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WEAKLY LOCALLY THERMAL STABILIZATION OF BRESSE SYSTEMS

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Abstract. Fatori and Rivera [7] studied the stability of the Bresse system with one distributed temperature dissipation law operating on the angle displacement equation. They proved that, in general, the energy of the system does not decay exponentially and they established the rate of $t^{-1/3}$. In this article, our goal is to extend their results, by taking into consideration the important case when the thermal dissipation is locally distributed and to improve the polynomial energy decay rate. We then study the energy decay rate of Bresse system with one locally thermal dissipation law. Under the equal speed wave propagation condition, we establish an exponential energy decay rate. On the contrary, we prove that the energy of the system decays, in general, at the rate $t^{-1/2}$.

1. Introduction and statement of main result

In this article, we study the energy decay rate of the Bresse system subject to one locally temperature dissipation law operating on the angle displacement equation. The system is governed by the partial differential equations

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + l\omega)_x - \kappa_0 l(\omega_x - l\varphi) = 0 \quad \text{in } (0,L)\times(0,\infty), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + l\omega) + \alpha(x)\theta_x = 0 \quad \text{in } (0,L)\times(0,\infty), \tag{1.2}$$

$$\rho_1\omega_{tt} - \kappa_0(\omega_x - l\varphi)_x + \kappa l(\varphi_x + \psi + l\omega) = 0 \quad \text{in } (0,L)\times(0,\infty), \tag{1.3}$$

$$\rho_3\theta_t - \theta_{xx} + T_0(\alpha\psi_t)_x = 0 \quad \text{in } (0,L)\times(0,\infty) \tag{1.4}$$

with the boundary conditions

$$\omega_x(t,x) = \varphi(t,x) = \psi_x(t,x) = \theta(t,x) = 0 \quad \text{for } x = 0, L, \tag{1.5}$$

$$\omega(t,x) = \varphi(t,x) = \psi(t,x) = \theta(t,x) = 0 \quad \text{for } x = 0, L, \tag{1.6}$$

and initial conditions

$$\begin{gathered} \omega(0,x) = \omega_0(x), \quad \omega_t(0,x) = \omega_1(x), \quad \psi(0,x) = \psi_0(x), \quad \psi_t(0,x) = \psi_1(x) \\ \varphi(0,x) = \varphi_0(x), \quad \varphi_t(0,x) = \varphi_1(x), \quad \theta(0,x) = \theta_0(x) \end{gathered} \tag{1.7}$$

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where φ, ψ, ω are the vertical, shear angle and longitudinal displacements; θ is the temperature deviation from the reference temperature T_0 along the shear angle displacement and $\alpha \in W^{2,\infty}(0;L)$ is a function verifying the following condition

$$\alpha \geq 0 \text{ on }]0;L[\quad \text{and} \quad \alpha \geq \alpha_0 > 0 \text{ on }]a_0;b_0[\subset]0;L[. \tag{1.8}$$

Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\rho_3 = \rho c$, $\kappa_0 = EA$, $\kappa = \kappa' GA$, $b = EI$ and $l = R^{-1}$ are positive constants for the elastic and thermal material properties. To be more precise, ρ for density, E for the modulus of elasticity, G for the shear modulus, κ' for the shear factor, A for the cross-sectional area, I for the second moment of area of cross-section, R for the radius of the curvature and c for the thermal material property (for more details see Lagnese et al. [9]). The velocities of waves propagations are, respectively, $v_1 = \frac{\kappa}{\rho_1}$, $v_2 = \frac{b}{\rho_2}$, $v_3 = \frac{\kappa_0}{\rho_1}$.

The energy of solutions of the system (1.1)-(1.4) subject to initial state (1.7) to either the boundary conditions (1.5) or (1.6) is defined by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^L \{ &\kappa|\psi + \varphi_x + l\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - l\varphi|^2 + \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 \\ &+ \rho_1|\omega_t|^2 + \frac{\rho_3}{T_0}|\theta|^2\}dx. \end{aligned} \tag{1.9}$$

then a straightforward computation gives

$$\frac{d}{dt}E(t) = -\frac{1}{T_0}\int_0^L |\theta_x|^2 dx \leq 0. \tag{1.10}$$

Then the thermoelastic Bresse system is dissipative in the sense that its energy is non increasing with respect to the time t. Our goal is to study the effect of this dissipation on the Bresse system.

Different types of damping have been introduced to Bresse system and several uniform and polynomial stability results have been obtained. We start by recall some results related to the stabilization of elastic Bresse system. Wehbe and Youssef [18], considered elastic Bresse system subject to two locally internal dissipation laws. They proved that the system is exponentially stable if and only if the wave propagation speeds are equal. Otherwise, only a polynomial stability holds. Alabau-Boussouira et al. [1], considered the same system with one globally distributed dissipation law. The authors proved that, in general, the system is not exponentially stable but there exists polynomial decay with rates that depend on some particular relation between the coefficients. Using boundary conditions of Dirichlet-Dirichlet-Dirichlet type, they proved that the energy of the system decays at a rate $t^{-1/3}$ and at the rate $t^{-\frac{2}{3}}$ if $\kappa = \kappa_0$. These results are completed by Fatori and Montiero [6]. Using boundary conditions of Dirichlet-Neumann-Neumann type, the authors showed that the energy of the elastic Bresse system decays polynomially at the rate $t^{-1/2}$ and at the rate t^{-1}if $\kappa = \kappa_0$. Noun and Wehbe [14] extended the results of [1] and [6]. The authors considered the elastic Bresse system subject to one locally distributed feedback with Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions type. They proved that the exponentially decay rate is preserved when the wave propagation speeds are equal. On the contrary, the authors established a polynomial energy decay with rates that depend on some particular relation between the coefficients and they obtained the rate of $t^{-1/2}$ or t^{-1}. Finally, see [17] for the stabilization of elastic Bresse system with internal indefinite damping and [10] for the stabilization of elastic Bresse system with a nonlinear damping acting in the equation of the shear angle displacement, and nonlinear localized damping in other equations.

For the thermoelastic Bresse system, subject of this paper, there exist two important results. The first result is due to Liu and Rao [12], when they considered the Bresse system with two thermal dissipation laws. The authors showed that the energy decays exponentially when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, they found polynomial decay rates depending on the boundary conditions. When the system is subject to Dirichlet-Neumann-Neumann boundary conditions, they showed that the energy decays at the rate $t^{-1/2}$ and for fully Dirichlet boundary conditions, they proved that the energy of the system decays as $t^{-\frac{1}{4}}$. This result has been recently improved by Fatori and Rivera [7] in the sense that the authors considered only one globally dissipative mechanism given by one temperature, and they established the rate of decay $t^{-1/3}$ for Dirichlet-Neumann- Neumann and Dirichlet-Dirichlet-Dirichlet boundary conditions type. The main result of this paper is to extend the results from [7], by taking into consideration the important case when the thermal dissipation law is locally distributed on the angle displacement equation i.e the damping coefficient α is not constant but it is a positive function in $W^{2,\infty}(0,L)$ and strictly positive in an open subinterval $]a,b[\subset]0,L[$ (the cases $a=0$ or $b=L$ are not excluded) and to improve the polynomial energy decay rate. Then, in this paper, we consider the Bresse system damped by one thermal dissipation law acting locally on the angle displacement equation with Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions types. Under the equal speed wave propagation condition, $\kappa=\kappa_0$ and $\frac{\rho_1}{\rho_2}=\frac{\kappa}{b}$, using a frequency domain approach combining with a piecewise multiplier method, we establish an exponential energy decay rate for usual initial data. On the contrary, in the natural case, when $\kappa\neq\kappa_0$ and $\frac{\rho_1}{\rho_2}\neq\frac{\kappa}{b}$, we establish a new polynomial energy decay rate of type $t^{-1/2}$ for smooth solution. Finally, if $\kappa=\kappa_0$ and $\frac{\rho_1}{\rho_2}\neq\frac{\kappa}{b}$, we establish a new polynomial energy decay rate of type t^{-1} for the smooth solution.

We now outline briefly the content of this paper. In section 2, in a convenable Hilbert space, we formulate system (1.1)-(1.4) with either boundary condition (1.5) or (1.6) into an evolution equation. We recall the well-posedness of the problem by the semigroup approach and by a spectrum method we prove that system (1.1)-(1.4) is strongly stable for usual initial data. In section 3, we consider the particular case when the speed of the three waves are equal and we establish an exponential energy decay rate for usual initial data. In section 4, we consider the natural general case when the speed wave propagations are different two by two and we establish a new polynomial energy decay rate for smooth initial data.

2. Well-posedness and strong stability

In this section we study the existence, uniqueness and the strong stability of the solution of (1.1)-(1.7).

2.1. **The semigroup setting.** We start by study the existence and uniqueness of the solution of the thermoelastic Bresse system. We first, define the following energy spaces

$$\mathcal{H}_1=H_0^1\times(H_*^1)^2\times(L^2)^2\times L_*^2\times L^2 \quad\text{and}\quad \mathcal{H}_2=(H_0^1)^3\times(L^2)^4,$$

where

$$L^2_* = \{f \in L^2(0,L) : \int_0^L f(x)dx = 0\}, \quad H^1_* = \{f \in H^1(0,L) : \int_0^L f(x)dx = 0\}.$$

Both spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are equipped with the inner product which induces the energy norm

$$\begin{aligned} \|U\|^2_{\mathcal{H}_j} &= \kappa\|\varphi_x + \psi + l\omega\|^2 + b\|\psi_x\|^2 + \kappa_0\|\omega_x - l\varphi\|^2 \\ &\quad + \rho_1\|u\|^2 + \rho_2\|v\|^2 + \rho_1\|z\|^2 + \frac{\rho_3}{T_0}\|\theta\|^2. \end{aligned} \tag{2.1}$$

Here and after, $\|\cdot\|$ denotes the $L^2(0,L)$ norm.

Remark 2.1. In the case of boundary condition (1.6), it is easy to see that expression (2.1) define a norm on the energy space $\mathcal{H}_2$. But in the case of boundary condition (1.5) the expression (2.1) define a norm on the energy space $\mathcal{H}_1$ if $L \neq \frac{n\pi}{l}$ for all positive integer n. Then, here and after, we assume that there exist no $n \in \mathbb{N}$ such that $L = \frac{n\pi}{l}$ when $j = 1$.

Next, define a linear unbounded operator $\mathcal{A}_j : D(\mathcal{A}_j) \to \mathcal{H}_j$ by

$$D(\mathcal{A}_1) = \{U \in \mathcal{H}_1 : \varphi, \theta \in H^1_0 \cap H^2,\ \psi, \omega \in H^1_* \cap H^2,\ u, \psi_x, \omega_x \in H^1_0,\ v, z \in H^1_*\} \tag{2.2}$$

$$D(\mathcal{A}_2) = \{U \in \mathcal{H}_2 : \varphi, \psi, \omega, \theta \in H^1_0 \cap H^2,\ u, v, z \in H^1_0\} \tag{2.3}$$

$$\mathcal{A}_j(\varphi, \psi, \omega, u, v, z, \theta) = \begin{pmatrix} u \\ v \\ z \\ \frac{\kappa}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{\kappa_0 l}{\rho_1}(\omega_x - l\varphi) \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi + l\omega) - \frac{1}{\rho_2}\alpha(x)\theta_x \\ \frac{\kappa_0}{\rho_1}(\omega_x - l\varphi)_x - \frac{\kappa l}{\rho_1}(\varphi_x + \psi + l\omega) \\ \frac{1}{\rho_3}\theta_{xx} - \frac{T_0}{\rho_3}(\alpha v)_x \end{pmatrix} \tag{2.4}$$

for all $U = (\varphi, \psi, \omega, u, v, z, \theta) \in D(\mathcal{A}_j)$, $j = 1, 2$. Thus, if $U = (\varphi, \psi, \omega, \varphi_t, \psi_t, \omega_t, \theta)$ is a smooth solution of system (1.1)-(1.7), then the thermoelastic Bresse system is transformed into a first order evolution equation on the Hilbert space $\mathcal{H}_j$:

$$U_t = \mathcal{A}_j U, \quad U(0) = U_0 \tag{2.5}$$

with $j = 1, 2$ corresponding to the boundary conditions (1.6) and (1.7), respectively.

It is easy to see that the operator $\mathcal{A}_j$ is m-dissipative in the energy space $\mathcal{H}_j$, $j = 1, 2$, then we have the following results concerning existence and uniqueness of solution of the problem (2.5) (see [15], [13]).

Theorem 2.2. *The operator $\mathcal{A}_j$ generates a C_0-semigroup e^{tA_j} of contractions on $\mathcal{H}_j$ for $j = 1, 2$. Thus for any initial data $U^0 \in \mathcal{H}_j$, the problem* (2.5) *has a unique weak solution $U \in \mathcal{C}^0([0,\infty), \mathcal{H}_j)$. Moreover, if $U^0 \in D(\mathcal{A}_j)$, then U is a strong solution of* (2.5)*, i. e $U \in \mathcal{C}^1([0,\infty), \mathcal{H}_j) \cap \mathcal{C}^0([0,\infty), D(\mathcal{A}_j))$.*

2.2. **Strong stability.** In this part, using a spectrum method, we will prove the strong stability of the C_0-semigroup $e^{t\mathcal{A}_j}$.

Theorem 2.3. *The semigroup e^{tA_j} is strongly stable in the energy space $\mathcal{H}_j$. In other words*

$$\lim_{t\to+\infty} \|e^{tA_j}U_0\|_{\mathcal{H}_j} = 0 \quad j = 1,\ 2,\ \forall U_0 \in \mathcal{H}_j. \tag{2.6}$$

Proof. Since the resolvent of $\mathcal{A}_j$ is compact in $\mathcal{H}_j$, $j = 1, 2$, then using a result due to Benchimol [3], the system (1.1)-(1.4) is strongly stable if and only if $\mathcal{A}_j$ does not have pure imaginary eigenvalues. By contradiction argument, let $0 \neq U = (\varphi, \psi, \omega, u, v, z, \theta) \in D(\mathcal{A}_j)$, $i\lambda \in i\mathbb{R}$, such that

$$\mathcal{A}_j U = i\lambda U.$$

Our goal is to find a contradiction by proving that $U = 0$. Taking the real part of the inner product in $\mathcal{H}_j$ of $\mathcal{A}_j U$ and U, we obtain

$$0 = Re(i\lambda \|U\|^2_{\mathcal{H}_j}) = Re((\mathcal{A}_j U, U)_{\mathcal{H}_j}) = -\frac{1}{T_0} \int_0^L |\theta_x|^2 dx.$$

It follows that

$$\theta = \theta_x = 0 \quad \text{a.e. in } (0, L).$$

Now, detailing the equation $\mathcal{A}_j U = i\lambda U$, and using the fact that $\theta = 0$, we obtain

$$u = i\lambda\varphi, \tag{2.7}$$
$$v = i\lambda\psi, \tag{2.8}$$
$$z = i\lambda\omega, \tag{2.9}$$
$$\frac{\kappa}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{\kappa_0 l}{\rho_1}(\omega_x - l\varphi) = i\lambda u, \tag{2.10}$$
$$\frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi + l\omega) = i\lambda v, \tag{2.11}$$
$$\frac{\kappa_0}{\rho_1}(\omega_x - l\varphi)_x - \frac{\kappa l}{\rho_1}(\varphi_x + \psi + l\omega) = i\lambda z, \tag{2.12}$$
$$(\alpha v)_x = 0. \tag{2.13}$$

If $\lambda = 0$, then $u = v = z = 0$ and using Lax-Milgram theorem (see [5]), it is clear to see that the system (2.10)-(2.12) has the unique trivial solution $\varphi = \psi = \omega = 0$. This implies that $U = 0$ and the desired contradiction is proved.

Now, assume that $\lambda \neq 0$. Then let $\xi(x) = \int_0^x v(s)ds$, multiply (2.13) by $-\overline{\xi(x)}$, and integrate by parts, to obtain

$$\int_0^L \alpha |v|^2 dx - \alpha(L)v(L) \int_0^L v(s)ds = 0.$$

In the case of Dirichlet-Neumann-Neumann conditions, we have $v \in H^1_*(0, L)$ then $\int_0^L v(s)ds = 0$, and in the case of Dirichlet- Dirichlet-Dirichlet conditions, we have $v \in H^1_0(0, L)$ then $v(L) = 0$. This together with condition (1.8), implies that

$$\sqrt{\alpha} v = 0 \text{ a.e. in } (0, L) \quad \text{and} \quad v = 0 \text{ a.e. in } (a_0, b_0). \tag{2.14}$$

Now, combining equations (2.8), (2.11) and (2.14), we obtain

$$\psi = 0 \quad \text{and} \quad \varphi_x + l\omega = 0 \quad \text{a.e. in } (a_0, b_0). \tag{2.15}$$

Combining equations (2.7), (2.10) and (2.15), we obtain

$$\rho_1 \lambda^2 \varphi + \kappa_0 l(\omega_x - l\varphi) = 0, \quad \text{a.e. in } (a_0, b_0). \tag{2.16}$$

Similarly, combining equations (2.9), (2.12) and (2.15), we obtain

$$\rho_1 \lambda^2 \omega + \kappa_0 (\omega_x - l\varphi)_x = 0, \quad \text{a.e. in } (a_0, b_0). \tag{2.17}$$

By a direct calculation we deduce that system (2.15)-(2.17) has the solution

$$\varphi = c, \quad \psi = 0, \quad \omega = 0, \quad \text{a.e. in } (a_0, b_0).$$

Then, from (2.16) we deduce that

$$(\lambda^2 \rho_1 - \kappa_0 l^2)\varphi = 0, \quad \text{a.e. in } (a_0, b_0).$$

We have then two cases to discuss: $\lambda = l\sqrt{\frac{\kappa_0}{\rho_1}}$, and $\lambda \neq l\sqrt{\frac{\kappa_0}{\rho_1}}$.

Case 1. Suppose that $\lambda \neq l\sqrt{\frac{\kappa_0}{\rho_1}}$, then

$$\varphi = 0 \quad \text{a.e. in } (a_0, b_0).$$

Let $X = (\varphi, \varphi_x, \psi, \psi_x, \omega, \omega_x)^T$ and

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-\rho_1}{\kappa}\lambda^2 + \frac{\kappa_0}{\kappa}l^2 & 0 & 0 & -1 & 0 & -l - \frac{\kappa_0}{\kappa}l \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\kappa}{b} & \frac{-\rho_2}{b}\lambda^2 + \frac{\kappa}{b} & 0 & \frac{\kappa}{b}l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & l + \frac{\kappa}{\kappa_0}l & \frac{\kappa}{\kappa_0}l & 0 & \frac{-\rho_1}{\kappa_0}\lambda^2 + \frac{\kappa}{\kappa_0}l^2 & 0 \end{pmatrix}.$$

Then system (2.10)-(2.12) can be written as

$$\begin{gathered} X' = MX, \quad \text{in } (0, a_0), \\ X(a_0) = 0. \end{gathered} \tag{2.18}$$

Using ordinary differential equation theory, we deduce that system (2.18) has the unique trivial solution $X = 0$ in $(0, a_0)$ and $\varphi = \psi = \omega = 0$ a.e in $(0, a_0)$. Same argument as above leads us to prove that $\varphi = \psi = \omega = 0$ a.e. in (b_0, L) and therefore $U = 0$.

Case 2. Suppose that $\lambda = l\sqrt{\frac{\kappa_0}{\rho_1}}$. Then (2.10) can be rewritten as

$$\kappa(\varphi_x + \psi + l\omega)_x + \frac{\kappa_0 l}{\kappa}\omega_x = 0 \quad \text{a.e. in } (0, a_0). \tag{2.19}$$

Let $X = (\varphi_x, \psi, \psi_x, \omega, \omega_x)^T$ and

$$M = \begin{pmatrix} 0 & 0 & -1 & 0 & -l - \frac{\kappa_0}{\kappa}l \\ 0 & 0 & 1 & 0 & 0 \\ \frac{\kappa}{b} & \frac{-\rho_2}{b}\lambda^2 + \frac{\kappa}{b} & 0 & \frac{\kappa}{b}l & 0 \\ 0 & 0 & 0 & 0 & 1 \\ l + \frac{\kappa}{\kappa_0}l & \frac{\kappa}{\kappa_0}l & 0 & \frac{-\rho_1}{\kappa_0}\lambda^2 + \frac{\kappa}{\kappa_0}l^2 & 0 \end{pmatrix}$$

Then system (2.10)-(2.12) could be given as

$$\begin{gathered} X' = MX, \quad \text{in } (0, a_0), \\ X(a_0) = 0. \end{gathered} \tag{2.20}$$

Using ordinary differential equation theory, we deduce that system (2.20) has the unique trivial solution $X = 0$ in $(0, a_0)$. This implies that $\varphi = c$, $\psi = 0$ and $\omega = 0$ a.e in $(0, a_0)$. Since $\varphi \in H^2(0, L) \subset C^1([0, L])$ and $\varphi(0) = 0$, we conclude that $\varphi = 0$ a.e in $(0, a_0)$. Same argument as above leads us to prove that $\varphi = \psi = \omega = 0$ a.e in (b_0, L) and therefore $U = 0$. The proof is complete. $\square$

3. Exponential Stability, in the case $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$

In this section, we consider system (1.1)-(1.4) under the equal speed propagation conditions i.e. $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$. We prove the following exponential stability result.

Theorem 3.1. *If $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ then the semigroup $e^{t\mathcal{A}_j}$ is exponentially stable, i.e., there exist constant $M \geq 1$, and $\epsilon > 0$ independent of U_0 such that*

$$\|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} \leq M e^{-\epsilon t} \|U_0\|_{\mathcal{H}_j}, \quad t \geq 0,\ j = 1, 2. \tag{3.1}$$

For this aim, we will use the frequency domain method. More precisely, using Huang [8] and Pruss [16], inequality (3.1) hold if and only if the following two conditions are satisfied:

(H1) $i\mathbb{R} \subset \rho(\mathcal{A}_j)$,
(H2) $\sup_{\lambda \in \mathbb{R}} \|(i\lambda - \mathcal{A}_j)^{-1}\| = O(1)$.

We first check condition (H1). Since $(I - \mathcal{A}_j)^{-1}$ is compact and $\mathcal{A}_j$ has no pure imaginary eigenvalues (Theorem 2.3), we deduce that condition (H1) is true. We will prove condition (H2) by contradiction argument. Suppose that there exist a sequence $\lambda_n \in \mathbb{R}$ and a sequence $U^n = (\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta^n) \in D(\mathcal{A}_j)$, verifying the following conditions

$$|\lambda_n| \to +\infty, \tag{3.2}$$

$$\|U^n\|_{\mathcal{H}_j} = 1, \tag{3.3}$$

$$(i\lambda_n I - \mathcal{A}_j) U^n = (f_1^n, f_2^n, f_3^n, g_1^n, g_2^n, g_3^n, g_4^n) \to 0 \quad \text{in } \mathcal{H}_j,\ j = 1,\ 2. \tag{3.4}$$

Equation (3.4) can be written as

$$i\lambda_n \varphi^n - u^n = f_1^n \tag{3.5}$$

$$i\lambda_n \psi^n - v^n = f_2^n \tag{3.6}$$

$$i\lambda_n \omega^n - z^n = f_3^n \tag{3.7}$$

$$\lambda_n^2 \varphi^n + \frac{\kappa}{\rho_1}(\varphi_{xx}^n + \psi_x^n + l\omega_x^n) + \frac{\kappa_0 l}{\rho_1}(\omega_x^n - l\varphi^n) = -g_1^n - i\lambda_n f_1^n, \tag{3.8}$$

$$\lambda_n^2 \psi^n + \frac{b}{\rho_2}\psi_{xx}^n - \frac{\kappa}{\rho_2}(\varphi_x^n + \psi^n + l\omega^n) - \frac{1}{\rho_2}\alpha(x)\theta_x^n = -g_2^n - i\lambda_n f_2^n, \tag{3.9}$$

$$\lambda_n^2 \omega^n + \frac{\kappa_0}{\rho_1}(\omega_{xx}^n - l\varphi_x^n) - \frac{\kappa l}{\rho_1}(\varphi_x^n + \psi^n + l\omega^n) = -g_3^n - i\lambda_n f_3^n \tag{3.10}$$

$$i\lambda_n \theta^n - \frac{1}{\rho_3}\theta_{xx}^n + i\frac{T_0}{\rho_3}\lambda_n (\alpha\psi^n)_x = g_4^n + T_0 \rho_3^{-1} (\alpha f_2^n)_x. \tag{3.11}$$

Our goal is, using a multiplier method, to prove that $\|U\|_{\mathcal{H}_j} = o(1)$. This contradicts equation (3.3). We will establish the proof by several Lemmas. For simplicity, here and after we drop the index n.

Consider the function $\eta \in C^1([0, L])$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $[a_0 + \varepsilon, b_0 - \varepsilon]$ and $\eta = 0$ on $[0, a_0] \cup [b_0, L]$, where $0 < a_0 + \varepsilon < b_0 - \varepsilon < L$. We have the first information.

Lemma 3.2. *With the above notation, we have*

$$\|\psi_x\| = O(1), \quad \|\psi\| = \frac{O(1)}{\lambda}, \quad \|\eta\psi_{xx}\| = O(\lambda). \tag{3.12}$$

The proof of the above lemma follows from equations (3.5), (3.6), (3.7) and (3.9), which lead to equations (3.12).

Lemma 3.3 (Dissipation). *With the above notation, we have*

$$\int_0^L |\theta_x|^2 dx = o(1), \quad \int_0^L |\theta|^2 dx = o(1). \tag{3.13}$$

Proof. Multiplying (3.7) by the uniformly bounded sequence $U = (\varphi, \psi, \omega, u, v, z, \theta)$, we obtain

$$\int_0^L |\theta_x|^2 dx = -\operatorname{Re}((i\lambda - \mathcal{A}_j)U, U)_{\mathcal{H}_j} = o(1). \tag{3.14}$$

Finally, using Poincaré inequality, it follows the second asymptotic equality. The proof is complete. □

Lemma 3.4. *With the above notation, if* $\|U\| = o(1)$ *on* $]a_1; b_1[\subset]0, L[$*, then* $\|U\| = o(1)$ *on* $]0; L[$.

Proof. Let $h \in H_0^1(0; L)$ be a given function.

(i) Multiply equation (3.8) by $2\rho_1 h\overline{\varphi_x}$ and integrate over $[0; L]$, we obtain

$$\begin{aligned} &-\rho_1 \int_0^L h'|\lambda\varphi|^2 + \rho_1[h|\lambda\varphi|^2]_0^L - \kappa \int_0^L h'|\varphi_x|^2 + \kappa[h|\varphi_x|^2]_0^L \\ &+ 2\operatorname{Re}\Big\{\kappa \int_0^L h\psi_x\overline{\varphi_x} + l(\kappa+\kappa_0)\int_0^L h\omega_x\overline{\varphi_x} - \kappa_0 l^2 \int_0^L h\varphi\overline{\varphi_x}\Big\} \\ &= 2\rho_1 \operatorname{Re}\Big\{\int_0^h g_1\overline{\varphi_x} + i\int_0^L (f_{1x}h + f_1h')\lambda\overline{\varphi} - i\lambda[f_1h\overline{\varphi}]_0^L\Big\}. \end{aligned} \tag{3.15}$$

Using (3.3) and (3.5), we deduce that $\|\varphi\| = \frac{O(1)}{\lambda}$ and $\|\varphi_x\| = O(1)$. Then using the fact that $\varphi(0) = \varphi(L) = 0$, $h(0) = h(L) = 0$, $\|g_1\| = o(1)$, $\|f_1\| = o(1)$ and $\|f_{1x}\| = o(1)$ in (3.15), we obtain

$$\begin{aligned} &-\rho_1 \int_0^L h'|\lambda\varphi|^2 - \kappa\int_0^L h'|\varphi_x|^2 + 2\operatorname{Re}\Big\{\kappa\int_0^L h\psi_x\overline{\varphi_x} + l(\kappa+\kappa_0)\int_0^L h\omega_x\overline{\varphi_x}\Big\} \\ &= o(1). \end{aligned} \tag{3.16}$$

(ii) Multiply (3.9) by $2\rho_2 h\overline{\psi_x}$ and integrate over $[0; L]$, we obtain

$$\begin{aligned} &-\rho_2 \int_0^L h'|\lambda\psi|^2 + \rho_2[h|\lambda\psi|^2]_0^L - b\int_0^L h'|\psi_x|^2 + b[h|\psi_x|^2]_0^L \\ &- 2\operatorname{Re}\Big\{\kappa\int_0^L h\varphi_x\overline{\psi_x} + \kappa\int_0^L h\psi\overline{\psi_x} + \kappa l\int_0^L h\omega\overline{\psi_x} + \int_0^L h\alpha(x)\theta_x\overline{\psi_x}\Big\} \\ &= 2\rho_2\operatorname{Re}\Big\{-\int_0^L hg_2\overline{\psi_x} + i\int_0^L (f_{2x}h + f_2h')\lambda\overline{\psi} - i\lambda[f_2h\overline{\psi}]_0^L\Big\}. \end{aligned} \tag{3.17}$$

Using (3.3), (3.6) and (3.7) we deduce that $\|\psi\| = \frac{O(1)}{\lambda}$, $\|\omega\| = \frac{O(1)}{\lambda}$ and $\|\psi_x\| = O(1)$. Then using the fact that $h(0) = h(L) = 0$, $\|\theta_x\| = o(1)$, $\|g_2\| = o(1)$, $\|f_2\| = o(1)$ and $\|f_{2x}\| = o(1)$ in (3.17), we obtain

$$-\rho_2\int_0^L h'|\lambda\psi|^2 - b\int_0^L h'|\psi_x|^2 - 2\kappa\operatorname{Re}\Big\{\int_0^L h\varphi_x\overline{\psi_x}\Big\} = o(1). \tag{3.18}$$

(iii) Similarly, multiply (3.10) by $2\rho_1 h\overline{\omega_x}$ and integrate over $[0;L]$, we obtain

$$
\begin{aligned}
&-\rho_1\int_0^L h'|\lambda\omega|^2+\rho_1[h|\lambda\omega|^2]_0^L-\kappa_0\int_0^L h'|\omega_x|^2+\kappa_0[h|\omega_x|^2]_0^L\\
&-2l\operatorname{Re}\Big\{\kappa_0\int_0^L h\varphi_x\overline{\omega_x}+\kappa\int_0^L h\varphi_x\overline{\omega_x}+\kappa\int_0^L h(\psi+l\omega)\overline{\omega_x}\Big\}\\
&=2\rho_1\operatorname{Re}\Big\{-\int_0^L hg_3\overline{\omega_x}+i\int_0^L (f_{3x}h+f_3h')\lambda\overline{\omega}-i\lambda[f_3h\overline{\omega}]_0^L\Big\}.
\end{aligned}
\tag{3.19}
$$

By a similar way as in (i) and (ii), it follows that

$$
-\rho_1\int_0^L h'|\lambda\omega|^2-\kappa_0\int_0^L h'|\omega_x|^2-2l(\kappa+\kappa_0)\operatorname{Re}\Big\{\int_0^L h\varphi_x\overline{\omega_x}\Big\}=o(1). \tag{3.20}
$$

(iv) Adding (3.16), (3.18) and (3.20), we obtain

$$
\begin{aligned}
&-\rho_1\int_0^L h'|\lambda\varphi|^2-\kappa\int_0^L h'|\varphi_x|^2-\rho_2\int_0^L h'|\lambda\psi|^2\\
&-b\int_0^L h'|\psi_x|^2-\rho_1\int_0^L h'|\lambda\omega|^2-\kappa_0\int_0^L h'|\omega_x|^2=o(1).
\end{aligned}
\tag{3.21}
$$

(v) Let $\varepsilon>0$ such that $a_1+\varepsilon<b_1$ and define the function $\widehat{\eta}$ in $C^1([0;L])$ by

$$
0\le\widehat{\eta}\le 1,\quad \widehat{\eta}=1 \text{ on } [0;a_1] \quad\text{and}\quad \widehat{\eta}=0 \text{ on } [a_1+\varepsilon;L]
$$

Then take $h=x\widehat{\eta}$ in (3.21) and using the fact that $\|U\|_{\mathcal{H}_j}=o(1)$ on $]a_1,b_1[$, we obtain

$$
\begin{aligned}
&-\rho_1\int_0^{a_1}|\lambda\varphi|^2-\kappa\int_0^{a_1}|\varphi_x|^2-\rho_2\int_0^{a_1}|\lambda\psi|^2\\
&-b\int_0^{a_1}|\psi_x|^2-\rho_1\int_0^{a_1}|\lambda\omega|^2-\kappa_0\int_0^{a_1}|\omega_x|^2=o(1).
\end{aligned}
\tag{3.22}
$$

It follows that $\|U\|_{\mathcal{H}_j}=o(1)$ on $]0,a_1[$.

(vi) Let $\varepsilon>0$ such that $b_1-\varepsilon>a_1$ and define the function $\widetilde{\eta}$ in $C^1([0;L])$ by

$$
0\le\widetilde{\eta}\le 1,\quad \widetilde{\eta}=1 \text{ on } [b_1,L] \quad\text{and}\quad \widetilde{\eta}=0 \text{ on } [0,b_1-\varepsilon].
$$

Then, as in (v), take $h=(x-L)\widetilde{\eta}$ in (3.21) and using the fact that $\|U\|_{\mathcal{H}_j}=o(1)$ on $]a_1,b_1[$, we obtain

$$
\|U\|_{\mathcal{H}_j}=o(1) \quad \text{on }]b_1,L[.
$$

The proof is complete. $\square$

Now we have information on ψ and ψ_x.

Lemma 3.5. *With the above notation, we have*

$$
\int_0^L \eta|\psi|^2=\frac{o(1)}{\lambda^2},\quad \int_0^L \eta|\psi_x|^2=o(1). \tag{3.23}
$$

Proof. First, multiplying (3.11) by $\eta\bar{\psi}_x$, we obtain

$$
\begin{aligned}
T_0\int_0^L \eta\alpha|\psi_x|^2&=\frac{T_0}{2}\int_0^L(\eta\alpha')'|\psi|^2+\operatorname{Re}\Big\{\rho_3\int_0^L(\eta'\theta+\eta\theta_x)\bar{\psi}\\
&\quad+i\int_0^L\theta_x\lambda^{-1}\eta\bar{\psi}_{xx}+\frac{i}{\lambda}\int_0^L\eta'\theta_x\bar{\psi}_x\Big\}+\frac{o(1)}{\lambda}.
\end{aligned}
\tag{3.24}
$$

Using (3.13) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$, $\|\psi_x\| = O(1)$ and $\|\eta\psi_{xx}\| = O(\lambda)$ in (3.24), we obtain

$$\int_0^L \eta|\psi_x|^2 = o(1). \tag{3.25}$$

Next, multiplying (3.9) by $\eta\bar{\psi}$, we obtain

$$\begin{aligned}\rho_2\int_0^L \eta|\lambda\psi|^2 = b\int_0^L \eta|\psi_x|^2 + b\int_0^L \eta'\psi_x\bar{\psi} + \int_0^1 [\kappa(\psi + l\omega) + \alpha\theta_x]\eta\bar{\psi}\\ - \int_0^1 \kappa(\eta'\varphi\psi + \eta\varphi\psi_x) + o(1).\end{aligned} \tag{3.26}$$

Using (3.13), (3.25) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$ and $\|\omega\| = \frac{O(1)}{\lambda}$ in equation (3.26), we obtain

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^2}. \tag{3.27}$$

□

Now we have information on φ and φ_x.

Lemma 3.6. *With the above notation, if $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$, then*

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\varphi_x|^2 = o(1). \tag{3.28}$$

Proof. (i) First, multiplying (3.8) by $\eta\overline{\psi_x}$ and integrating over $]0, L[$, we obtain

$$\begin{aligned}&\int_0^L \eta\lambda^2\varphi\overline{\psi_x} + \frac{\kappa}{\rho_1}\int_0^L \eta\varphi_{xx}\overline{\psi_x} + \frac{\kappa}{\rho_1}\int_0^L \eta|\psi_x|^2 + \frac{\kappa l}{\rho_1}\int_0^L \eta\omega_x\overline{\psi_x}\\ &+ \frac{\kappa_0 l}{\rho_1}\int_0^L (\omega_x - l\varphi)\eta\overline{\psi_x}\\ &= \int_0^L (-g_1\eta\overline{\psi_x} + i\lambda f_{1x}\eta\overline{\psi} + i\lambda f_1\eta'\overline{\psi}) - [i\lambda f_1\eta\overline{\psi}]_0^L.\end{aligned} \tag{3.29}$$

From (3.3), (3.5) and (3.6) it is clear to see that sequences ω_x, $(\omega_x - l\varphi)$, $\lambda\psi$ are uniformly bounded in $L^2(0, L)$. Then using Lemma 3.5 and the fact that $\|f_1\| = o(1)$, $\|f_{1x}\| = o(1)$, $\|g_1\| = o(1)$, and that $f_1(0) = f_1(L) = 0$, we obtain that

$$-\int_0^L \eta\lambda^2\varphi\overline{\psi_x} - \frac{\kappa}{\rho_1}\int_0^L \eta\varphi_{xx}\overline{\psi_x} = o(1). \tag{3.30}$$

(ii) Multiply (3.9) by $\eta\overline{\varphi_x}$ and integrate over $]0, L[$, we obtain

$$\begin{aligned}&-\int_0^L \lambda^2\psi_x\eta\overline{\varphi} - \int_0^L \lambda^2\psi\eta'\overline{\varphi} + [\lambda^2\psi\eta\overline{\varphi}]_0^L - \frac{b}{\rho_2}\int_0^L \psi_x\eta\overline{\varphi_{xx}}\\ &- \frac{b}{\rho_2}\int_0^L \psi_x\eta'\overline{\varphi_x} + \frac{b}{\rho_2}[\psi_x\eta\varphi_x]_0^L - \frac{\kappa}{\rho I}\int_0^L \eta|\varphi_x|^2\\ &+ \frac{Gh}{\rho_2}\int_0^L (\psi + l\omega)\eta\overline{\varphi_x} + \frac{1}{\rho_2}\int_0^L \eta\alpha(x)\theta_x\overline{\varphi_x}\\ &= \int_0^L (-g_2\eta\overline{\varphi_x} + i\lambda f_{2x}\eta\overline{\varphi} + i\lambda f_2\eta'\overline{\varphi}) - [i\lambda f_2\eta\overline{\varphi}]_0^L.\end{aligned} \tag{3.31}$$

Using Lemma 3.5 and the fact that the sequences $\lambda\varphi$, φ_x, $\alpha(x)\varphi_x$ are uniformly bounded in $L^2(0,L)$, we obtain

$$\int_0^L \lambda^2\psi_x\eta\overline{\varphi} + \frac{b}{\rho_2}\int_0^L \psi_x\eta\overline{\varphi_{xx}} + \frac{\kappa}{\rho_2}\int_0^L \eta|\varphi_x|^2 = o(1). \tag{3.32}$$

(iii) Adding the real parts of (3.30) and (3.32) and using the condition $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ we obtain

$$\int_0^L \eta|\varphi_x|^2 = o(1) \tag{3.33}$$

Multiplying (3.8) by $\eta\overline{\varphi}$ and integrating over $]0,L[$, we obtain

$$\begin{aligned}\rho_1\int_0^L \eta|\lambda\varphi|^2 = \kappa\int_0^L \eta|\varphi_x|^2 + \kappa\int_0^L \eta'\varphi_x\overline{\varphi} - \kappa\int_0^L (\psi_x + l\omega_x)\eta\overline{\varphi}\\ - \kappa_0 l\int_0^L (\omega_x - l\varphi)\eta\overline{\varphi} + o(1).\end{aligned} \tag{3.34}$$

Using (3.33), (3.25), the fact that $\|\varphi\| = \frac{O(1)}{\lambda}$ and the sequences φ_x, $(\psi_x - l\omega_x)$, $(\omega_x - l\varphi)$ are uniformly bounded in $L^2(0,L)$ in (3.34), we obtain

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^2}. \tag{3.35}$$

The proof is complete. □

Now we have information on ω and ω_x.

Lemma 3.7. *With the above notation, if $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$, then*

$$\int_0^L \eta|\omega|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\omega_x|^2 = o(1). \tag{3.36}$$

Proof. (i) First, multiply (3.8) by $\rho_1\eta\overline{\omega_x}$ and integrate over $]0,L[$, to obtain

$$\begin{aligned}&-\rho_1\int_0^L \lambda^2\eta\varphi_x\overline{\omega} - \kappa\int_0^L \varphi_x\eta\overline{\omega_{xx}} - \kappa\int_0^L \varphi_x\eta'\overline{\omega_x}\\ &+\kappa\int_0^L \psi_x\eta\overline{\omega_x} + (\kappa+\kappa_0)l\int_0^L \eta|\omega_x|^2 - \kappa_0 l^2\int_0^L \varphi\eta\overline{\omega_x} = o(1)\end{aligned} \tag{3.37}$$

Using Lemmas 3.5 and 3.6 and the fact that $\|\omega_x\| = O(1)$ in (3.37), we obtain

$$-\rho_1\int_0^L \lambda^2\eta\varphi_x\overline{\omega} + (\kappa+\kappa_0)l\int_0^L \eta|\omega_x|^2 - \kappa\int_0^L \varphi_x\eta\overline{\omega_{xx}} = o(1). \tag{3.38}$$

(ii) Next, multiplying (3.10) by $\rho_1\eta\overline{\varphi_x}$ and integrating over $]0,L[$, we obtain

$$\rho_1\int_0^L \lambda^2\eta\omega\overline{\varphi_x} + \kappa_0\int_0^L \eta\omega_{xx}\overline{\varphi_x} - (\kappa+\kappa_0)l\int_0^L \eta|\varphi_x|^2 - \kappa l\int_0^L (\psi + l\omega)\eta\overline{\varphi_x} = o(1). \tag{3.39}$$

Using Lemmas 3.5 and 3.6, and the fact that $\|\omega\| = \frac{O(1)}{\lambda}$ in (3.39), we obtain

$$\rho_1\int_0^L \lambda^2\eta\omega\overline{\varphi_x} + \kappa_0\int_0^L \eta\omega_{xx}\overline{\varphi_x} = o(1). \tag{3.40}$$

(iii) Adding the real parts of equations (3.38) and (3.40), and using the fact that $\kappa = \kappa_0$, we deduce that

$$\int_0^L \eta|\omega_x|^2 = o(1) \tag{3.41}$$

Finally, as in (iii), Lemma 3.6, multiplying (3.10) by $\eta\bar{\omega}$, we deduce the first asymptotic behavior equation in (3.36). The proof is complete. □

Proof of Theorem 3.1. Using Lemmas 3.3, 3.5, 3.6 and 3.7, we deduce that $\|U\|_{\mathcal{H}_j} = o(1)$ on the subinterval $[a_0; b_0]$. Then using Lemma 3.4 we deduce that $\|U\| = o(1)$ on the interval $[0; L]$, this contradicts equality (3.3). We deduce that the resolvent of the operator $\mathcal{A}_j$ is uniformly bounded on the imaginary axis $i\mathbb{R}$. This together with the fact that $i\mathbb{R} \subset \rho(\mathcal{A}_j)$ implies, under the equal speed propagation conditions, the exponential stability of system (1.1)-(1.4) with either boundary Dirichlet-Dirichlet- Dirichlet or Dirichlet-Neumann-Neumann conditions types. The proof is complete. □

Remark 3.8. From the theory of elasticity, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa_0 = EA$, $\kappa = \kappa' GA$, and $b = EI$, where ρ for density, E denotes the Young's modulus of elasticity, G for the shear modulus, κ' for the shear factor, A for the cross-sectional area and I for the second moment of area of cross-section. Then the equal speed propagation conditions $\kappa = \kappa_0$ or $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ are equivalent to $\kappa' G = E$. But the two elastic modulus are not equal since $\kappa' G = \frac{E}{2(1+\mu)}$ where $\mu \in (0, 1/2)$ is the Poisson's ratio. Thus, the exponential stability is only mathematically sound.

4. Polynomial stability in the general case

The thermoelastic Bresse system (1.1)-(1.4) with the boundary condition (1.5) is not exponentially stable when $\kappa \neq \kappa_0$ or $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$ (see [18], [7], [1]). The idea is to find a real sequence (λ_n) with $|\lambda_n| \to \infty$ and a sequence U^n of elements of $D(\mathcal{A}_1)$ with $\|U^n\| = 1$ such that $\|(i\lambda_n - \mathcal{A}_1)U^n\| = o(1)$. Then the resolvent of the operator $\mathcal{A}_1$ is not uniformly bounded on the imaginary axes and the system is not exponentially stable (see [8], [16]). Our main results are the following polynomial-type decay rate.

Theorem 4.1. *Assume that $\kappa \neq \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$. Then there exists a constant $C > 0$ such that for every initial data $U_0 = (\varphi_0, \psi_0, \omega_0, \varphi_1, \psi_1, \omega_1, \theta_0) \in D(\mathcal{A}_j)$, $j = 1, 2$, the energy of system* (1.1)-(1.4) *with boundary conditions* (1.5) *or* (1.6) *verify the following estimation:*

$$E(t) \leq C \frac{1}{\sqrt{t}} \|U_0\|^2_{D(\mathcal{A}_j)} \quad \forall t > 0. \tag{4.1}$$

Following Borichev and Tomilov [4], (see also [11], [2]), a C_0 semigroup of contractions $e^{t\mathcal{A}_j}$ on a Hilbert space $\mathcal{H}_j$ verify (4.1) if (H1) and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^4} \|(i\lambda I - \mathcal{A}_j)^{-1}\| < +\infty \tag{4.2}$$

are satisfied. Condition (H1) was already proved in Theorems 2.3 and 3.1. Our goal is to prove that $\|(i\lambda - \mathcal{A}_j)^{-1}\| = O(|\lambda^4|)$. By contradiction argument, suppose that there exist a sequence $\lambda_n \in \mathbb{R}$ and a sequence $U^n = (\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta^n) \in D(\mathcal{A}_j)$, verifying the following conditions:

$$|\lambda_n| \to +\infty, \quad \|U^n\| = \|(\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta^n)\|_{\mathcal{H}_j} = 1, \tag{4.3}$$

$$\lambda_n^4 (i\lambda_n I - \mathcal{A}_j) U^n = (f_1^n, f_2^n, f_3^n, g_1^n, g_2^n, g_3^n, g_4^n) \to 0 \quad \text{in } \mathcal{H}_j,\ j = 1, 2. \tag{4.4}$$

Equation (4.4) can be written as

$$i\lambda_n \varphi^n - u^n = \frac{f_1^n}{\lambda_n^4} \tag{4.5}$$

$$i\lambda_n \psi^n - v^n = \frac{f_2^n}{\lambda_n^4} \tag{4.6}$$

$$i\lambda_n \omega^n - z^n = \frac{f_3^n}{\lambda_n^4} \tag{4.7}$$

$$\lambda_n^2 \varphi^n + \frac{\kappa}{\rho_1}(\varphi_{xx}^n + \psi_x^n + l\omega_x^n) + \frac{\kappa_0 l}{\rho_1}(\omega_x^n - l\varphi^n) = -\frac{g_1^n + i\lambda_n f_1^n}{\lambda_n^4}, \tag{4.8}$$

$$\lambda_n^2 \psi^n + \frac{b}{\rho_2}\psi_{xx}^n - \frac{\kappa}{\rho_2}(\varphi_x^n + \psi^n + l\omega^n) - \frac{1}{\rho_2}\alpha(x)\theta_x^n = -\frac{g_2^n + i\lambda_n f_2^n}{\lambda_n^4}, \tag{4.9}$$

$$\lambda_n^2 \omega^n + \frac{\kappa_0}{\rho_1}(\omega_{xx}^n - l\varphi_x^n) - \frac{\kappa l}{\rho_1}(\varphi_x^n + \psi^n + l\omega^n) = -\frac{g_3^n + i\lambda_n f_3^n}{\lambda_n^4}, \tag{4.10}$$

$$i\lambda_n \theta^n - \frac{1}{\rho_3}\theta_{xx}^n + i\frac{T_0}{\rho_3}\lambda_n(\alpha\psi^n)_x = \frac{g_4^n + T_0\rho_3^{-1}(\alpha f_2^n)_x}{\lambda_n^4}. \tag{4.11}$$

Our goal is, using a multiplier method, to prove that $\|U^n\|_{\mathcal{H}_j} = o(1)$, this contradicts equation (4.3). We will establish the proof by several Lemmas. For simplicity, here and after we drop the index n.

Using (4.3), (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) we deduce that

$$\begin{gathered}
\|\varphi_x\| = O(1), \quad \|\varphi\| = \frac{O(1)}{\lambda}, \quad \|\varphi_{xx}\| = O(\lambda), \\
\|\psi_x\| = O(1), \quad \|\psi\| = \frac{O(1)}{\lambda}, \quad \|\psi_{xx}\| = O(\lambda), \\
\|\omega_x\| = O(1), \quad \|\omega\| = \frac{O(1)}{\lambda}, \quad \|\omega_{xx}\| = O(\lambda).
\end{gathered}$$

Lemma 4.2 (Dissipation). *With the above notation, we have*

$$\int_0^L |\theta_x|^2 dx = \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_0^L |\theta|^2 dx = \frac{o(1)}{\lambda^4}. \tag{4.12}$$

Proof. Multiplying (4.4) by the uniformly bounded sequence $U = (\varphi, \psi, \omega, u, v, z, \theta)$, we obtain

$$\int_0^L |\theta_x|^2 dx = -\operatorname{Re}((i\lambda - \mathcal{A}_j)U, U)_{\mathcal{H}_j} = \frac{o(1)}{\lambda^4}. \tag{4.13}$$

Finally, using Poincaré inequality, it follows the second asymptotic equality. $\square$

Now we have the first information on ψ and ψ_x.

Lemma 4.3. *With the above notation, we have*

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^3}, \tag{4.14}$$

where η is the function defined in Theorem 3.1

Proof. (i) We start by multiplying (4.11) by $\eta\bar{\psi}_x$, we obtain

$$\begin{aligned} T_0\int_0^L \eta\alpha|\psi_x|^2 = \frac{T_0}{2}\int_0^L (\eta\alpha')'|\psi|^2 + \mathrm{Re}\Big\{\rho_3\int_0^L(\eta'\theta+\eta\theta_x)\bar{\psi} \\ + i\int_0^L \theta_x\lambda^{-1}\eta\bar{\psi_{xx}} + \frac{i}{\lambda}\int_0^L \eta'\theta_x\bar{\psi}_x\Big\} + \frac{o(1)}{\lambda^5}. \end{aligned} \tag{4.15}$$

Using equation (4.12) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$, $\|\psi_x\| = O(1)$ and $\|\eta\psi_{xx}\| = O(\lambda)$ in (4.15), we obtain

$$\int_0^L \eta|\psi_x|^2 = o(1). \tag{4.16}$$

Next, multiplying (4.9) by $\eta\bar{\psi}$, we obtain

$$\begin{aligned} \rho_2\int_0^L \eta|\lambda\psi|^2 = b\int_0^L \eta|\psi_x|^2 + b\int_0^L \eta'\psi_x\bar{\psi} + \int_0^1 [\kappa(\psi+l\omega)+\alpha\theta_x]\eta\bar{\psi} \\ - \int_0^1 \kappa(\eta'\varphi\psi+\eta\varphi\psi_x) + \frac{o(1)}{\lambda^4}. \end{aligned} \tag{4.17}$$

Using (4.12), (4.16) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$ and $\|\omega\| = \frac{O(1)}{\lambda}$ in (4.17), we obtain

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^2}. \tag{4.18}$$

(ii) Multiplying (4.15) by λ^2 and using (4.12), (4.18) and the fact that $\|\psi_{xx}\| = O(\lambda)$, we obtain

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^2}. \tag{4.19}$$

(iii) Multiplying (4.17) by λ^2 and using (4.12), (4.18), (4.19) and the fact that $\|\lambda\omega\| = O(1)$, $\|\lambda\varphi\| = O(1)$, we obtain

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^4}. \tag{4.20}$$

In addition, using (4.12), (4.20) and the fact that $\|\omega\| = \frac{O(1)}{\lambda}$, $\|\varphi_x\| = O(1)$ in (4.9), we obtain

$$\int_0^L |\eta\psi_{xx}|^2 = O(1). \tag{4.21}$$

Finally, multiplying (4.15) by λ^3, and using (4.20), (4.21) we deduce the second asymptotic behavior equation in (4.14). $\square$

Now we have the relation between φ and ψ.

Lemma 4.4. *Let $1/2 \le \gamma \le 1$. With the above notation, assume that*

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\gamma}}. \tag{4.22}$$

Then

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{2\gamma}} \quad \textit{and} \quad \int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^{2+2\gamma}}. \tag{4.23}$$

Proof. Let $l_N = \sum_{k=0}^{N} \frac{1}{2^k}$, we will prove by induction on $N \in \mathbb{N}$ that

$$\int_0^L \eta|\varphi_x^n|^2 = \frac{o(1)}{\lambda^{\gamma l_N}}. \tag{4.24}$$

(i) **Verification for** $N = 0$**.** Multiplying (4.9) by $\eta\bar{\varphi}_x$ and integrating over $]0, L[$, we obtain

$$\begin{aligned}\kappa \int_0^L \eta|\varphi_x|^2 &= -\rho_2 \int_0^L \lambda^2(\eta\psi)_x\bar{\varphi} - b\int_0^L \lambda\eta\psi_x\lambda^{-1}\bar{\varphi}_{xx} \\ &\quad - \int_0^L (\kappa\psi + \kappa l\omega + \alpha\theta_x)\eta\bar{\varphi}_x - b\int_0^L \psi_x\eta'\bar{\varphi}_x + \frac{o(1)}{\lambda^4}\end{aligned} \tag{4.25}$$

Using equations (4.12), (4.14) and the fact that $\|\varphi_{xx}\| = O(\lambda)$, $\|\varphi_x\| = O(1)$, $\|\varphi\| = \frac{O(1)}{\lambda}$ and $\|\omega\| = \frac{O(1)}{\lambda}$) in (4.25), we obtain

$$\int_0^L \eta|\varphi_x|^2 = o(1). \tag{4.26}$$

Now, multiplying (4.25) by λ^γ. Since $\gamma \le 1$, then $\|\lambda^\gamma\omega\| = O(1)$ and $\|\lambda^\gamma\varphi\| = O(1)$. Using (4.12), (4.14), (4.22), (4.26) and the fact that $\|\varphi_{xx}\| = O(\lambda)$, we obtain

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^\gamma}. \tag{4.27}$$

Hence, the asymptotic behavior formula (4.24) is true for $N = 0$.

(ii) **Information on** φ**.** In addition, multiplying (4.8) by $\eta\bar{\varphi}$ and integrating over $]0, L[$, we obtain

$$\begin{aligned}\rho_1 \int_0^L \eta|\lambda\varphi|^2 &= \kappa \int_0^L (\eta|\varphi_x|^2 + (\eta'\varphi_x - \eta\psi_x)\bar{\varphi}) \\ &\quad + l\int_0^L (\kappa + \kappa_0)\omega(\eta\bar{\varphi})_x + l^2\kappa_0 \int_0^L \eta|\varphi|^2 + \frac{o(1)}{\lambda^4}.\end{aligned} \tag{4.28}$$

Multiplying (4.28) by λ^γ. Then, using (4.27) and the fact that $\|\lambda^\gamma\omega\| = O(1)$, we obtain

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^{2+\gamma}}. \tag{4.29}$$

(iii) **Induction.** Suppose that the asymptotic behavior formula (4.24) is true for the order $N - 1$, then we have

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{\gamma l_{N-1}}}. \tag{4.30}$$

Now, multiplying (4.28) by $\lambda^{\gamma l_{N-1}}$. Since $\lambda^{\gamma l_{N-1}} \le 2$, then $\|\lambda^{\frac{\gamma}{2} l_{N-1}}\omega\| = O(1)$. This implies that, using (4.14), (4.29), (4.30) and the fact that $\|\varphi_{xx}\| = O(\lambda)$, we obtain

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^{2+\gamma l_{N-1}}}. \tag{4.31}$$

On the other hand, using (4.31) and the fact that $\|\omega_x\| = O(1)$ in (4.8), we obtain

$$\int_0^L \eta|\varphi_{xx}|^2 = O(\lambda^{1-\frac{\gamma}{2} l_{N-1}}). \tag{4.32}$$

Noting that $\gamma + \frac{\gamma}{2} l_{N-1} = \gamma l_N$ and multiplying (4.25) by $\lambda^{\gamma+\frac{\gamma}{2} l_{N-1}}$. Then, using (4.14), (4.22), (4.30), (4.31), (4.32), we obtain

$$\int_0^L \eta |\varphi_x|^2 = \frac{o(1)}{\lambda^{\gamma l_N}}.$$

As a consequence, the asymptotic behavior (4.24) is true for all $N \geq 0$.

(iv) **Result on φ_x.** Since $\lim_{N\to+\infty} l_N = \sum_{k=0}^{+\infty} \frac{1}{2^k} = 2$, we deduce the first desired asymptotic behavior equation:

$$\int_0^L \eta |\varphi_x|^2 = \frac{o(1)}{\lambda^{2\gamma}}. \tag{4.33}$$

(v) **Result on φ.** Multiplying equation (4.28) by $\lambda^{2\gamma}$. Then, using equations (4.29), (4.33) and the fact that $\|\lambda\omega\| = O(1)$, we deduce the second desired asymptotic behavior equation in (4.23). The proof is complete. □

Now we have the relation between ψ and ψ_x.

Lemma 4.5. *Let $\frac{1}{2} \leq \gamma \leq 1$. With the above notation, assume that*

$$\int_0^L \eta |\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\gamma}}. \tag{4.34}$$

Then we have

$$\int_0^L \eta |\psi|^2 = \frac{o(1)}{\lambda^{4+2\gamma}}. \tag{4.35}$$

Proof. Let $l_N = \sum_{k=0}^{N} \frac{1}{2^k}$, we will prove by induction on $N \in \mathbb{N}$ that

$$\int_0^L \eta |\psi|^2 = \frac{o(1)}{\lambda^{4+\gamma l_N}}. \tag{4.36}$$

(i) **Verification for $N = 0$.** Multiplying (4.17) by $\lambda^{2+\gamma}$. Then, using (4.14), (4.34), Lemma 4.4 and the fact that $\|\omega\| = \frac{O(1)}{\lambda}$, we obtain

$$\int_0^L \eta |\psi|^2 = \frac{o(1)}{\lambda^{4+\gamma}}. \tag{4.37}$$

Hence, the asymptotic behavior formula (4.36) is true for $N = 0$.

(ii) **Induction.** Suppose that the asymptotic behavior formula (4.36) is true for the order $N - 1$, then we have

$$\int_0^L \eta |\psi|^2 = \frac{o(1)}{\lambda^{4+\gamma l_{N-1}}}. \tag{4.38}$$

Multiplying (4.17) by $\lambda^{2+(\gamma+\frac{\gamma}{2} l_{N-1})}$. Since $\gamma + \frac{\gamma}{2} l_{N-1} \leq 2 + 2\gamma$ and $\gamma \leq 1$, then using (4.12), (4.34), (4.38), Lemma 4.4, and the fact that $\|\lambda\omega\| = O(1)$, we obtain

$$\int_0^L \eta |\psi|^2 = \frac{o(1)}{\lambda^{4+(\gamma+\frac{\gamma}{2} l_{N-1})}}. \tag{4.39}$$

Since $\gamma + \frac{\gamma}{2} l_{N-1} = \gamma l_N$, we deduce the asymptotic behavior formula (4.35).

(iii) **Result.** Using the fact that $\lim_{N\to+\infty} l_N = \sum_{k=0}^{+\infty} \frac{1}{2^k} = 2$, we deduce the asymptotic behavior result (4.35). □

Now we have the final information on ψ and ψ_x.

Lemma 4.6. *With the above notation, we have*

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^6}. \tag{4.40}$$

Proof. Let $\hat{l}_N = \sum_{k=1}^N \frac{1}{2^k}$. We will prove by induction on $N \in \mathbb{N}^\star$, that

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\hat{l}_N}}. \tag{4.41}$$

(i) **Verification for $N = 1$.** Using Lemma 4.3 we deduce that the asymptotic behavior equality (4.41) is true for $N = 1$.

(ii) **Induction.** Suppose that the asymptotic behavior equality (4.41) is true for $N-1$, then we have

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\hat{l}_{N-1}}}. \tag{4.42}$$

Then, applying Lemma 4.4 and Lemma 4.5 with $\gamma = \hat{l}_{N-1}$, we obtain

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{2\hat{l}_{N-1}}}, \quad \int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+2\hat{l}_{N-1}}}. \tag{4.43}$$

On the other hand, using (4.43) and the fact that $\hat{l}_{N-1} \le 1$, $\|\lambda^{\hat{l}_{N-1}}\omega\| = O(1)$ in (4.9), we obtain

$$\|\psi_{xx}\| = \frac{O(1)}{\lambda^{\hat{l}_{N-1}}}. \tag{4.44}$$

Now, multiplying (4.15) by $\lambda^{3+\hat{l}_{N-1}}$. Then, using (4.12), (4.43) and (4.44), we obtain

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{3+\hat{l}_{N-1}}}. \tag{4.45}$$

Using the fact that $3+\hat{l}_{N-1} = 2+2\hat{l}_N$, we deduce the asymptotic behavior formula (4.41) for all $N \in \mathbb{N}^\star$.

(iii) **Result.** Using the fact that $\lim_{N\to+\infty} \hat{l}_{N-1} = \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1$, we deduce the first asymptotic behavior equation in (4.40). Then applying Lemma 4.5 with $\gamma = 1$, we deduce the second asymptotic behavior equation in (4.40). The proof is complete. □

Now we have information on φ and φ_x.

Lemma 4.7. *With the above notation, we have*

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^4}. \tag{4.46}$$

Proof. Using Lemma 4.6 we deduce the asymptotic behavior equations (4.46) by applying Lemma 4.4 with $\gamma = 1$. the proof is complete. □

now we have information on ω and ω_x.

Lemma 4.8. *With the above notation, we have*

$$\int_0^L \eta|\omega_x|^2 = o(1) \quad \text{and} \quad \int_0^L \eta|\omega|^2 = \frac{o(1)}{\lambda^2}. \tag{4.47}$$

Proof. Multiply (4.8) by $\eta\overline{\omega_x}$. Then, using (4.40), (4.46) and the fact that $\|\omega_x\| = O(1)$, we obtain

$$(\kappa+\kappa_0)l\int_0^L \eta|\omega_x|^2 = \kappa\int_0^L \lambda\varphi_x\lambda^{-1}\eta\overline{\omega_{xx}} + o(1) \tag{4.48}$$

Then, using (4.46) and the fact that $\|\omega_{xx}\| = O(\lambda)$ in (4.48), we deduce the first asymptotic behavior equation in (4.47). Finally, multiplying equation (4.10) by $\eta\bar{\omega}$, we deduce the second asymptotic behavior equation in (4.47). The proof is complete. □

Proof of Theorem 4.1. Using lemmas 4.6, 4.7 and 4.8, we obtain $\|U\|_{\mathcal{H}_j} = o(1)$ over (a_0, b_0). Then by applying lemma 3.4, we deduce that $\|U\|_{\mathcal{H}_j} = o(1)$, over $(0, L)$ which is a contradiction to (4.3). This implies that $\|(i\lambda - \mathcal{A}_j)^{-1}\| = O(\lambda^4)$. This together with the fact that $i\mathbb{R} \subset \rho(\mathcal{A}_j)$ imply (4.1) (see [2, 4, 11]). The proof is complete. □

Remark 4.9. The conditions $\kappa \neq \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$ considered in Theorem 4.1 describe the natural physical problem. All other speed wave conditions have only mathematical sound. However, they do provide useful insight to the study of similar models arising from other applications.

Remark 4.10. In the case $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$, by a similar way used in Theorem 4.1, we can prove that

$$E(t) \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A}_j)} \quad \forall t > 0. \tag{4.49}$$

Noting that, in this case, technically, the process of the proof is much easier to that of the natural general case of Theorem 4.1. In fact, we need to prove

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{\lambda^2}\|(i\lambda I - \mathcal{A}_j)^{-1}\| < \infty.$$

From dissipation law we obtain

$$\int_0^L |\theta_x|^2 dx = \frac{o(1)}{\lambda^2}, \quad \int_0^L |\theta|^2 dx = \frac{o(1)}{\lambda^2}.$$

This leads to

$$\int_0^L |\eta\psi_x|^2 dx = \frac{o(1)}{\lambda^2}, \quad \int_0^L |\eta\psi|^2 dx = \frac{o(1)}{\lambda^4}.$$

This implies

$$\int_0^L |\eta\varphi_x|^2 dx = o(1), \quad \int_0^L |\eta\varphi_x|^2 dx = \frac{o(1)}{\lambda^2}.$$

Here, we can use the condition $\kappa = \kappa_0$ in order to obtain

$$\int_0^L |\eta\omega_x|^2 dx = o(1), \quad \int_0^L |\eta\omega_x|^2 dx = \frac{o(1)}{\lambda^2}.$$

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