|T_i v - T_i w\|_{i, \infty} \leq |\lambda| \int_{z_i}^{z_{i+1}} \frac{ds}{p(s)} \int_{z_i}^{s} (\alpha(\tau) + \beta(\tau)) d\tau \|v - w\|_{i, \infty}
\]

\[
\leq k_i \|v - w\|_{i, \infty}.
\]

So, \( T_i \) is a \( k_i \)-contraction.

For \( i \in \{0, 1, \ldots, (n - 1)\} \), let \( u_i \) be the restriction of \( u \) to the interval \( J_i \). We have that \( u_0 \) is a fixed point of \( T_0 \). Indeed, since \( L^1_G(\xi, \eta) \subset L^1_{\text{loc}}(\xi, \eta) \) and \( u \) is continuous on \( [t_0, t_*] \), \( (pu')' \in L^1[0, t_*] \). So integrating the differential equation in (4.4) over \( [z_0, s] \subset [z_0, z_1] \) we obtain from the initial value condition

\[
p(s)u_0'(s) = -\int_{z_0}^{s} (\alpha(\tau)u_0^+(\tau) - \beta(\tau)u_0^-(\tau)) d\tau
\]

from which for all \( t \in [z_0, z_1] \) we have

\[
u_0(t) = -\int_{z_0}^{t} \left( \frac{\lambda}{p(s)} \int_{z_i}^{s} (\alpha(\tau)u_0^+(\tau) - \beta(\tau)u_0^-(\tau)) d\tau \right) ds = T_0 u_0(t).
\]
Thus, the fact that $T_0$ is a contraction and the trivial function is a fixed point of $T_0$ lead to $u_0 \equiv 0$, and in particular, we have
\[
u_0(z_1) = u_0^{[1]}(z_1) = u_1(z_1) = u_1^{[1]}(z_1) = 0. \tag{4.5}
\]
From this equality, $u_1$ is a fixed of $T_1$. Then for the same reasons $u_1 \equiv 0$, and in particular,
\[
u_1(z_2) = u_1^{[1]}(z_2) = u_2(z_2) = u_2^{[1]}(z_2) = 0.
\]
Repeating the above process, we obtain that $u_i \equiv 0$ for all $i \in \{0, 1, \ldots, (n - 1)\}$; that is, $u \equiv 0$ on $[t_0, t_*]$.

- $t_0 = \xi$: In this case by Lemma 4.2 we can suppose that $u > 0$ in $(\xi, t^*)$. We distinguish two cases:
  
  (a) $\alpha \in L^1[\xi, t^*]$. Let $t_+ \in (\xi, t^*)$ be such that
  
  \[
k_+ = |\lambda| \int_{\xi}^{t_+} \frac{d\tau}{p(\tau)} \int_{\xi}^{\tau} \alpha(\tau) d\tau < 1.
  \]
  
  Set $J_+ = [\xi, t_+]$, $X_+ = C(J_+)$ equipped with the sup-norm $\|\cdot\|_{+\infty}$ and $T_+: X_+ \to X_+$
  
  \[
  T_+ v(t) = -\int_{\xi}^{t} \int_{t}^{\xi} \alpha(s) v(\tau) d\tau ds.
  \]
  It is easy to see that $T_+$ is a $k_+$-contraction and $u_+$ the restriction of $u$ to $[\xi, t_+]$ is a fixed point of $T_+$, so $u_+ \equiv 0$ in $[\xi, t_+]$, and in particular, $u(t_+) = u_1^{[1]}(t_+) = 0$.
  
  Thus, we conclude from the above step that $u \equiv 0$ on its interval of definition contradicting the beginning of this step.

  (b) $\alpha \notin L^1[\xi, t^*]$: In this case $b = 0$ and $\int_{\xi}^{t^*} (\alpha(t) \int_{\xi}^{t} \frac{ds}{p(s)}) dt < \infty$. Thus, let $t_\infty \in (\xi, t^*)$ be such that
  
  \[
k_\infty = |\lambda| \int_{\xi}^{t_\infty} (\alpha(t) \int_{\xi}^{t} \frac{ds}{p(s)}) dt < 1.
  \]
  Set
  
  \[
  L^1_+[\xi, t_\infty] = \{v: (\xi, t_\infty) \to \mathbb{R} \text{ measurable and } \int_{\xi}^{t_\infty} \alpha(s) |v(s)| ds < \infty\}
  \]
  equipped with the norm
  
  \[
  \|v\|_{L^1_+[\xi, t_\infty]} = \int_{\xi}^{t_\infty} \alpha(s) |v(s)| ds.
  \]
  Let $u_\infty$ be the restriction of $u$ to the interval $[\xi, t_\infty]$. We claim that $u_\infty$ belongs to $L^1_+[\xi, t_\infty]$. Indeed, integrating the differential equation in \[4.4\] over $[\epsilon, t_\infty] \subset (\xi, t^*)$, we obtain
  
  \[
u^{[1]}_\infty(\epsilon) - u^{[1]}_\infty(t_\infty) = \lambda \int_{\epsilon}^{t_\infty} \alpha(s) u_\infty(s) ds. \tag{4.6}
  \]
  Letting $\epsilon \to \xi$ in \[4.6\], we obtain
  
  \[
  |\lambda| \int_{\xi}^{t_\infty} \alpha(s) u_\infty(s) ds = |u^{[1]}_\infty(t_\infty)|;
  \]
  and then
  
  \[
  \int_{\xi}^{t_\infty} \alpha(s) |u_\infty(s)| ds = \int_{\xi}^{t_\infty} \alpha(s) u_\infty(s) ds = \frac{|u^{[1]}_\infty(t_\infty)|}{\lambda} < \infty.
  \]
Now, let $T_\infty: L^1_\alpha[\xi, t_\infty] \to L^1_\alpha[\xi, t_\infty]$ be defined by

$$T_\infty v(t) = \int_\xi^t \left( -\frac{\lambda}{p(s)} \int_\xi^s \alpha(\tau)v(\tau)d\tau \right)ds.$$  

It is easy to see that $T_\infty$ is a $k_\infty$-contraction and $u_\infty$ is a fixed point of $T_\infty$, so $u_\infty \equiv 0$ in $[\xi, t_\infty]$, and in particular, $u(t_\infty) = u^{[1]}(t_\infty) = 0$. Thus, we conclude from the above step that $u \equiv 0$ on its interval of definition contradicting the beginning of this step. This completes the proof. □

**Lemma 4.4.** Assume that (2.2) and (2.3) hold. If $\lambda$ is a half-eigenvalue of bvp (2.7) associated with an eigenvector $u$ then $u \in S_k$ for some $k \geq 1$.

**Proof.** If $u(t_0) = u^{[1]}(t_0) = 0$ for some $t_0 \in [\xi, \eta]$ then we have from Lemma 4.3 that $u \equiv 0$, contradicting $(\lambda, u)$ is a nontrivial solution of (2.7). This shows that $u$ has only simple zeros.

Now, to the contrary, assume that $u$ has an infinite sequence of consecutive zeros $(t_n)$ converging to some $t^* \in [\xi, \eta]$. We have from the continuity of $u$, $u(t^*) = 0$ and so from Lemma 4.2 $t^* \in (\xi, \eta)$. Because of the simplicity of zeros of $u$, we have that $(t_n) = (t_n^1) \cup (t_n^2)$ with $u^{[1]}(t_n^1) > 0$ and $u^{[1]}(t_n^2) < 0$. Since $u \in C^1[t^* - \varepsilon, t^* + \varepsilon]$ for some $\varepsilon > 0$ small enough, we obtain that

$$0 \leq \lim_{n \to \infty} u^{[1]}(t_n^1) = u^{[1]}(t^*) = \lim_{n \to \infty} u^{[1]}(t_n^2) \leq 0.$$

Again by Lemma 4.3, $u \equiv 0$, contradicting $(\lambda, u)$ is a nontrivial solution of (2.7). □

**Lemma 4.5.** Assume that (2.2) and (2.3) hold and $\alpha, \beta \in K_G \cap L^1[\xi, \eta]$. Then, for each integer $k \geq 1$ and $\nu = + or -$, bvp (2.7) admits at most one simple half-eigenvalue having an eigenvector in $S_k^\nu$.

**Proof.** To the contrary, suppose that $(\lambda_i, \phi_i) \in \mathbb{R} \times S_k^\nu$ satisfy (2.7) for $i = 1, 2$. Then the integrability of $1/p, \alpha$ and $\beta$ implies that $\phi_i \in S_k^\nu \cap AC[\xi, \eta]$ and $p\phi_i' \in AC[\xi, \eta]$. Let $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$ be the intervals given by Theorem 3.2. Since $\phi_1$ and $\phi_2$ have the same sign in each of $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$, we have

$$0 \leq \int_{\xi_1}^{\eta_1} \phi_2 \ell_1 \phi_1 - \phi_1 \ell_1 \phi_2 = (\lambda_1 - \lambda_2) \int_{\xi_1}^{\eta_1} \alpha \phi_1^+ \phi_2^- + \beta \phi_1^- \phi_2^+,$$

$$0 \geq \int_{\xi_2}^{\eta_2} \phi_2 \ell_1 \phi_1 - \phi_1 \ell_1 \phi_2 = (\lambda_1 - \lambda_2) \int_{\xi_2}^{\eta_2} \alpha \phi_1^+ \phi_2^- + \beta \phi_1^- \phi_2^+,$$

leading to $\lambda_1 = \lambda_2$.

Now, suppose that $\lambda$ is a half-eigenvalue of (2.7) having two eigenvectors $\phi_1$ and $\phi_2$ with $\phi_1 \phi_2 > 0$ in a right neighborhood of $\xi$, $\phi_1, \phi_2 \in AC[\xi, \eta]$ and $p\phi_1', p\phi_2' \in AC[\xi, \eta]$. Because of the positive homogeneity of bvp (2.7), there exists two eigenvectors $\psi_1$ and $\psi_2$ associated with $\lambda$ such that $\psi_1 \psi_2 > 0$ in a right neighborhood of $\xi$, $\psi_1, \psi_2 \in AC[\xi, \eta]$, $p\psi_1', p\psi_2' \in AC[\xi, \eta]$ and

$$\psi_1(\xi) = \psi_2(\xi) = b, \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.$$  

Indeed; Without loss of generality, suppose that $\phi_1 > 0$ and $\phi_2 > 0$ in a right neighborhood of $\xi$. Then we distinguish the following three cases.

• $\phi_1(\xi) = 0$. In this case we have $b = 0$ and from (2.3) that $a > 0$ (otherwise if $b \neq 0$ we obtain from the boundary condition at $\xi$ that $\phi_1^{[1]}(\xi) = 0$ and Lemma
leads to $\phi_1 = 0$). The positivity of $\phi_1$ near $\xi$ leads to $\phi_1^{[1]}(\xi) > 0$. Since $a > 0$, $b = 0$ and $\phi_2 > 0$ near $\xi$, we have $\phi_2(\xi) = 0$ and $\phi_2^{[1]}(\xi) > 0$. Thus,

$$
\psi_1 = \frac{a\phi_1}{\phi_1^{[1]}(\xi)}, \quad \psi_2 = \frac{a\phi_2}{\phi_2^{[1]}(\xi)}
$$

are eigenvectors associated with $\lambda$ satisfying

$$
\psi_1(\xi) = \psi_2(\xi) = b \quad \text{and} \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.
$$

- $\phi_1^{[1]}(\xi) = 0$. In this case we have $a = 0$ and from (2.3) that $b > 0$ (otherwise if $a \neq 0$ we obtain from the boundary condition at $\xi$ that $\phi_1(\xi) = 0$ and Lemma 4.3 leads to $\phi_1 = 0$). The positivity of $\phi_1$ near $\xi$ leads to $\phi_1(\xi) > 0$. Since $b > 0$, $a = 0$ and $\phi_2 > 0$ near $\xi$, we have $\phi_2^{[1]}(\xi) = 0$ and $\phi_2(\xi) > 0$. Thus,

$$
\psi_1 = \frac{b\phi_1}{\phi_1(\xi)}, \quad \psi_2 = \frac{b\phi_2}{\phi_2(\xi)}
$$

are eigenvectors associated with $\lambda$ satisfying

$$
\psi_1(\xi) = \psi_2(\xi) = b \quad \text{and} \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.
$$

- $\phi_1^{[1]}(\xi) > 0$. This happens only in the case $a > 0$ and $b > 0$ and we have the boundary condition at $\xi$, $\phi_1(\xi) > 0$ and $\phi_1^{[1]}(\xi) > 0$. Thus,

$$
\psi_1 = \frac{a\phi_1}{\phi_1^{[1]}(\xi)} = \frac{b\phi_1}{\phi_1(\xi)}, \quad \psi_2 = \frac{a\phi_2}{\phi_2^{[1]}(\xi)} = \frac{b\phi_2}{\phi_2(\xi)}
$$

are eigenvectors associated with $\lambda$ satisfying

$$
\psi_1(\xi) = \psi_2(\xi) = b \quad \text{and} \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.
$$

At this stage, $\psi = \psi_1 - \psi_2$ satisfies

$$
-(pu')' (t) = \lambda (\alpha (t) \psi^+ (t) - \beta (t) \psi^- (t))
$$

$$
\psi(\xi) = \psi^{[1]}(\xi) = 0,
$$

and we have from Lemma 4.3 $\psi = 0$. That is, $\psi_1 = \psi_2$, and then $\phi_1 = \omega \phi_2$ with $\omega > 0$. This shows that the half-eigenvalue $\lambda$ is simple and completes the proof of Lemma 4.5.

For $q \in K_G$ we define the linear compact operator $L_q : E \rightarrow E$ by

$$
L_q u(t) = \int_\xi^n G(t, s)q(s)u(s)ds.
$$

Since, we will use the global bifurcation theorem of Rabinowitz to prove the main result of this paper, we need to discuss the geometric and algebraic multiplicities of characteristic values of $L_q$ (which are also eigenvalues of bvp (2.8)). Let $\mu_0$ be a characteristic value of $L_q$ and note that $N(\mu_0 L_q - I) \subset N(\mu_0 L_q - I)^2$. Thus, if $\mu_0$ is not simple then $\mu_0$ is of algebraic multiplicity greater than 1. We know from Theorem 3.6 that if $q \in K_G \cap L^1[\xi, \eta]$ then all characteristic values of $L_q$ have the geometric multiplicity equal to one, so let us see what can happens with the algebraic multiplicity.

**Lemma 4.6.** Assume that (2.2) and (2.3) hold and $q \in K_G \cap L^1[\xi, \eta]$. Then, all characteristic values of $L_q$ are of algebraic multiplicity one.
Proof. Let \((\mu_k(q))\) be the sequence of characteristic values of \(L_q\) given by Theorem 3.6. Thus the eigenvector \(\phi_k\) associated with \(\mu_k(q)\) satisfies
\[-(p\phi_k')(t) = \mu_k(q)q(t)\phi_k(t), \quad \text{a.e. } t \in (\xi, \eta),\]
\[a\phi_k(\xi) - b \lim_{t \to \xi} p(t)\phi_k'(t) = 0,\]
\[c\phi_k(\eta) + d \lim_{t \to \eta} p(t)\phi_k'(t) = 0.\]

Multiplying by \(\phi_k\) and integrating over \([\xi, \eta]\) we obtain
\[
\int_{\xi}^{\eta} p(\phi_k')^2 = \mu_k(q) \int_{\xi}^{\eta} q\phi_k^2 \tag{4.7}
\]
leading to
\[
\mu_k(q) > 0, \quad \text{and} \quad \int_{\xi}^{\eta} q\phi_k^2 > 0. \tag{4.8}
\]

Now, let \(u \in N((\mu_k(q)L_q - I)^2)\) and set \(v = (\mu_k(q)Lq_0 - I)(u) = \mu_k(q)L_qu - u.\) We have \(\mu_k(q)Lqv - v = 0\) leading to \(v = x\phi\) and
\[
\mu_k(q)L_qu - u = x\phi_k. \tag{4.9}
\]

On the other hand we have that \(u\) satisfies the bvp
\[-(pu')' = \mu_k(q)q(t)u(t) - x\mu_k(q)q(t)\phi_k(t), \quad \text{a.e. } t \in (\xi, \eta),\]
\[au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,\]
\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0. \tag{4.10}
\]

Multiplying the differential equation in (4.10) by \(\phi_k\) and integrating on \((\xi, \eta)\) we obtain
\[
x\mu_k(q) \int_{\xi}^{\eta} q\phi_k^2 = 0.
\]
Because of (4.7), the above equality leads to \(x = 0.\) Therefore, we obtain from (4.9) that \(u = \omega\phi_k \in N(\mu_k(q)L_q - I)\) with \(\omega \in \mathbb{R}.\) This completes the proof. \(\square\)

It remains to discuss the geometric and algebraic multiplicities of characteristic values of \(L_q\) when \(q \in (K_G \setminus L^1[\xi, \eta]).\) We need the following lemma which is a version of L’Hospital’s rule.

Lemma 4.7. Let \(f\) and \(g\) be two differentiables functions on \((\xi, \xi + \epsilon)\) with \(\epsilon > 0\) such that \(\lim_{t \to \xi} f(t) = \lim_{t \to \xi} g(t) = +\infty.\) If \(\lim_{t \to \xi} \frac{f(t)}{g(t)} = l\) then \(\lim_{t \to \xi} \frac{t^{(l)}}{g(t)} = 1.\)

Lemma 4.8. Assume that [2.2] and [2.3] hold and \(q \in K_G \setminus L^1[\xi, \eta].\) Let \(\mu\) be a characteristic value of \(L_q\) associated with an eigenvector \(\phi.\) We have

1. If \(\phi\) does not change sign then \(\mu\) is double
2. If \(\phi\) has more than one zero in \((\xi, \eta)\) then \(\mu\) is simple.

Proof. Suppose that \(\psi\) is another eigenvector associated with the characteristic value \(\mu\) and let \(W = W(\phi, \psi)\) be the Wronskian of \(\phi\) and \(\psi.\) By simple computations follows \((pW)' = 0,\) from which we obtain
\[
\phi\psi' - \psi'\phi = \frac{B}{p}, \quad B \in \mathbb{R}. \tag{4.11}
\]
Considering (4.11) as a linear first order differential equation where the unknown is \( \psi \), we obtain that \( \psi \) takes the form
\[
\psi(t) = A\phi(t) + B\psi(t) \quad \text{with } A, B \in \mathbb{R}, \ v \in (\xi, \eta),
\]
\[
\psi_{\varepsilon}(t) = \phi(t) \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi^2(s)}.
\]

Thus, we have to examine for \( \varepsilon \in (\xi, \eta) \) the ability of the function \( \psi_{\varepsilon} \) to be an eigenvector associated with \( \mu \) or not. Without loss of generality, suppose that \( q \) is not integrable at \( \xi \) and \( \eta \) (the other cases can be checked similarly). This occurs if \( b = d = 0 \) and in this case the boundary conditions in bvp (2.1) become the Dirichlet conditions
\[
u(\xi) = \psi(\eta) = 0. \quad (4.12)
\]

1. Suppose that \( \phi \) is positive in \( (\xi, \eta) \) and let \( \varepsilon \in (\xi, \eta) \) be fixed. We have by simple computations
\[
p(t)\psi'_{\varepsilon}(t) = \frac{1}{\phi(t)} + p(t)\phi'(t) \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi^2(s)}, \quad \text{for all } t \in (\xi, \eta) \quad (4.13)
\]
then
\[
- (p\psi'_{\varepsilon})'(t) = \lambda q(t)\psi_{\varepsilon}(t), \quad \text{a.e. } t \in (\xi, \eta). \quad (4.14)
\]

Moreover, since \( q \) is not integrable at \( \xi \) and \( \eta \), from Lemma [4.2 and 36 Theorem 2.3.1] we have
\[
\lim_{t \to \xi} p(t)\phi'(t) = \infty, \quad \lim_{t \to \eta} p(t)\phi'(t) = \infty.
\]

Thus, from Lemma [4.7 when \( \lim_{t \to \xi} \int_{\xi}^{t} \frac{ds}{p(s)\phi^2(s)} = \infty \) (the case \( \int_{\xi}^{\eta} \frac{ds}{p(s)\phi^2(s)} < \infty \) is obvious), we have
\[
\lim_{t \to \xi} \psi_{\varepsilon}(t) = \lim_{t \to \xi} \left( \int_{\xi}^{t} \frac{ds}{p(s)\phi^2(s)} \right)' = \lim_{t \to \xi} \frac{1}{\phi(t)} = - \frac{1}{\lim_{t \to \xi} p(t)\phi'(t)} = 0
\]
and also
\[
\lim_{t \to \eta} \psi_{\varepsilon}(t) = - \frac{1}{\lim_{t \to \eta} p(t)\phi'(t)} = 0.
\]

That is, \( \psi_{\varepsilon} \) satisfies the boundary conditions (4.12) and all the above shows that \( \psi_{\varepsilon} \) is an eigenvector of the characteristic value \( \mu \) of \( L_q \). Moreover, since the function \( \phi_{\varepsilon}(t) = A + B \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi^2(s)} \) vanishes at most once in \( (\xi, \eta) \), the eigenvector \( \psi \) lies in \( S_1 \cup S_2 \) and this shows that \( \mu \) is double.

2. Note that if \( \phi(t_1) = 0 \) for some \( t_1 \in (\xi, \eta) \), we obtain from (4.13) following \( t_1 > \varepsilon \) and \( t_1 < \varepsilon \) that at least one of the limits
\[
\lim_{t \to t_1, t \to \varepsilon} p(t)\psi_{\varepsilon}(t), \quad \lim_{t < t_1, t \to \varepsilon} p(t)\psi_{\varepsilon}(t)
\]
are infinite and this means that \( \psi_{\varepsilon} \notin Y_{\#} \). So, the function \( \psi_{\varepsilon} \) can not be an eigenvector associated with the characteristic value \( \mu \) of \( L_q \).

By the contrary suppose that the characteristic value \( \mu \) is not simple and there exists another eigenvector of \( \mu, \phi_{\varepsilon} \in Y_{\#} \). In this case, arguing as in the discussion in the beginning of this proof we obtain
\[
\phi(t) = A\phi(t) + B\phi_{\varepsilon}(t) \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi^2(s)}
\]
and
\[ p(t)\phi'(t) = A\phi_1(t) + \frac{B}{\phi_1(t)} + p(t)\phi_1'(t) \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi_1^2(s)}, \quad \text{for all } t \in (\xi, \eta) \]
where \( A, B \in \mathbb{R} \) and \( \varepsilon \in (\xi, \eta) \).

Thus, arguing as in the beginning of part 2 of this proof we obtain that the
eigenvector \( \phi_1 \) must be positive. Therefore, from
\[ \phi(t) = \phi_1(t) \left( A + B \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi_1^2(s)} \right) \]
yields that \( \phi \) has at most one zero in \((\xi, \eta)\). Contradicting \( \phi \) has more than one
zero. This completes the proof.

\[ \square \]

**Lemma 4.9.** Assume that \( 2.2 \) and \( 2.3 \) hold and \( q \in K_G \setminus L^1[\xi, \eta] \). If \( \mu \) is a
characteristic value of \( L_q \) associated with an eigenvector \( \phi \) vanishing more than one
time in \((\xi, \eta)\) then \( \mu \) is of algebraic multiplicity one.

**Proof.** In the same way as in the proof of Lemma 4.6, let us show that if
\( u \) is a solution to
\[ -(pu')' = pu' - x\mu q(t)\phi(t), \quad \text{a.e. } t \in (\xi, \eta), \]
\[ au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0, \]
\[ cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0. \]  \hspace{1cm} (4.15)
then \( u = \omega \phi \) with \( \omega \in \mathbb{R} \). Let \( u \) be a solution to (4.15) and \( W = W(\phi, u) \) be the
Wronksian of \( \phi \) and \( u \). We have that \( W \) satisfies
\[ (pW)' = (pu' - pu\phi')' = xpq \phi^2 \]
leading to
\[ u'\phi - u\phi' = \frac{B}{p} + \frac{x\mu}{p} \int_{\varepsilon}^{s} q(\tau)(\phi(\tau))^2 d\tau, \quad B \in \mathbb{R}. \]  \hspace{1cm} (4.16)
Considering (4.16) as a linear first order differential equation where the unknown
is \( u \), we obtain that \( u \) takes the form
\[ u(t) = A\phi(t) + B\phi(t) \int_{\varepsilon}^{t} \frac{ds}{p(s)\phi^2(s)} + x\mu \phi(t) \int_{\varepsilon}^{t} \left( \frac{1}{p(s)\phi^2(s)} \int_{\varepsilon}^{s} q(\tau)(\phi(\tau))^2 d\tau \right) ds, \] \hspace{1cm} (4.17)
for \( A, B \in \mathbb{R} \) and \( \varepsilon \in (\xi, \eta) \).

Arguing as in 2 of the proof of Lemma 4.8 we see that the expression for \( u \)
given in (4.17) is a solution of (4.10) if and only if \( B = x = 0 \). This completes the
proof.

\[ \square \]

**Lemma 4.10.** Assume that \( 2.2 \) and \( 2.3 \) hold and let \( (q_n) \) be a sequence in \( L^1_G \)
converging to \( q \in L^1_G \). Then \( L_{q_n} \to L_q \) as \( n \to \infty \) in operator norm. Moreover, for
all \([\xi_0, \eta_0] \subset (\xi, \eta)\)
\[ p(L_{q_n}u)' \to p(L_qu)' \quad \text{in } C[\xi_0, \eta_0] \quad \text{for all } u \in E. \]
Proof. It is easy to check that for all $[\xi_0, \eta_0] \subset (\xi, \eta)$, $q_n \rightarrow q$ in $L^1[\xi_0, \eta_0]$, $\Phi_{ab}q_n \rightarrow 0$ in $L^1[\xi, \eta]$ and $\Psi_{cd}q_n \rightarrow \Psi_{cd}q$ in $L^1[\eta_0, \eta]$. Thus, we have, for all $[\xi_0, \eta_0] \subset (\xi, \eta)$ and $u \in E$ with $\|u\|_\infty = 1$,

$$\sup_{t \in [\xi, \eta]} |L_{q_n}u(t) - L_qu(t)| \leq \int_{\xi}^{\eta} G(s, s)|q_n(s) - q(s)|ds = \|q_n - q\|_G,$$

leading to $L_{q_n} \rightarrow L_q$ as $n \rightarrow \infty$.

Also, for all $[\xi_0, \eta_0] \subset (\xi, \eta)$ and $u \in E$ with $\|u\|_\infty = 1$, we have

$$|p(t)(L_{q_n}u)'(t) - p(t)(L_qu)'+(t)| \leq \int_{\xi_0}^{\eta_0} \frac{c\Phi_{ab}(s) + a\Psi_{cd}(s)}{\Delta}|q_n(s) - q(s)|ds$$

$$+ \int_{\xi}^{\eta_0} a\Psi_{cd}(q_n(s) - q(s))ds + \int_{\eta_0}^{\eta} a\Psi_{cd}(q_n(s) - q(s))ds$$

$$= \frac{c}{\Delta}\|\Phi_{ab}(q_n - q)\|_{L^1[\xi, \eta]} + \frac{a}{\Delta}\|\Psi_{cd}(q_n - q)\|_{L^1[\eta_0, \eta]} + \|q_n - q\|_{L^1[\xi_0, \eta_0]}$$

leading to $p(L_{q_n}u)' \rightarrow p(L_qu)'$ in $C[\xi_0, \eta_0]$. The proof is complete. \qed

Lemma 4.11. Assume that (2.2) and (2.3) hold and let $(\alpha_n)$ and $(\beta_n)$ be two sequences in $K_G$ converging, respectively, to $\alpha$ and $\beta$ in $L^1_G[\xi, \eta]$. Assume also that for all integers $n \geq 1$, there exist $\lambda_n > 0$ and $\phi_n \in E \setminus \{0\}$ satisfying

$$-(p\phi_n)'(t) = \lambda_n(\phi_n(t) + (t) - \beta_n(\phi_n(t)), \quad t \in (\xi, \eta),$$

$$a\phi_n(\xi) - b\lim_{t \to \xi} p(t)\phi_n(t) = 0,$$

$$c\phi_n(\eta) + d\lim_{t \to \eta} p(t)\phi_n(t) = 0.$$ 

We have $\phi_n = \lambda_nA\phi_n$ where $A_n = L_{\alpha_n}I^+ - L_{\beta_n}I^-.$

If $(\lambda_n)$ converges to $\lambda > 0$, then there exists $\psi \in E \setminus \{0\}$ such that $\psi = \lambda\tilde{A}\psi$ where $\tilde{A} = L_{\alpha}I^+ - L_{\beta}I^-$ (i.e. $\lambda$ is a half-eigenvalue to bvp (2.7)).

Proof. First, note that Lemma 4.10 guarantee that $L_{\alpha_n} \rightarrow L_{\alpha}$ and $L_{\beta_n} \rightarrow L_{\beta}$ in operator norm. Let $\phi_n$ be the eigenvector corresponding to $\lambda_n$ with $\|\phi_n\|_\infty = 1$ and set $\psi_n = \lambda_nA\phi_n$ and $\psi = \lim \psi_n$ (up to a subsequence). We have

$$\|\phi_n - \psi\|_\infty \leq \lambda_n\|A\phi_n - \psi\|_\infty + \|\lambda_nA\phi_n - \psi\|_\infty$$

$$\leq \lambda_n\|L_{\alpha_n} - L_{\alpha}\| + \|\lambda_n\|\|L_{\beta_n} - L_{\beta}\| + \|\lambda_n\|\|A\phi_n - \psi\|_\infty,$$

leading to $\lim \phi_n = \psi$ and $\|\psi\|_\infty = 1$. Also we have

$$\lambda_nA\phi_n - \lambda\tilde{A}\psi \leq \lambda_n\|A\phi_n - \psi\|_\infty + \|\lambda_nA\psi - \lambda\tilde{A}\psi\|_\infty$$

$$+ \|\lambda_n\|\|A\phi_n - \psi\|_\infty,$$

leading to $\lambda_nA\phi_n = \lambda\tilde{A}\psi$. At the end, letting $n \rightarrow \infty$ in the equation $\phi_n = \lambda_nA\phi_n$ we obtain $\psi = \lambda\tilde{A}\psi$. \qed

Remark 4.12. Arguing as in the proof of Lemma 4.10 one can prove that $p\phi_n \rightarrow p\psi'$ in $C[\xi_0, \eta_0]$ for all $[\xi_0, \eta_0] \subset (\xi, \eta)$ where $\phi_n$ and $\psi$ are those of the above proof.
sequences of simple half-eigenvalues and the functions $\alpha$.

**Corollary 4.13.** Assume that $p \equiv 1$ and $\alpha$ and $\beta$ are positive constants. Then bvp (2.7) admits two sequences of half-eigenvalues ($\lambda^+_k$) and ($\lambda^-_k$) such that

- $\lambda^+_1$ is the unique solution of $\Lambda_{a,b,c,d}(\alpha \sigma) = \eta - \xi$,
- $\lambda^-_1$ is the unique solution of $\Lambda_{a,b,c,d}(\beta \sigma) = \eta - \xi$,
- $\lambda^-_{2l}$ with $l \geq 1$ is the unique solution of
  $$
  \Lambda_{a,b}(\alpha \sigma) + \Lambda_{c,d}(\beta \sigma) + \pi(l - 1)\left(\frac{1}{\sqrt{\alpha \sigma}} + \frac{1}{\sqrt{\beta \sigma}}\right) = \eta - \xi,
  $$
- $\lambda^+_{2l}$ with $l \geq 1$ is the unique solution of
  $$
  \Lambda_{a,b}(\beta \sigma) + \Lambda_{c,d}(\alpha \sigma) + \pi(l - 1)\left(\frac{1}{\sqrt{\alpha \sigma}} + \frac{1}{\sqrt{\beta \sigma}}\right) = \eta - \xi,
  $$
- $\lambda^+_{2l+1}$ with $l \geq 1$ is the unique solution of
  $$
  \Lambda_{a,b}(\alpha \sigma) + \Lambda_{c,d}(\alpha \sigma) + \frac{\pi(l - 1)}{\sqrt{\alpha \sigma}} + \frac{\pi l}{\sqrt{\beta \sigma}} = \eta - \xi,
  $$
- $\lambda^-_{2l+1}$ with $l \geq 1$ is the unique solution of
  $$
  \Lambda_{a,b}(\beta \sigma) + \Lambda_{c,d}(\beta \sigma) + \frac{\pi(l - 1)}{\sqrt{\beta \sigma}} + \frac{\pi l}{\sqrt{\alpha \sigma}} = \eta - \xi.
  $$

**Proposition 4.14.** Assume that $p \equiv 1$ and $\alpha$ and $\beta$ are positive and continuous on $[\xi, \eta]$. Then the set of half-eigenvalues of bvp (2.7) consists of two increasing sequences of simple half-eigenvalues ($\lambda^+_k$) and ($\lambda^-_k$), such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda^+_k\} \times S^\nu_k$. Moreover for all $k \geq 1$ and $\nu = +$ or $-$, $\lambda^+_k$ is a decreasing function with respect to the weights $\alpha$ and $\beta$ lying in $C[\xi, \eta]$.

**Proof.** Consider the bvp

$$
-u''(t) = \theta(\alpha(t) + \beta(t))u^+(t) + \lambda \alpha(t)u^-(t) - \lambda \beta(t)u^+(t), \quad t \in (\xi, \eta),
$$

$$
au(\xi) - b \lim_{t \to \xi^-} p(t)u'(t) = 0,
$$

$$
au(\eta) + d \lim_{t \to \eta^-} p(t)u'(t) = 0.
$$

From Theorem 3.9 we have that for each integer $k \geq 1$, $\nu = +$ or $-$ and all $\lambda \geq 0$, there exists a unique $\theta^*_k(\lambda)$ such that (4.18) has a solution in $S^\nu_k$.

Note that $\theta^*_k(0) = \mu_k(\alpha + \beta) > 0$. Now we claim that there exists $\lambda_0 > 0$ such that $\theta^*_k(\lambda_0) \leq \lambda_0$. To the contrary, assume that for all $\lambda \geq 0$, $\theta^*_k(\lambda) > \lambda$. Thus we have from Proposition 3.10 that

$$
\lambda < \theta^*_k(\lambda) < \theta^*_k(\lambda, \alpha_+, \beta_+) = \theta^*(\lambda)
$$

where for $k \geq 1$ and $\nu = +$ or $-$, $\theta^*_k(\lambda, \alpha_+, \beta_+)$ is the unique real number for which

$$
-u''(t) = \theta(\alpha_+ + \beta_+)u^+(t) + \lambda \alpha_+ u^-(t) - \lambda \beta_+ u^+(t), \quad t \in (\xi, \eta),
$$

$$
au(\xi) - b \lim_{t \to \xi^-} p(t)u'(t) = 0,
$$

$$
au(\eta) + d \lim_{t \to \eta^-} p(t)u'(t) = 0.
$$
has a solution in $S_k^\nu$.

Assume that $k = 2l$ with $l \geq 1$ and $\nu = +$ or $-$ (the other cases can be checked similarly). We have from Corollary 4.13 that
\[
\Lambda_{a,b}((\theta^*(\lambda) - \lambda)\beta_+ + \theta^*(\lambda)\alpha_+) + \Lambda_{c,d}((\theta^*(\lambda) - \lambda)\alpha_+ + \theta^*(\lambda)\beta_+)
\]
\[
+ \frac{\pi(l-1)}{\sqrt{(\theta^*(\lambda) - \lambda)\beta_+ + \theta^*(\lambda)\alpha_+}} + \frac{\pi(l-1)}{\sqrt{(\theta^*(\lambda) - \lambda)\alpha_+ + \theta^*(\lambda)\beta_+}} = \eta - \xi. \quad (4.19)
\]
Taking into consideration the fact that $\Lambda_{a,b}$ and $\Lambda_{c,d}$ are decreasing functions, we obtain from (4.19) that
\[
\eta - \xi \leq \Lambda_{a,b}(\theta^*(\lambda)\alpha_+) + \Lambda_{c,d}(\theta^*(\lambda)\beta_+) + \frac{\pi(l-1)}{\sqrt{\theta^*(\lambda)\alpha_+}} + \frac{\pi(l-1)}{\sqrt{\theta^*(\lambda)\beta_+}}
\]
leading to
\[
\theta^*(\lambda) \leq \pi^2 \xi^2 (\eta - \xi)^2 (\frac{1}{\sqrt{\alpha_+}} + \frac{1}{\sqrt{\beta_+}})^2,
\]
which contradicts $\lim_{\lambda \to +\infty} \theta^*(\lambda) = +\infty$.

Thus there exists $\lambda_k^\nu$ such that $\theta_k^*(\lambda_k^\nu) = \lambda_k^\nu$ and $\lambda_k^\nu$ is a half-eigenvalue of (2.7). Uniqueness and simplicity of $\lambda_k^\nu$ follow from Lemma 4.5. Finally, the monotonicity of $\lambda_k^\nu$ with respect of the weights $\alpha$ and $\beta$ follows directly from Proposition 3.10.

**Proposition 4.15.** Assume that $p \equiv 1$, $\alpha, \beta$ are nonnegative and continuous on $[\xi, \eta]$ and the set $\{ t \in [\xi, \eta] : \alpha(t)\beta(t) > 0 \}$ has positive measure. Then the set of half-eigenvalues of bvp (2.7) consists of two increasing sequences of simple half-eigenvalues $(\lambda_k^+)^{k\geq1}$ and $(\lambda_k^-)^{k\geq1}$, such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda_k^\nu\} \times S_k^\nu$. Moreover for all $k \geq 1$ and $\nu = +$ or $-$, $\lambda_k^\nu$ is a decreasing function with respect to the weights $\alpha$ and $\beta$ lying in $C[\xi, \eta]$.

**Proof.** For $n \geq 1$, $\alpha_n = \alpha + \frac{1}{n}$ and $\beta_n = \beta + \frac{1}{n}$, let $\lambda_{k,n}^\nu = \lambda_k^\nu(\alpha_n, \beta_n)$ be the half-eigenvalue given by Proposition 4.14 associated with the eigenvector $\phi_n \in \Theta_k^\nu$. Because $(\alpha_n)$ and $(\beta_n)$ are decreasing sequences, we have from Proposition 4.14 that $(\lambda_{k,n}^\nu)n$ is nondecreasing. Now let $I_0 = [\xi_0, \eta_0] \subset (\xi, \eta)$ be such that $\alpha \beta > 0$ in $I_0$ and set $\vartheta = \min(\alpha, \beta)$. Hence, we have $\vartheta_n = \min(\alpha_n, \beta_n) = \vartheta + \frac{1}{n} \geq \vartheta$ and we deduce, from the monotonicity property in Proposition 4.14 and Properties 5 and 6 in Theorem 2.7, that
\[
\lambda_{k,n}^\nu = \lambda_k^\nu(\alpha_n, \beta_n) \leq \lambda_k^\nu(\vartheta_n, \vartheta_n) = \mu_k(\vartheta_n, [\xi, \eta]) \leq \mu_k(\vartheta, I_0) \leq \mu_k(\vartheta, I_0),
\]
and the sequence $(\lambda_{k,n})_n$ converges to some $\lambda_k^\nu > 0$, which is by Lemma 4.11 and Lemma 4.5 a simple half-eigenvalue of bvp (2.7) having an eigenvector $\phi = \lim \phi_n \in \Theta_k^\nu$ (up to a subsequence). Because the functions $u \in \partial \Theta_k^\nu$ have a double zero, Lemma 4.4 guarantees that $\phi \in \Theta_k^\nu$.

Let $\alpha_1$ be a nonnegative continuous function such that the set $\{ t \in [\xi, \eta] : \alpha_1(t)\beta(t) > 0 \}$ has a positive measure and $\alpha \leq \alpha_1$. We have from Proposition 4.14 that
\[
\lambda_k^\nu(\alpha + \frac{1}{n}, \beta + \frac{1}{n}) \leq \lambda_k^\nu(\alpha_1 + \frac{1}{n}, \beta + \frac{1}{n}).
\]
Letting $n \to \infty$ we obtain $\lambda_k^\nu(\alpha, \beta) \leq \lambda_k^\nu(\alpha_1, \beta)$. Similarly we prove that $\lambda_k^\nu$ is nonincreasing with respect to the weight $\beta$. The proof is complete. \hfill $\Box$

4.2. **Proof of Theorem 2.4.** Let $\varphi$ and $\rho_0$ be as in Section 2 and note that $\lambda$ is a half-eigenvalue with an eigenvector $u$ of (2.7) if and only if $\lambda/\rho_0$ is a half-eigenvalue with the eigenvector $v = u \circ \varphi^{-1}$ of the bvp

$$
-v''(t) = \frac{\lambda}{\rho_0} \left( \tilde{a}(t)v^+(t) - \tilde{b}(t)v^-(t) \right), \quad t \in (\xi, \eta),
$$

where

$$
\tilde{a}(t) = p(\varphi^{-1}(t))\alpha(\varphi^{-1}(t)), \quad \tilde{b}(t) = p(\varphi^{-1}(t))\beta(\varphi^{-1}(t))
$$

are integrable functions. So, it suffices to prove Theorem 2.4 with $p \equiv 1$. To this aim, let $(\alpha_n)$ and $(\beta_n)$ be two sequences in $C_c(\xi, \eta)$ such that $\lim \alpha_n = \alpha$ and $\lim \beta_n = \beta$ in $L^1(\xi, \eta)$, and let $\lambda^\nu_{k,n} = \lambda^\nu_{k}(\alpha_n, \beta_n)$ be the half-eigenvalue given by Proposition 4.15 associated with an eigenvector $\phi_n$. Let $\vartheta_n = \inf(\alpha_n, \beta_n)$, $\theta_n = \sup(\alpha_n, \beta_n)$, and $\vartheta = \sup(\alpha, \beta) \geq \vartheta = \inf(\alpha, \beta) > 0$ on some interval $I_0 = [\xi_0, \eta_0] \subset (\xi, \eta)$. We have that $\lim \vartheta_n = \vartheta$ in $L^1(\xi, \eta)$. Then, we deduce from the monotonicity property in Proposition 4.15 and Property 5 in Theorem 3.6 that

$$
0 < \mu_k(\vartheta) - \epsilon = \mu_k(\vartheta_n)
$$

where the constant $\epsilon$ and $C$ are respectively small enough and large enough. Let

$$
\lambda^\nu_{k,s} = \sup \lambda^\nu_{k,n} \quad \text{and} \quad \lambda^\nu_{k,i} = \liminf \lambda^\nu_{k,n}. \quad \text{We have from Lemma 4.11 that} \quad \lambda^\nu_{k,s} \quad \text{and} \quad \lambda^\nu_{k,i} \quad \text{are half-eigenvalues of} \quad (2.7). \quad \text{Then we deduce from Lemma 4.5 that} \quad \lambda^\nu_{k} = \lambda^\nu_{k,s} = \lambda^\nu_{k,i} \quad \text{is the unique and simple half-eigenvalue of} \quad (2.7). \quad \text{The same arguments as those used in the proof of Proposition 4.15 show that the eigenvector associated with} \quad \lambda^\nu_{k} \quad \text{belongs to} \quad S^\nu_{k} \cap Y.
$$

4.3. **Proof of Theorem 2.5**

4.3.1. **Proof of uniqueness of $\lambda^\nu_{k}$.**

**Lemma 4.16.** Assume that Hypotheses (2.2) and (2.3) hold and $\alpha, \beta \in K_G$. Then for $\nu = +$ or $-$, bvp (2.7) admits at most one half-eigenvalue having an eigenvector in $S^\nu_{k}$.

**Proof.** Suppose that $\lambda^+_{k}$ is a half-eigenvalue having an eigenvector $\phi_1 \in S^+_{k}$ (uniqueness of $\lambda^+_{k}$ can be proved in the same way), then $1/\lambda^+_{k}$ is a positive eigenvalue of the positive operator $L_\alpha : E \to E$ defined by

$$
L_\alpha u(t) = \int_{\xi}^{t} G(t,s)\alpha(s)u(s)ds.
$$

So, we have that $r(L_\alpha) > 0$ and since the cone of nonnegative functions is total in $E$, $r(L_\alpha)$ is a positive eigenvalue of $L_\alpha$ and

$$
\lambda^+_k \geq 1/r(L_\alpha).
$$
Let \((\xi_n)\) and \((\eta_n)\) be two sequences in \((\xi, \eta)\) such that \(\lim \xi_n = \xi, \lim \eta_n = \eta, \)
\((\xi_n)\) is decreasing, \((\eta_n)\) is increasing, and set
\[
\alpha_n(t) = \begin{cases} 
\inf(\alpha(t), \alpha(\xi_n)), & \text{if } t \leq \xi_n, \\
\alpha(t), & \text{if } t \in (\xi_n, \eta_n), \\
\inf(\alpha(t), \alpha(\eta_n)), & \text{if } t \geq \eta_n,
\end{cases}
\]
and let \(L_n : E \to E\) be the linear operator defined by
\[
L_n u(t) = \int_\xi^\eta G(t, s)\alpha_n(s)u(s)\,ds.
\] 
(4.22)
We see that for all \(n \in \mathbb{N}, L_n \leq L_\alpha.\) Then from Lemmas 4.10 and 3.5 we have \(r(L_n) \leq r(L_\alpha).\)

We have that for all \(n \in \mathbb{N}, \lambda^+_1 = 1/r(L_n) > 0\) is the unique positive eigenvalue associated with a positive eigenvector \(\phi^+_1\) to the linear bvp
\[-(pu')'(t) = \lambda \alpha_n(t)u(t), \quad \text{a.e. } t \in (\xi, \eta),
\]
\[au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,
\]
\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0.
\]

Moreover, \(\alpha_n \to \alpha\) in \(L^1_\alpha[\xi, \eta].\) So, we have from Lemmas 4.10 and 3.5 that
\[
\lim \lambda^+_1 = 1/r(L_\alpha) \leq \lambda^+_1.
\] 
(4.23)
Before proving uniqueness, note that, if \(\lambda\) is a positive eigenvalue of bvp (2.8) associated with an eigenvector \(\phi,\) then there exists a subinterval \([\gamma, \delta] \subset (\xi, \eta)\) such that \(\alpha(t)\phi(t) > 0\) for almost all \(t \in [\gamma, \delta].\) Indeed if this does not occur, we obtain the contradiction
\[
\phi(t) = \lambda \int_\xi^\eta G(t, s)\alpha(s)\phi(s)\,ds = 0 \quad \text{for all } t \in (\xi, \eta).
\]
This means also that
\[
\phi(t) = \lambda \int_\xi^\eta G(t, s)\alpha(s)\phi(s)\,ds > 0 \quad \text{for all } t \in (\xi, \eta).
\]
Set
\[
\psi_n = L_n \phi_1 \leq L_\alpha \phi_1 = (\lambda^+_1)^{-1} \phi_1.
\]
Observe that \(\psi_n\) satisfies
\[-(p'\psi_n)'(t) = \alpha_n(t)\phi_1(t) \geq \lambda^+_1 \alpha_n(t)\psi_n(t) \quad \text{a.e. } t \in (\xi, \eta),
\]
\[a\psi_n(\xi) - b \lim_{t \to \xi} p(t)\psi_n'(t) = 0,
\]
\[c\psi_n(\eta) + d \lim_{t \to \eta} p(t)\psi_n'(t) = 0.
\] 
(4.24)
Multiplying the differential inequality in (4.24) by \(\phi^+_1\) (the eigenvector of \(\lambda^+_1\)) and integrating over \([\xi, \eta]\) we obtain
\[
\int_\xi^\eta -(p\psi_n)'\phi^+_1 \geq \lambda^+_1 \int_\xi^\eta \alpha_n \psi_n \phi^+_1.
\]
We find, after two integration by parts of the left hand side,
\[
\lambda^+_1 \int_\xi^\eta \alpha_n \psi_n \phi^+_1 \geq \lambda^+_1 \int_\xi^\eta \alpha_n \psi_n \phi^+_1.
\]
leading to $\lambda_1^+ \leq \lambda_n^-$ for all $n \geq 1$, from which we have

$$\lambda_1^+ \leq \lim_{n \to \infty} \lambda_n^-. \tag{4.25}$$

At the end, combining $[4.25]$ with $[4.23]$, we obtain $\lambda_1^+ = 1/r(L_n)$, that is $1/r(L_n)$ is the unique half eigenvalue of $(2.7)$ having an eigenvector in $S_1^+$.

4.3.2. Proof of uniqueness of $\lambda_n^+$, $k \geq 2$.

**Lemma 4.17.** Assume that Hypotheses $[2.2]$ and $[2.3]$ hold and $\alpha, \beta \in K_G$. Then for each integer $k \geq 1$ and $\nu = +$ or $-$, bvp $(2.7)$ admits at most one half-eigenvalue having an eigenvector in $S_k^+.$

**Proof.** To the contrary, assume that $\lambda_1$ and $\lambda_2$ are two half-eigenvalues having, respectively, the eigenvectors $\phi_1, \phi_2 \in S_k^+$ with the sequences of simple zeros $(x_i)_{1 \leq i \leq k}$ and $(y_i)_{1 \leq i \leq k}.$ In the spirit of Theorem 3.1, assume that $x_1 \leq y_1$ and let $i_0, j_0 \in \{1, \ldots, k\}$ such that $x_{i_0} \leq z_{j_0} \leq z_{j_0} + 1 \leq x_{i_0} + 1$, and without loss of generality, suppose that $\phi_1 \geq 0$ and $\phi_2 \geq 0$ in each of the intervals $[\xi, x_1]$ and $[z_{j_0}, z_{j_0} + 1].$ Let $(\xi_n)$ and $(\eta_n)$ be the sequences given in the proof of Lemma 4.16 and set

$$\alpha_n^1(t) = \begin{cases} \inf(\alpha(t), \alpha(\xi_n)), & \text{if } t \leq \xi_n, \\ \alpha(t), & \text{if } t \in (\xi_n, \eta_n). \end{cases}$$

From Lemma 4.16 we have $\lambda_1 = \lim_{n \to \infty} \mu_1(\alpha_n^1, [\xi, x_1])$ and $\lambda_2 = \lim_{n \to \infty} \mu_1(\alpha_n^1, [\xi, \eta_n])$, and from Property 4 of Theorem 3.6, that for all $n \geq 1$, $\mu_1(\alpha_n^1, [\xi, x_1]) \geq \mu_1(\alpha_n^1, [\xi, \eta_n]).$ Letting $n \to \infty$ we obtain $\lambda_1 \geq \lambda_2$.

Now we will discuss the cases $z_{j_0} + 1 < \eta$ and $z_{j_0} + 1 = \eta$. If $z_{j_0} + 1 < \eta$, then integrating on $[z_{j_0}, z_{j_0} + 1]$, we obtain

$$0 \geq \int_{z_{j_0}}^{z_{j_0} + 1} - (p\phi_1')\phi_2 + (p\phi_2')\phi_1 = (\lambda_1 - \lambda_2) \int_{z_{j_0}}^{z_{j_0} + 1} \alpha \phi_1 \phi_2,$$

leading to $\lambda_1 = \lambda_2$.

If $z_{j_0} + 1 = \eta$, then considering

$$\alpha_n^2(t) = \begin{cases} \alpha(t), & \text{if } t \in (x_{i_0}, \eta_n), \\ \inf(\alpha(t), \alpha(\nu_n)), & \text{if } t \geq \eta_n, \end{cases}$$

we have that $\lambda_1 = \lim_{n \to \infty} \mu_1(\alpha_n^2, [x_{i_0}, \eta])$ and $\lambda_2 = \lim_{n \to \infty} \mu_1(\alpha_n^2, [z_{j_0}, \eta]),$ and from Property 3 of Theorem 2.5, that for all $n \geq 1$, $\mu_1(\alpha_n^2, [x_{i_0}, \eta]) \geq \mu_1(\alpha_n^2, [z_{j_0}, \eta]).$ So, letting $n \to \infty$ we obtain also in this case $\lambda_1 = \lambda_2$. This completes the proof. □

4.3.3. Proof of existence of $(\lambda_k^+)_{k \geq 1}$. Let $(\xi_n)$ and $(\eta_n)$ be the sequences introduced in the proof of Lemma 4.16 and consider

$$\alpha_n(t) = \begin{cases} \alpha(t), & \text{if } t \in (\xi_n, \eta_n), \\ 0, & \text{if } t \notin (\xi_n, \eta_n), \end{cases}$$

$$\beta_n(t) = \begin{cases} \beta(t), & \text{if } t \in (\xi_n, \eta_n), \\ 0, & \text{if } t \notin (\xi_n, \eta_n). \end{cases}$$

For all $n, k \geq 1$ and $\nu = +$ or $-$, let $\lambda_{k,n}^+$ be the unique half-eigenvalue of

$$-(pu')'(t) = \lambda_n^+(t)u^+(t) - \beta_n(t)u^-(t) \quad \text{a.e. } t \in (\xi, \eta),$$

$$au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,$$
we have from Property 2 in Theorem 2.4 that, for all \( n \) in some closed interval \( I_0 \subset (\xi, \eta) \). Since \( (\alpha_n) \) and \( (\beta_n) \) are nondecreasing sequences, we have from Property 2 in Theorem 2.4 that, for all \( n \geq 1 \),

\[
\lambda_{k,n}^\nu \geq \lambda_{k,n+1}^\nu,
\]

and

\[
\lambda_{k,n}^\nu = \lambda_{k}^\nu(\alpha_n, \beta_n) \geq \lambda_{k}^\nu(\theta_n) = \mu_k(\theta_n) = \mu_1(\theta_n). \tag{4.26}
\]

Because \( \mu_1(\theta_n) = 1/r(L_{\theta_n}) \), \( \mu(\theta) = 1/r(L_\theta) \) and \( L_{\theta_n} \to L_\theta \) in operator norm, it follows from Lemma 3.5 that, for \( \epsilon > 0 \) small enough,

\[
\lambda_{k,n}^\nu \geq \lambda_{k,n+1}^\nu \geq \mu_1(\theta) - \epsilon > 0.
\]

Thus, from Lemma 4.11 we have that \( \lambda_{k}^\nu = \lim_{n \to \infty} \lambda_{k,n}^\nu \) is a half-eigenvalue of (2.7) having an eigenvector \( \psi \) (as it is shown in proof of Lemma 4.11 \( \psi = \lim \phi_n \)).

In view of Lemma 4.11, it remains to show that \( \psi \in S_k^\nu \). To the contrary, assume that \( \psi \in S_l^\nu \) with \( l \neq k \) and let \( (z_j)_{j=1}^{l-1} \) be the sequence of interior zeros of \( \psi \) and \( [\xi_1, \eta_1] \subset (\xi, \eta) \) such that

\[
\xi_1 < z_1 < z_2 < \cdots < z_{l-1} < \eta_1.
\]

Choose \( \delta > 0 \) small enough and set \( I_j = (z_j - \delta, z_j + \delta) \) for \( j \in \{1, \ldots, l-1\} \). There exists \( n_* \in \mathbb{N} \) such that for all \( n \geq n_* \), \( \phi_n \psi > 0 \) in all the intervals \([\xi_1, z_1 - \delta], [z_{k-1} + \delta, \eta_1], [z_j + \delta, z_{j+1} - \delta], j \in \{1, \ldots, l-2\} \).

Fix \( j \in \{1, \ldots, l-1\} \). There exists \( n_j \geq n_* \) such that the function \( \phi_n \) has exactly one zero in \( I_j \). Otherwise if there is a subsequence \( (\phi_{n_i}) \) such that for all \( i \geq 1 \), \( \phi_{n_i} \) has at least two zeros, then we can choose \( x_{n_i}^1 \) and \( x_{n_i}^2 \) in \( I_j \) such that

\[
\phi_{n_i}^{[1]}(x_{n_i}^1) \leq \phi_{n_i}^{[1]}(x_{n_i}^2).
\]

Let

\[
x_{n_i}^1 = \liminf x_{n_i}^1, \quad x_{n_i}^1 = \limsup x_{n_i}^1, \quad x_{n_i}^2 = \liminf x_{n_i}^2, \quad x_{n_i}^2 = \limsup x_{n_i}^2.
\]

Hence, since \( \psi = \lim \phi_n / \|\phi_n\| \) we have

\[
\psi(x_{n_i}^1) = \psi(x_{n_i}^2) = \psi(x_{n_i}^1) = \psi(x_{n_i}^2) = 0
\]

leading to

\[
\lim x_{n_i}^1 = \lim x_{n_i}^2 = z_j.
\]

Moreover, from Remark 4.12 it follows that

\[
\psi^{[1]}(z_j) = \lim \phi_{n_i}^{[1]}(x_{n_i}^1) = \lim \phi_{n_i}^{[1]}(x_{n_i}^2) = 0,
\]

contradicting the simplicity of \( z_j \).

Now, we claim that there exists \( n^* \in \mathbb{N} \) such that for all \( n \geq n^* \), \( \phi_n \) does not vanish in the intervals \((\xi, \xi_1) \) and \((\eta_1, \eta) \). Again, to the contrary, assume that there is a subsequence \( (\phi_{n_i}) \) such that for all \( i \geq 1 \), \( \phi_{n_i} \) has at least one zero. Let \( x_{n_i} \in (\xi, \xi_1) \) be the first zero of \( \phi_{n_i} \). In this case, we have that

\[
\lim x_{n_i} = \xi, \quad \psi(\xi) = \phi_{n_i}(\xi) = 0.
\]
Moreover, for all \(i \geq 1\), \(\phi_n\) satisfies
\[
-(p\phi_n')' = \mu_n \alpha_n(t) \phi_n(t) \quad \text{a.e. } t \in (\xi, x_{n_i}),
\]
\[
\phi_n(\xi) = \phi_n(x_{n_i}) = 0 \quad \text{and } \phi_n > 0 \text{ in } (\xi, x_{n_i}).
\]
(4.27)

Clearly, Equation (4.27) implies that \(\mu_n = \mu_1(\alpha_n, [\xi, x_{n_i}])\). Taking into consideration \(\lim \alpha_n = \alpha\) in \(L^1_{\alpha}[\xi, \eta]\) and \(\mu_n = \mu_1(\alpha_n, [\xi, x_{n_i}]) = 1/r(L^n_{\alpha})\) where \(L^n_{\alpha} : C[\xi, x_{n_i}] \to C[\xi, x_{n_i}]\) is defined by
\[
L^n_{\alpha}u(t) = \int_\xi^{x_{n_i}} G_n(t, s) \alpha(s) u(s) ds,
\]
from Lemma 3.5, we obtain
\[
\mu_n = \mu_1(\alpha_n, [\xi, x_{n_i}]) = \frac{1}{r(L^n_{\alpha})} \geq \frac{1}{r(L^\alpha)} = \mu_1(\alpha, [\xi, x_{n_i}]).
\]
(4.28)

Thus, combining Lemma 4.1 with (4.28), we obtain the contradiction
\[
\widetilde{\mu} = \lim \mu_n \geq \lim \mu_1(\alpha, [\xi, x_{n_i}]) = \lim \frac{1}{r(L^\alpha)} = +\infty.
\]

Hence, we conclude that for all \(n \geq n_\infty = \max\{n_*, n^*, n_1, \ldots, n_{k-1}\}\), \(\phi_n\) has exactly \((l-1)\) simple zeros in \((\xi, \eta)\) contradicting \(\phi_n \in S_k^+\).

Finally, letting \(n \to \infty\) in \(\lambda^\nu_k < \lambda^\nu_{k+1}\) we obtain \(\lambda^\nu_k \leq \lambda^\nu_{k+1}\).

4.4. Proof of Theorem 2.7

The existence of \((\mu_k(q))_{k \geq 1}\) as a nondecreasing sequence follows from Theorem 2.5 when taking \(\alpha = \beta = q\) in bvp (2.7) and for all \(k \geq 1\), \(\mu_k(q)\) has an eigenvector \(\phi_k \in S_k\). We have from Lemma 4.3 and assertion 2 in Lemma 4.8 that \(\mu_k(q)\) is simple for all \(k \geq 3\) and Assertion 1 is proved. Assertion 2 follows from the monotonicity of the sequence \((\mu_k(q))_{k \geq 1}\) and the simplicity of \(\mu_k(q)\) for \(k \geq 3\). Assertion 3 follows from Lemma 4.5 and assertion 1 in Lemma 4.8 and Lemma 4.16. Assertion 4 is obvious.

Assertion 5 follows from the monotonicity property of half-eigenvalues in Theorem 2.5 and Assertion 6 is obtained when letting \(n \to \infty\) in the relation
\[
\mu_k(q_n, [\xi, \eta]) \leq \mu_k(q_n, [\xi_1, \eta_1]),
\]
where
\[
q_n(t) = \begin{cases} q(t), & \text{if } t \in (\xi_n, \eta_n), \\ 0, & \text{if } t \notin (\xi_n, \eta_n), \end{cases}
\]
and \((\xi_n)_{n \geq 1}, (\eta_n)_{n \geq 1}\) are those in the proof of Lemma 4.16.

It remains to prove Assertion 7. Let \((q_n) \subset K_G\) be a sequence converging to \(q \in K_G\) in \(L^1_{\alpha}[\xi, \eta]\), and \([\xi_0, \eta_0] \subset (\xi, \eta)\) such that \(q > 0\) in \([\xi_0, \eta_0]\). We have then from Property 6 and Property 5 in Theorem 3.6
\[
\mu_k(q_n, [\xi, \eta]) \leq \mu_k(q_n, [\xi_0, \eta_0]) \leq \mu_k(q, [\xi_0, \eta_0]) + C.
\]

Set \(\mu^1 = \lim \inf \mu_k(q_n, [\xi, \eta])\) and \(\mu^2 = \lim \sup \mu_k(q_n, [\xi, \eta])\). There exist two subsequences \((\mu_k(q_n, [\xi, \eta]))\) and \((\mu_k(q_n, [\xi, \eta]))\) of \((\mu_k(q_n, [\xi, \eta]))\) converging respectively to \(\mu^1\) and \(\mu^2\). Applying Lemma 4.11 we obtain that \(\mu^1\) and \(\mu^2\) are eigenvalues of (2.8). Furthermore, arguing as in Subsection 4.3.3, we see that the eigenvectors associated with \(\mu^1\) and \(\mu^2\) belongs to \(S_k\). Thus, we deduce from Lemmas 4.16 and 4.17 that
\[
\mu^1 = \mu^2 = \lim \mu_k(q_n, [\xi, \eta]) = \mu_k(q, [\xi, \eta]).
\]
This completes the proof.

4.5. **Proof of Theorem 2.9** Consider the bifurcation bvp associated with bvp (2.1),

\[-(pu')'(t) = \mu q(t)u(t) + \mu g(t, u(t)), \quad \text{a.e. } t \in (\xi, \eta),
au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,
\]

\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0,
\]

where \( \mu \) is a real parameter and \( g(t, u) = f(t, u) - q_0(t)u \) and in all that follows, we denote by \( (\mu_k(q_0))_{k \geq 1} \) the sequence of eigenvalues obtained from Theorem 2.7

for the bvp

\[Lu(t) = \mu q(t)u(t), \quad \text{a.e. } t \in (\xi, \eta),
\]

\[au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,
\]

\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0,
\]

and by \( (\chi_k^\nu)_{k \geq 1} \), with \( \nu = + \) or \( - \), the two sequences of half-eigenvalues of the bvp

\[-(pu')'(t) = \chi(\alpha_\infty(t)u^+(t) - \beta_\infty(t)u^-(t)), \quad \text{a.e. } t \in (\xi, \eta),
\]

\[au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0,
\]

\[cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0,
\]

given by Theorem 2.5.

Applying \( L^{-1} \), we obtain that bvp (4.29) is equivalent to the equation

\[u = \mu L_\eta u + \mu H(u)
\]

where \( H : E \to E \) is defined by

\[Hu(s) = \int_{\xi}^{\eta} G(t, s)g(s, u(s))ds
\]

and is completely continuous.

**Lemma 4.18.** Assume that (2.2)-(2.4) hold. Then from each \( \mu_\xi(q_0) \), with \( k \geq 3 \), bifurcate two unbounded components (in \( \mathbb{R} \times E \)), \( \Gamma^+_k \) and \( \Gamma^-_k \) such that for all \( k \geq 3 \) and \( \nu = + \) or \( - \), \( \Gamma^\nu_k \subset \mathbb{R} \times S^\nu_k \).

**Proof.** First, note that Hypothesis (2.4) implies that \( H(u) = o(\|u\|_\infty) \) near 0. Indeed, we have for \( (u_n) \subset E \) with \( \lim \|u_n\|_\infty = 0 \)

\[
\frac{|Hu_n(t)|}{\|u_n\|_\infty} \leq \int_{\xi}^{\eta} R_n(s)ds \quad \text{where } R_n(s) = G(s, s)\left|\frac{f(s, u_n(s)) - q_0(s)u_n(s)}{\|u_n\|_\infty}\right|.
\]

Then, it follows from (2.5) that

\[R_n(s) \leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + q_0(s)) \frac{|u_n(s)|}{\|u_n\|_\infty}
\leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + q_0(s)) \in L^1[\xi, \eta],

and from (2.4) that

\[R_n(s) \leq G(s, s)\frac{f(s, u_n(s))}{u_n(s)} - q_0(s)\frac{|u_n(s)|}{\|u_n\|_\infty}
\]
as \( n \to +\infty, s \in [\xi, \eta] \) a.e..

So, by the Lebesgue dominated convergence theorem, we have

\[
\lim_{n \to \infty} \frac{H(u_n)}{\|u_n\|_\infty} = 0;
\]

that is, \( H(u) = o(\|u\|_\infty) \) at 0.

Since for all \( k \geq 3, \mu_k(q_0) \) is of algebraic multiplicity one, from \cite{23} Theorem 1.40 we conclude that from each \( (\mu_k(q_0), 0) \) with \( k \geq 3 \), bifurcate two components \( \Gamma_k^1 \) and \( \Gamma_k^2 \) of nontrivial solutions of bvp \((4.29)\) such that for \( i = 1, 2, \Gamma_k^i \) is either unbounded in \( \mathbb{R} \times E \) or meets \( (\mu_l(q_0), 0) \) where \( l \neq k \).

Now, note that if \( (\lambda, u) \in \Gamma_k^i \) \( i = 1, 2 \), then all zeros of \( u \) are simple. This is due to the fact that \( (\lambda, u) \) satisfies also the bvp

\[
-(pu')' = \lambda q_u(t)v(t), \quad t \in (\xi, \eta),
\]

\[
\alpha v(\xi) - b \lim_{t \to \xi^-} p(t)v(t) = 0,
\]

\[
\alpha v(\eta) + d \lim_{t \to \eta^-} p(t)v(t) = 0,
\]

with

\[
q_u(t) = \frac{f(t, u)}{u}.
\]

Since Hypothesis \cite{23} guarantees that \( q_0 \in L^2[\xi, \eta] \), we deduce from Theorem \cite{2} that there exists an integer \( j \geq 1 \) such that \( u \in S_j \).

Also, we claim that for all \( k \geq 3 \) and \( i = 1, 2 \), there exists a neighborhood \( V_k^i \) of \( (\mu_k(q_0), 0) \) such that \( \Gamma_k^1 \cap V_k^1 \subset \mathbb{R} \times S_k \). Let \( (\mu_n, u_n)_{n \geq 1} \subset \Gamma_k^i \) be a sequence converging to \( (\mu_k, 0) \). Thus, \( v_n = \frac{u_n}{\|u_n\|_\infty} \) satisfies

\[
v_n = \mu_n L_{q_0}(v_n) + \mu_n \frac{H(u_n)}{\|u_n\|_\infty} \quad \text{and} \quad \|v_n\|_\infty = 1.
\]

Since \( L_{q_0} \) is compact and \( H(u) = o(\|u\|_\infty) \) near 0, there exists a subsequence \( (v_{n_j}) \) of \( (v_n) \) converging to \( v \) in \( E \) satisfying

\[
v = \mu_k(q_0)L_{q_0}v \quad \text{and} \quad \|v\|_\infty = 1.
\]

So, from Theorem \cite{2} we have \( v \in S_k \).

Let \( (z_j)_{j=1}^{k-1} \) be the sequence of interior zeros of \( v \) and \( [\xi_1, \eta_1] \subset (\xi, \eta) \) such that

\[
\xi_1 < z_1 < z_2 < \cdots < z_{k-1} < \eta_1.
\]

Choose \( \delta > 0 \) small enough and set \( I_j = (z_j - \delta, z_j + \delta) \) for \( j \in \{1, \ldots, k-1\} \). There exists \( n_* \in \mathbb{N} \) such that for all \( n_j \geq n_* \), \( v_{n_j} \) has exactly one zero in \( I_j \). Otherwise if there is a subsequence \( (v_{n_l}) \) such that for all \( l \geq 1 \), \( v_{n_l} \) has at least two zeros, then we can choose \( x_{n_l}^1 \) and \( x_{n_l}^2 \) in \( I_j \) such that

\[
v_{n_l}^{[1]}(x_{n_l}^1) \leq v_{n_l}^{[1]}(x_{n_l}^2).
\]

Hence, we obtain

\[
\lim_{n \to \infty} x_{n_l}^1 = \lim_{n \to \infty} x_{n_l}^2 = z_j.
\]
and from Lemma 4.10 that
\[ v^{[1]}(z_j) = \lim_{n \to \infty} v^{[1]}_{n_i}(x_{n_i}^1) = \lim_{n \to \infty} v^{[1]}_{n_i}(x_{n_i}^2) = 0, \]
contradicting the simplicity of \( z_j \).

Now, we claim that there exists \( n^* \in \mathbb{N} \) such that for all \( n \geq n^* \), \( v_n \) does not vanish in the intervals \( (\xi, \xi_1) \) and \( (\eta_1, \eta) \). To the contrary, assume that there is a subsequence \( (v_{n_i}) \) such that, for all \( l \geq 1 \), \( v_{n_i} \) has at least one zero. Let \( x_{n_i} \in (\xi, \xi_1) \) be the first zero of \( v_{n_i} \). In this case, we have that
\[ \lim_{n \to \infty} x_{n_i} = \xi, \quad v(\xi) = v_n(\xi) = 0. \]
Moreover, for all \( l \geq 1 \), \( u_{n_i} \) satisfies
\[ -(pu_{n_i}'(t))' = \mu_{n_i} q_{n_i}(t) u_{n_i}(t), \quad t \in (\xi, x_{n_i}), \]
\[ u_{n_i}(\xi) = u_{n_i}(x_{n_i}) = 0, \]
\[ u_{n_i} > 0 \quad \text{in} \quad (\xi, x_{n_i}), \]
where \( q_{n_i}(t) = (f(t, u_{n_i}(t))) u_{n_i}(t) \). Clearly, Equation (4.30) implies that \( \mu_{n_i} = \mu_1(q_{n_i}, [\xi, x_{n_i}]) \). Taking into consideration Hypothesis (2.5), from Property 3 in Theorem 2.7 we obtain that
\[ \mu_{n_i} = \mu_1(q_{n_i}, [\xi, x_{n_i}]) \geq \mu_1(\gamma_\infty, [\xi, x_{n_i}]). \]
So, we obtain as in Subsection 4.3.3 the contradiction
\[ \mu_k(q_0) = \lim \mu_{n_i} \geq \lim_{n \to \infty} \mu_1(\gamma_\infty, [\xi, x_{n_i}]) = +\infty. \]

At this stage we conclude that, for all \( n \geq n_\infty = \max\{n_*, n^*, n_1, \ldots, n_{k-1}\} \), \( v_n \) has exactly \( (k - 1) \) simple zeros in \( (\xi, \eta) \) and so the existence of the neighborhood \( V_k \).

Using the same arguments as those used above, we see that for all \( (\mu_0, u_0) \in \Gamma_k^\nu \) with \( u_0 \in S_{k_1} \), there exists a neighborhood \( W_0 \) of \( (\mu_0, u_0) \) such that \( W_0 \cap \Gamma_k^\nu \subset \mathbb{R} \times S_{k_1} \). This shows that the number of zeros of functions \( u \) lying in the projection of \( \Gamma_k^\nu \) onto the space \( E \) is locally constant, so it is constant and it is equal to \((k - 1).\)

Thus, \( \Gamma_k^\nu \subset \mathbb{R} \times S_k \). Set
\[ \Gamma_k^+ = (\Gamma_k^\nu \cap \mathbb{R} \times S_k^+) \cup (\Gamma_k^2 \cap \mathbb{R} \times S_k^+) \]
and let \( \epsilon > 0 \) and \( \kappa \in (0, 1) \). We have from Theorem 1.25 in [25] that, for \( i = 1, 2 \), there exists a sequence \((\mu_{n_i}, u_{n_i})_{n \geq 1} \subset \Gamma_k^\nu \) such that \( |\mu_{n_i} - \mu_k(q_0)| < \epsilon \), \( u_{n_i} = t_{n_i} v_k + w_{n_i} \), \( |t_{n_i}'| = 0 \) and \( t_{n_i}' > \kappa ||u_{n_i}^2||, \ t_{n_i}' < -\kappa ||u_{n_i}^2|| \). Moreover, from [25] Lemma 1.24 we have that \( w_{n_i} = o(|t_{n_i}^2|) \). Arguing as above, we see that \( \lim (u_{n_i} ||u_{n_i}||_\infty) = v^1 \) (up to a subsequence) where \( v^1 \) and \( v^2 \) are eigenvectors associated with \( \mu_k(q_0) \) with \( v^1 > 0 \) near \( \xi \) and \( v^2 < 0 \) near \( \xi \). So \( v^1 \in S_k^+ \) and \( v^2 \in S_k^- \). Since the limits are in \( E = C[\xi, \eta] \), arguing as in the proof of existence of neighborhood \( V_k^\nu \), in the beginning of the proof, we obtain that \( u_{n_i}^1 \in S_k^+ \) and \( u_{n_i}^2 \in S_k^- \) for \( n \) large enough. This shows that for all \( k \geq 3 \) and \( \nu = + \text{ or } - \), \( \Gamma_k^\nu \neq \emptyset \), and again because of the topology of \( E \) (if \( v_n \to v \in E \) and \( v > 0 \) near \( \xi \) then \( v_n > 0 \) near \( \xi \) for \( n \) large) and functions \( u \) lying in the projection of \( \Gamma_k^\nu \) onto the space \( E \) have only simple zeros and all have the same number of zeros, \( \Gamma_k^\nu \) does not leave \( \mathbb{R} \times S_k^\nu \).

Finally, taking into consideration the claim in the beginning of the proof, and the fact that \( \Gamma_k^\nu \) does not leave \( \mathbb{R} \times S_k^\nu \) we understand that for all \( k \geq 3 \) and \( \nu = + \text{ or } - \), \( \Gamma_k^\nu \) is unbounded in \( \mathbb{R} \times E \). \( \Box \)
Lemma 4.19. Assume that (2.2)–(2.5) hold. Then for all \( k \geq 3 \) and \( \nu = + \) or \( - \), the component \( \Gamma_k^\nu \) rejoins the point \((\chi_k^\nu, \infty)\).

Proof. Because \( u \equiv 0 \) is the unique solution of bvp (4.29) for \( \mu = 0 \), we have from Lemma 4.18 that for all \( k \geq 1 \) and \( \nu = + \) or \( - \), \( \{0\} \times E \cap \Gamma_k^\nu = \emptyset \). Therefore, if \((\mu, u) \in \Gamma_k^\nu \) then \( \mu > 0 \).

Moreover, if \((\mu, u) \in \Gamma_k^\nu \), then \( \mu = \mu_k(\frac{L(t,u)}{u}, [\xi, \eta]) \), and this together with Hypothesis (2.5) and Property 3 of Theorem 2.7, leads to

\[
\mu = \mu_k \left( \frac{f(t, u)}{u}, [\xi, \eta] \right) \leq \mu_k(\delta_\infty, [\xi, \eta]).
\]

This shows that for all \( k \geq 1 \) and \( \nu = + \) or \( - \), the projection of \( \Gamma_k^\nu \) onto the real axis is bounded.

Now, let \((\mu_n, u_n)\) be a sequence in \( \Gamma_k^\nu \) such that \( \lim_{n \to +\infty} \|u_n\| = +\infty \). For contradiction purposes, suppose that \( \lim_{n \to +\infty} \mu_n \neq \chi_k^\nu \). Then there exist \( \varepsilon > 0 \) and a subsequence of \((\mu_n)\), which will be denoted for convenience by \((\mu_n)_{n \geq 1}\), such that

\[
|\mu_n - \chi_k^\nu| \geq \varepsilon.
\]

Denote by \((v_n)\) the sequence defined by \( v_n = \frac{u_n}{\|u_n\|} \). Note that \( \|v_n\|_\infty = 1 \) and \((\mu_n, v_n)\) satisfies

\[
v_n = \mu_n A_\infty v_n + \frac{\Omega u_n}{\|u_n\|_\infty}
\]

where \( A, K : E \to E \) are defined by

\[
A_\infty u(t) = \int_\xi^\eta G(t, s)(\alpha_\infty(s)u^+(s) - \beta_\infty(s)u^-(s))ds,
\]

\[
\Omega u(t) = \int_\xi^\eta G(t, s)g_\infty(s, u(s))ds,
\]

with \( g_\infty(t, x) = f(t, x) - \alpha_\infty(t)x^+ + \beta_\infty(t)x^- \). Note that Hypotheses (2.4) and (2.5) imply that \( \Omega u_n = o(\|u_n\|_\infty) \) at \( \infty \). Indeed, we have

\[
\frac{\|\Omega u_n(t)\|}{\|u_n\|_\infty} = \left| \int_\xi^\eta G(t, s)(\frac{g_\infty(s, u_n(s))}{\|u_n\|_\infty})ds \right| \leq \int_\xi^\eta G(s, s) \frac{f(s, u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)}{\|u_n\|_\infty} ds.
\]

From (2.4), we have

\[
G(s, s) \left| \frac{f(s, u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)}{\|u_n\|_\infty} \right| \leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + \alpha_\infty(s) + \beta_\infty(s)) \frac{\|u_n(s)\|}{\|u_n\|_\infty} \leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + \alpha_\infty(s) + \beta_\infty(s)) \in L^1[\xi, \eta].
\]

Now, set

\[
P_n(s) = G(s, s) \left| \frac{f(s, u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)}{\|u_n\|_\infty} \right|
\]

and let us prove that \( \lim P_n(s) = 0 \) for \( s \in [\xi, \eta] \) a.e..
Let \( s \in [\xi, \eta] \) (such \( s \) exists a.e.), such that
\[
\lim_{x \to +\infty} \frac{f(s, x)}{x} = \alpha_\infty(s) \quad \text{and} \quad \lim_{x \to -\infty} \frac{f(s, x)}{x} = \beta_\infty(s).
\]
For such an \( s \) we distinguish the following cases:

- **\( \lim u_n(s) = +\infty \)**: in this case we have
  \[
  P_n(s) = G(s, s) \frac{f(s, u_n(s))}{u_n(s)} - \alpha_\infty(s) \frac{u_n(s)}{\|u_n\|_{\infty}} 
  \leq G(s, s) \frac{f(s, u_n(s))}{u_n(s)} - \alpha_\infty(s) \to 0 \quad \text{as} \quad n \to +\infty.
  \]

- **\( \lim u_n(s) = -\infty \)**: in this case we have
  \[
  P_n(s) = G(s, s) \frac{f(s, u_n(s))}{u_n(s)} - \beta_\infty(s) \frac{u_n(s)}{\|u_n\|_{\infty}} 
  \leq G(s, s) \frac{f(s, u_n(s))}{u_n(s)} - \beta_\infty(s) \to 0 \quad \text{as} \quad n \to +\infty.
  \]

- **\( \lim u_n(s) \neq \pm \infty \)**: in this case there may exist subsequences \( (u_{n_1}(s)) \) and \( (u_{n_2}(s)) \) such that \( (u_{n_1}(s)) \) is bounded and \( \lim u_{n_2}(s) = \pm \infty \). Arguing as in the above two cases we obtain \( \lim P_{n_2}(s) = 0 \) and we have
  \[
  P_{n_1}(s) \leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + \alpha_\infty(s) + \beta_\infty(s)) \frac{|u_{n_1}(s)|}{\|u_{n_1}\|_{\infty}} \to 0 \quad \text{as} \quad k \to +\infty.
  \]

Thus, we have \( \lim P_n(s) = 0 \) for \( s \in [\xi, \eta] \) a.e. By the Lebesgue dominated convergence theorem, we conclude that \( \Omega u_n = o(\|u_n\|_{\infty}) \) at \( \infty \).

Now, because of the compactness of \( A_\infty \) and the boundedness of \( (v_n) \), there exists a subsequence \( (v_{n_k}) \) converging to \( v \in S_k^\nu \) (use the same arguments as in the proof of Lemma \( 4.18 \)) with \( \|v\|_{\infty} = 1 \) satisfying \( v = \chi_\infty Av \), where \( \chi_\infty \) is the limit of some subsequence of \( \langle \mu_n \rangle \) of \( (\mu_n) \). Thus, we have \( \chi_\infty = \chi_\nu^\nu \) and the contradiction
\[
0 = \lim |\mu_n - \chi_\nu^\nu| \geq \varepsilon > 0.
\]

Now, we are able to prove Theorem 2.9. Note that \( u \in S_k^\nu \) is a solution to \( (2.1) \) if and only if \( (1, u) \in \Gamma_\nu \), and this occurs if \( \lambda_\nu^\nu < 1 < \mu_i(q_0) \) or \( \mu_i(q_0) < 1 < \lambda_\nu^\nu \).

Assume that \( \mu_i(q_0) < 1 < \mu_k(\vartheta_\infty) \) with \( 2 < k < l \), and let \( i \in \{k, \ldots, l\} \). We have \( \mu_i(q_0) \leq \mu_i(q_0) < 1 \), and from the nondecreasing property of \( \lambda_\nu^\nu \) with respect to the functions \( \alpha \) and \( \beta \),
\[
\lambda_\nu^\nu = \lambda_\nu^\nu(\alpha_\infty, \beta_\infty) \geq \lambda_\nu^\nu(\vartheta_\infty, \vartheta_\infty) = \mu_i(\vartheta_\infty) \geq \mu_k(\vartheta_\infty) > 1.
\]

Now, assume that \( \mu_i(\vartheta_\infty) < 1 < \mu_k(q_0) \) with \( 2 < k < l \), and let \( i \in \{k, \ldots, l\} \). We have \( \mu_i(q_0) \geq \mu_k(q_0) > 1 \), and from the nondecreasing property of \( \lambda_\nu^\nu \) with respect to the functions \( \alpha \) and \( \beta \),
\[
\lambda_\nu^\nu = \lambda_\nu^\nu(\alpha_\infty, \beta_\infty) \leq \lambda_\nu^\nu(\vartheta_\infty, \vartheta_\infty) = \mu_i(\vartheta_\infty) \geq \mu_k(\vartheta_\infty) > 1.
\]

Thus, Theorem 2.9 is proved.

**Remark 4.20.** Note that if \( q_0 \in K_G \cap L^1[\xi, \eta] \), from Lemma 4.6 we have that for all \( n \geq 1 \), \( \mu_n(q_0) \) is of algebraic multiplicity one. Thus, Theorem 2.9 and Corollary 2.10 can be extended to the case \( 1 \leq k < l \).
Theorem 2.9 holds if we replace Hypothesis (2.4) by the following assumptions:

\[ |f(t, u) - q_0(t)u| \leq \hat{g}_0(t, |u|) \quad \text{for all } |u| \leq \varsigma_0, \ t \in [\xi, \eta] \ a.e., \text{ for some } \varsigma_0 > 0, \]
\[ |f(t, u) - \alpha_{\infty}(t)u^+ + \beta_{\infty}(t)u^-| \leq \hat{g}_{\infty}(t, |u|) \quad \text{for all } u \in \mathbb{R}, \ t \in [\xi, \eta] \ a.e., \]
\[ \lim_{u \to 0} \frac{\hat{g}_0(t, u)}{u} = 0 \quad \text{in } L_{G}^{1}([\xi, \eta]) \text{ for } \rho = 0, +\infty \]

where \( \hat{g}_0, \hat{g}_{\infty} : [\xi, \eta] \times \mathbb{R}^+ \to \mathbb{R}^+ \) are such that for \( \rho = 0, +\infty \), \( \hat{g}_0(t, u) \in L_{G}^{1}([\xi, \eta]) \) for \( u \) fixed and \( \hat{g}_0(t, \cdot) \) is nondecreasing for \( t \in [\xi, \eta] \) a.e. Indeed, we have for \( (u_n) \subset E \) with \( \lim \|u_n\|_{\infty} = 0 \)

\[ \frac{|Hu_n(t)|}{\|u_n\|_{\infty}} \leq \int_{\xi}^{\eta} G(s, s) \left| \frac{f(s, u_n(s)) - q_0(s)u_n(s)}{u_n(s)} \right| ds \]
\[ \leq \int_{\xi}^{\eta} G(s, s) \frac{\hat{g}_0(s, \|u_n\|_{\infty})}{\|u_n\|_{\infty}} ds \to 0 \quad \text{as } n \to +\infty \]
leading to \( Hu = o(\|u\|_{\infty}) \) at 0.

Also, for \( (u_n) \subset E \) with \( \lim \|u_n\|_{\infty} = +\infty \) we have

\[ \frac{|\Omega u_n(t)|}{\|u_n\|_{\infty}} \leq \int_{\xi}^{\eta} G(s, s) \left| \frac{f(s, u_n(s)) - \alpha_{\infty}(s)u^+_n(s) + \beta_{\infty}(s)u^-_n(s)}{u_n(s)} \right| ds \]
\[ \leq \int_{\xi}^{\eta} G(s, s) \frac{\hat{g}_{\infty}(s, \|u_n\|_{\infty})}{\|u_n\|_{\infty}} ds \to 0 \quad \text{as } n \to +\infty \]
leading to \( \Omega u = o(\|u\|_{\infty}) \) at \( \infty \).

The function \( f \) given in Example 2.2 satisfies the above condition with

\[ q_0(t) = At^{-3/2}(1 - t)^{-5/4}, \]
\[ \alpha_{\infty}(t) = At^{-3/2}(1 - t)^{-5/4} + Bt^{-7/6}(1 - t)^{-7/4}, \]
\[ \beta_{\infty}(t) = At^{-3/2}(1 - t)^{-5/4} + Ct^{-11/7}(1 - t)^{-13/10}, \]

and

\[ \hat{g}_0(t, u) = (Bt^{-7/6}(1 - t)^{-7/4} + Ct^{-11/7}(1 - t)^{-13/10})u^3, \]
\[ \hat{g}_{\infty}(t, u) = \begin{cases} 
MCt^{-11/7}(1 - t)^{-13/10} + Bt^{-7/6}(1 - t)^{-7/4}, & \text{for } u > 0, \\
Ct^{-11/7}(1 - t)^{-13/10} + MBt^{-7/6}(1 - t)^{-7/4}, & \text{for } u < 0,
\end{cases} \]

where

\[ M = \sup \left\{ \frac{x^3}{1 + x^2 + e^x} : x \geq 0 \right\}. \]

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