WEIGHTED PSEUDO PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

ZHINAN XIA

Abstract. In this article, we introduced and explore the properties of two sets of functions: weighted pseudo periodic functions of class $r$, and weighted Stepanov-like pseudo periodic functions of class $r$. We show the existence and uniqueness of weighted pseudo periodic solution of class $r$ that are solutions to neutral functional differential equations. Other applications to partial differential equations and scalar reaction-diffusion equations with delay are also given.

1. Introduction

The existence periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations. Many authors have made important contributions to this theory. Recently, in [1, 19], the concept of weighted pseudo periodicity, weighted Stepanov-like pseudo periodicity, is introduced and studied, respectively. On the other hand, to study issues related to delay differential equations, Diagana [5] introduce the functions called pseudo almost periodic of class $r$, for more on this topic and related applications in differential equations, we refer the reader to [2, 3, 4, 10, 11].

Motivated by the above mentioned papers, in this paper, we introduce new class of functions called weighted pseudo periodic of class $r$, weighted Stepanov-like pseudo periodic of class $r$, respectively. We systematically explore the properties of these functions in general Banach space including composition results and its applications in differential equations.

In recent years, neutral functional differential equations have attracted a great deal of attention of many mathematicians due to their significance and applications in physics, mathematical biology, control theory, and so on. The general asymptotic behavior of solutions have been one of the most attracting topics in the context of neutral functional differential equations [6, 9, 12, 13, 14, 15, 18]. However, to the best of our knowledge, the studies on the weighted pseudo periodic solutions of neutral functional differential equations is quite new and an untreated topic. This is one of the key motivations of this study.

2000 Mathematics Subject Classification. 35R10, 35B40.

Key words and phrases. Weighted pseudo periodicity; weighted Stepanov-like pseudo periodic; neutral functional differential equation; Banach contraction mapping principle.

©2014 Texas State University - San Marcos.
Submitted March 6, 2014. Published September 16, 2014.
The paper is organized as follows. In Section 2, some notations and preliminary results are presented. Next, we propose new class of functions called weighted pseudo periodic functions of class $r$, weighted Stepanov-like pseudo periodic functions of class $r$, explore the properties of these functions and establish the composition theorems. Sections 3 is devoted to the existence and uniqueness of weighted pseudo periodic solutions of class of $r$ to neutral functional differential equations. In section 4, we present applications to partial differential equations and scalar reaction-diffusion equations with delay.

2. Preliminaries and basic results

Let $(X, \| \cdot \|)$, $(Y, \| \cdot \|)$ be two Banach spaces and $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ be the stand sets of natural numbers, integers, real numbers, respectively. To facilitate the discussion below, we further introduce the following notation:

- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R} \times Y, X)$): the set of continuous functions from $\mathbb{R}$ to $X$ (resp. from $\mathbb{R} \times Y$ to $X$).
- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R} \times Y, X)$): the Banach space of bounded continuous functions from $\mathbb{R}$ to $X$ (resp. from $\mathbb{R} \times Y$ to $X$) with the supremum norm;
- $L^p(\mathbb{R}, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R});$
- $L^p_{loc}(\mathbb{R}, X)$: stand for the space of all classes of equivalence of measurable functions $f : \mathbb{R} \to X$ such that the restriction of $f$ to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, X)$.
- $C([-r, 0], X)$ endowed with the sup norm $\|\psi\|_C$ on $[-r, 0]$.
- $[D(A)]$: the domain of $A$ when it is endowed with graph norm, $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$ for each $x \in D(A)$.

2.1. Weighted pseudo periodic of class $r$. In this subsection, we introduce the new class of functions called weighted pseudo anti-periodic of class $r$, weighted pseudo periodic functions of class $r$, and investigate the properties of these functions.

Definition 2.1. A function $f \in C(\mathbb{R}, X)$ is said to be anti-periodic if there exists a $\omega \in \mathbb{R}\setminus\{0\}$ with the property that $f(t + \omega) = -f(t)$ for all $t \in \mathbb{R}$. The least positive $\omega$ with this property is called the anti-periodic of $f$. The collection of those functions is denoted by $P_{\omega ap}(\mathbb{R}, X)$.

Definition 2.2. A function $f \in C(\mathbb{R}, X)$ is said to be periodic if there exists a $\omega \in \mathbb{R}\setminus\{0\}$ with the property that $f(t + \omega) = f(t)$ for all $t \in \mathbb{R}$. The least positive $\omega$ with this property is called the periodic of $f$. The collection of those $\omega$-periodic functions is denoted by $P_{\omega}(\mathbb{R}, X)$.

Note that if $f \in P_{\omega ap}(\mathbb{R}, X)$, then $f \in P_{2\omega}(\mathbb{R}, X)$.

Let $U$ be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $T > 0$ and each $\rho \in U$, set

$$\mu(T, \rho) := \int_{-T}^{T} \rho(t)dt.$$ 

Define

$$U_{\infty} := \{ \rho \in U : \lim_{T \to \infty} \mu(T, \rho) = \infty \},$$
Definition 2.6. Let $\rho, \rho_1, \rho_2 \in U_{\infty}$. The function $\rho_1$ is said to be equivalent to $\rho_2$ (i.e., $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$.

It is trivial to show that “$\sim$” is a binary equivalence relation on $U_{\infty}$. The equivalence class of a given weight $\rho \in U_{\infty}$ is denoted by $\text{cl}(\rho) = \{g \in U_{\infty} : \rho \sim g\}$.

It is clear that $U_{\infty} = \bigcup_{\rho \in U_{\infty}} \text{cl}(\rho)$.

Let $\rho \in U_{\infty}$, $\tau \in \mathbb{R}$ be given, and defined $\rho^\tau$ by $\rho^\tau(t) = \rho(t + \tau)$ for $t \in \mathbb{R}$. Define

$$U_T = \{\rho \in U_{\infty} : \rho \sim \rho^\tau \text{ for each } \tau \in \mathbb{R}\}.$$ 

It is easy to see that $U_T$ contains many of weights, such as $1$, $(1 + t^2)/(2 + t^2)$, $e^t$, and $1 + |t|^n$ with $n \in \mathbb{N}$ etc. For $\rho_1, \rho_2 \in U_{\infty}$, define

$$WPP_0(\mathbb{R}, X, \rho_1, \rho_2) := \left\{ f \in BC(\mathbb{R}, X) : \lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \|f(t)\| \rho_2(t) dt = 0 \right\},$$

$$WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) := \left\{ f \in BC(\mathbb{R}, X) : \lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \sup_{\theta \in [t-r, t]} \|f(\theta)\| \rho_2(t) dt = 0 \right\},$$

$$WPP_0(\mathbb{R} \times Y, Y, r, \rho_1, \rho_2) := \left\{ f \in BC(\mathbb{R} \times Y, X) : \lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \sup_{\theta \in [t-r, t]} \|f(\theta, u)\| \rho_2(t) dt = 0 \right\}$$

uniformly for $u \in Y$.

Definition 2.4. Let $\rho_1, \rho_2 \in U_{\infty}$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo anti-periodic for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{wap}(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, \rho_1, \rho_2)$. Denote by $WPP_{wap}(\mathbb{R}, X, \rho_1, \rho_2)$ the set of such functions.

Definition 2.5. Let $\rho_1, \rho_2 \in U_{\infty}$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo periodic for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, X, \rho_1, \rho_2)$ the set of such functions.

If $\rho_1 \sim \rho_2$, $WPP_{wap}(\mathbb{R}, X, \rho_1, \rho_2)$, and $WPP_\omega(\mathbb{R}, X, \rho_1, \rho_2)$ coincide with the weighted pseudo anti-periodic, and the weighted pseudo periodic function respectively, introduce by $\Pi$.

Definition 2.6. Let $\rho_1, \rho_2 \in U_{\infty}$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo anti-periodic of class $r$ for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{wap}(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$. The set of these functions is denote by $WPP_{wap}(\mathbb{R}, X, r, \rho_1, \rho_2)$. 

Definition 2.7. Let $\rho_1, \rho_2 \in U_{\infty}$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo periodic of class $r$ for $\omega \in \mathbb{R} \setminus \{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ the set of such functions.
Remark 2.8. If \( r = 0 \), then the weighted pseudo anti-periodic function of class \( r \) reduces to the weighted pseudo anti-periodic function, the weighted pseudo periodic function of class \( r \) reduces to the weighted pseudo periodic function. That is, \( WPP_{\omega_{\text{ap}}}([0,1],\mathbb{R},X,0,\rho_1,\rho_2) = WPP_{\omega_{\text{p}}}([0,1],\mathbb{R},X,0,\rho_1,\rho_2) \) and \( WPP_{\omega}([0,1],\mathbb{R},X,0,\rho_1,\rho_2) = WPP_{\omega}([0,1],\mathbb{R},X,0,\rho_1,\rho_2) \).

Next, we show some properties of the space \( WPP_{\omega}([0,1],\mathbb{R},X,0,\rho_1,\rho_2) \). Similarly, results hold for \( WPP_{\omega_{\text{ap}}}([0,1],\mathbb{R},X,0,\rho_1,\rho_2) \).

Lemma 2.9. Let \( f \in BC([0,1],\mathbb{R}) \), then \( f \in WPP_0([0,1],\mathbb{R},X,0,\rho_1,\rho_2) \), \( \rho_1,\rho_2 \in \mathcal{U}_1 \), \( \sup_{t>0} \frac{\mu(t)}{\mu(t)} < \infty \) if and only if for every \( \varepsilon > 0 \),

\[
\lim_{T \to \infty} \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon)} \rho_2(t)dt = 0,
\]

where \( M(T,\varepsilon,\theta) := \{ t \in [T,1] : \sup_{\theta \in [T-\varepsilon,T]} \| f(\theta) \| \geq \varepsilon \} \).

Proof. The proof is similar as \([10]\).

Sufficiency: From the statement of the lemma it is clear that for any \( \varepsilon > 0 \), there exists \( T_0 > 0 \) such that for \( T > T_0 \),

\[
\frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon)} \rho_2(t)dt < \varepsilon
\]

Then

\[
\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [T-\varepsilon,T]} \| f(\theta) \| \right) \rho_2(t)dt
\]

\[
= \frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon)} \left( \sup_{\theta \in [T-\varepsilon,T]} \| f(\theta) \| \right) \rho_2(t)dt
\]

\[
+ \frac{1}{\mu(T,\rho_1)} \int_{[T,1]\setminus M(T,\varepsilon)} \left( \sup_{\theta \in [T-\varepsilon,T]} \| f(\theta) \| \right) \rho_2(t)dt
\]

\[
\leq \frac{\| f \|}{\mu(T,\rho_1)} \int_{M(T,\varepsilon)} \rho_2(t)dt + \frac{\varepsilon}{\mu(T,\rho_1)} \int_{-T}^{T} \rho_2(t)dt
\]

\[
\leq \varepsilon + \sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} \varepsilon,
\]

so

\[
\lim_{T \to \infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [T-\varepsilon,T]} \| f(\theta) \| \right) \rho_2(t)dt = 0.
\]

That is, \( f \in WPP_0([0,1],\mathbb{R},X,0,\rho_1,\rho_2) \).

Necessity: Suppose on the contrary that there exists \( \varepsilon_0 > 0 \) such that

\[
\frac{1}{\mu(T,\rho_1)} \int_{M(T,\varepsilon_0)} \rho_2(t)dt
\]

does not converge to 0 as \( T \to \infty \). Then there exists \( \delta > 0 \) such that for each \( n \),

\[
\frac{1}{\mu(T_n,\rho_1)} \int_{M(T_n,\varepsilon_0)} \rho_2(t)dt \geq \delta \quad \text{for some } T_n \geq n.
\]

Then

\[
\frac{1}{\mu(T_n,\rho_1)} \int_{-T_n}^{T_n} \rho_2(t) \left( \sup_{\theta \in [T-\varepsilon,T]} \| f(\theta) \| \right) dt
\]
\[ \frac{1}{\mu(T, \rho_1)} \int_{M(T, \varepsilon, f)} \rho_2(t) \left( \sup_{\theta \in [t-r, t]} ||f(\theta)|| \right) dt \geq \frac{\varepsilon_0}{\mu(T, \rho_1)} \int_{M(T, \varepsilon, f)} \rho_2(t) dt \geq \varepsilon_0 \delta, \]

which contradicts the fact that \( f \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \), and the proof is complete. \( \square \)

Let
\[ WPP_0(\mathbb{R}, \mathbb{R}^+, r, \rho_1, \rho_2) = \{ f \in WPP_0(\mathbb{R}, \mathbb{R}, r, \rho_1, \rho_2) : f(t) \geq 0, \forall t \in \mathbb{R} \}. \]

**Lemma 2.10.** Let \( \alpha > 0 \), then \( f \in WPP_0(\mathbb{R}, \mathbb{R}^+, r, \rho_1, \rho_2) \) if and only if \( f^\alpha \in WPP_0(\mathbb{R}, \mathbb{R}^+, r, \rho_1, \rho_2) \), where \( f^\alpha(t) = |f(t)|^\alpha \), \( \rho_1, \rho_2 \in U_\infty \), \( \sup_{T > 0} \mu(T, \rho_1) < \infty \).

**Proof.** By Lemma 2.9, \( f \in WPP_0(\mathbb{R}, \mathbb{R}^+, r, \rho_1, \rho_2) \) if and only if for every \( \varepsilon > 0 \),
\[ \lim_{T \to -\infty} \frac{1}{\mu(T, \rho_1)} \int_{M(T, \varepsilon, f)} \rho_2(t) dt = 0, \]
where \( M(T, \varepsilon, f) := \{ t \in [-T, T] : \sup_{\theta \in [t-r, t]} f(\theta) \geq \varepsilon \} \). It is equivalent to for every \( \varepsilon > 0 \),
\[ \lim_{T \to -\infty} \frac{1}{\mu(T, \rho_1)} \int_{M(T, \varepsilon, f^\alpha)} \rho_2(t) dt = 0, \]
where
\[ M(T, \varepsilon, f^\alpha) := \{ t \in [-T, T] : \sup_{\theta \in [t-r, t]} f^\alpha(\theta) \geq \varepsilon \}. \]

So \( f^\alpha \in WPP_0(\mathbb{R}, \mathbb{R}^+, r, \rho_1, \rho_2) \). \( \square \)

**Lemma 2.11.** Let \( \varphi_n \to \varphi \) uniformly on \( \mathbb{R} \) where each \( \varphi_n \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \), \( \rho_1, \rho_2 \in U_\infty \), if \( \sup_{T > 0} \mu(T, \rho_2) < \infty \), then \( \varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \).

**Proof.** For \( T > 0 \),
\[ \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} ||\varphi(\theta)|| \right) \rho_2(t) dt \leq \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} ||\varphi_n(\theta) - \varphi(\theta)|| \right) \rho_2(t) dt \]
\[ + \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} ||\varphi_n(\theta)|| \right) \rho_2(t) dt \leq \frac{\mu(T, \rho_2)}{\mu(T, \rho_1)} ||\varphi_n - \varphi|| + \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} ||\varphi_n(\theta)|| \right) \rho_2(t) dt \]
\[ \leq \frac{\mu(T, \rho_2)}{\mu(T, \rho_1)} ||\varphi_n - \varphi|| + \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} ||\varphi_n(\theta)|| \right) \rho_2(t) dt, \]
Let \( T \to \infty \) and then \( n \to \infty \) in the above inequality, it follows that \( \varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \). \( \square \)

By carrying out similar arguments as those in the proof of [21, Lemma 4.1], we conclude the following.
Lemma 2.12. Let $\rho_1, \rho_2 \in U_T$, $\varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$, then $\varphi(\cdot - \tau)$ belongs to $WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$ for $\tau \in \mathbb{R}$.

Using similar ideas as in [7 8], one can easily show the following result.

Lemma 2.13. If $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T, \rho_2)}{\mu(T, \rho_1)} = \delta_0 > 0$, then the decomposition of weighted pseudo periodic function of class $r$ is unique.

By Lemma 2.13, it is obvious that $(WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2), \| \cdot \|)$, $\rho_1, \rho_2 \in U_T$ and $\inf_{T>0} \frac{\mu(T, \rho_2)}{\mu(T, \rho_1)} = \delta_0 > 0$ is a Banach space when endowed with the sup norm.

Lemma 2.14. Let $\rho_1, \rho_2 \in U_T$, $u \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, then $u_t$ belongs to $WPP_\omega(\mathbb{R}, C, r, \rho_1, \rho_2)$.

Proof. Suppose that $u = \alpha + \beta$, where $\alpha \in P_\omega(\mathbb{R}, X)$ and $\beta \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$, then $u_t = \alpha_t + \beta_t$ and $\alpha_t \in P_\omega(\mathbb{R}, C, r, \rho_1, \rho_2)$. On the other hand, for $T > 0$, we see that

$$
\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left[ \sup_{\theta \in [t-r, t]} \left( \sup_{\tau \in [-r, 0]} \| \beta(\theta + \tau) \| \right) \right] \rho_2(t)dt 
\leq \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-2r, t]} \| \beta(\theta) \| \right) \rho_2(t)dt 
\leq \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| + \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t)dt 
\leq \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T-r} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t+r)dt 
+ \frac{1}{\mu(T, \rho_1)} \int_{T-r}^{T} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t)dt 
\leq \frac{\mu(T+r, \rho_1)}{\mu(T, \rho_1)} \frac{1}{\mu(T + r, \rho_1)} \int_{-T-r}^{T+r} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t) \frac{\rho_2(t+r)}{\rho_2(t)} dt 
+ \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t)dt.
$$

Since $\rho_1, \rho_2 \in U_T$ implies that there exists $\eta > 0$ such that $\rho_1(t+r)/\rho_1(t) \leq \eta$, $\rho_1(t-r)/\rho_1(t) \leq \eta$, $\rho_2(t+r)/\rho_2(t) \leq \eta$. For $T > r$,

$$
\mu(T+r, \rho_1) \int_{-T-r}^{T-r} \rho_1(t)dt + \int_{T-r}^{T+r} \rho_1(t)dt \leq \int_{-T-r}^{T-r} \rho_1(t)dt + \int_{T-r}^{T+r} \rho_1(t)dt 
= \int_{-T}^{T} \rho_1(t-r)dt + \int_{T}^{T+r} \rho_1(t+r)dt \leq 2\eta \mu(T, \rho_1),
$$

then

$$
\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left[ \sup_{\theta \in [t-r, t]} \left( \sup_{\tau \in [-r, 0]} \| \beta(\theta + \tau) \| \right) \right] \rho_2(t)dt 
\leq \frac{2\eta^2}{\mu(T+r, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t)dt 
+ \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| \beta(\theta) \| \right) \rho_2(t)dt.
$$
Note that $\beta \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$, $\rho_1, \rho_2 \in U_T$, then $\beta_t \in WPP_0(\mathbb{R}, C, r, \rho_1, \rho_2)$. Therefore, $u_t \in WPP_0(\mathbb{R}, C, r, \rho_1, \rho_2)$. □

Similarly as [3] Theorem 3.9, we have the following composition theorem for weighted pseudo periodic function of class $r$.

**Theorem 2.15.** Assume that $\rho_1, \rho_2 \in U_\infty$, $r \geq 0$, $f \in WPP_\omega(\mathbb{R} \times Y, X, r, \rho_1, \rho_2)$ and there exists a function $L_f : \mathbb{R} \to [0, +\infty)$ satisfying

1. $\|f(t, u) - f(t, v)\| \leq L_f(t)\|u - v\|$, for all $t \in \mathbb{R}$, $u, v \in Y$;
2. $\limsup_{t \to -\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} L_f(\theta) \right) \rho_2(t) dt < \infty$;
3. $\limsup_{t \to -\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} L_f(\theta) \right) \xi(t) \rho_2(t) dt = 0$ for each function $\xi \in WPP_\omega(\mathbb{R}, \mathbb{R}, \rho_1, \rho_2)$.

Then $f(., h(\cdot)) \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ if $h \in WPP_\omega(\mathbb{R}, Y, r, \rho_1, \rho_2)$.

**Remark 2.16.** Note that (A2) and (A3) are verified by many functions. Concrete examples include constant functions, and functions in $WPP_\omega(\mathbb{R}, \mathbb{R}, r, \rho_1, \rho_2)$.

### 2.2. Weighted Stepanov-like pseudo periodic of class $r$.

In this subsection, we introduce the new class of functions called weighted $S^p$-pseudo anti-periodic of class $r$, weighted $S^p$-pseudo periodic functions of class $r$, and investigate the properties of these functions.

Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent $p$, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0, 1]; X))$, where $f^b$ is the Bochner transform of $f$ defined by $f^b(t, s) := f(t + s), t \in \mathbb{R}, s \in [0, 1]$. $BS^p(\mathbb{R}, X)$ is a Banach space with the norm $\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left( \int_{t}^{t+1} \|f(\tau)\|^p d\tau \right)^{1/p}$. It is clear that $L^p(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X) \subset L^p_{loc}(\mathbb{R}, X)$ and $BS^q(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X)$ for $p \geq q \geq 1$.

For $\rho_1, \rho_2 \in U_\infty$, define the weighted ergodic space in $BS^p(\mathbb{R}, X)$

$$S^pWPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) := \left\{ f \in BS^p(\mathbb{R}, X) : \lim_{T \to -\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left( \sup_{\theta \in [t-r, t]} \left( \int_{\theta}^{\theta+1} \|f(s)\|^p ds \right)^{1/p} \right) dt = 0 \right\}.$$

**Definition 2.17.** Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be weighted Stepanov-like pseudo anti-periodic of class $r$ (or weighted $S^p$-pseudo anti-periodic of class $r$) if there exists $\varphi \in S^pWPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$ such that the function $g = f - \varphi$ satisfies $g(t + \omega) + g(t) = 0$ a.e. $t \in \mathbb{R}$. The collection of such functions is denoted by $S^pWPP_{wap}(\mathbb{R}, X, r, \rho_1, \rho_2)$

**Definition 2.18.** Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be weighted Stepanov-like pseudo periodic of class $r$ (or weighted $S^p$-pseudo periodic of class $r$) if there exists $\varphi \in S^pWPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$ such that the function $g = f - \varphi$ satisfies $g(t + \omega) - g(t) = 0$ a.e. $t \in \mathbb{R}$. Denote by $S^pWPP_{\omega}(\mathbb{R}, X, r, \rho_1, \rho_2)$ the collection of such functions.

Next, we show some properties of the space $S^pWPP_{\omega}(\mathbb{R}, X, r, \rho_1, \rho_2)$. Similarly results hold for $S^pWPP_{wap}(\mathbb{R}, X, r, \rho_1, \rho_2)$.
Lemma 2.19. Let $\rho_1, \rho_2 \in U_T$, then
\[
WPP_2(\mathbb{R}, X, r, \rho_1, \rho_2) \subset \mathbb{S}^pWPP_2(\mathbb{R}, X, r, \rho_1, \rho_2).
\]

Proof. If $f \in WPP_2(\mathbb{R}, X, r, \rho_1, \rho_2)$, let $f = f_1 + f_2$, where $f_1 \in P_2(\mathbb{R}, X), f_2 \in WPP_2(\mathbb{R}, X, r, \rho_1, \rho_2)$. Then $\|f_2(\cdot\cdot)\| \in WPP_2(\mathbb{R}, X^+, r, \rho_1, \rho_2).$ By Lemma 2.20 $\|f_2(\cdot\cdot)\|^p \in WPP_2(\mathbb{R}, X^+, r, \rho_1, \rho_2).$ Note that $\|f_2(\cdot + \sigma)\|^p \in WPP_2(\mathbb{R}, X^+, r, \rho_1, \rho_2)$ for each $\sigma \in [0, 1], then
\[
\lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \|f_2(\theta + \sigma)\|^p \rho_2(t) \right) dt = 0.
\]

by Lebesgue’s dominated convergence theorem, one has
\[
\int_{0}^{1} \left( \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \|f_2(\theta + \sigma)\|^p \rho_2(t) \right) dt \right) d\sigma \to 0, \quad T \to \infty,
\]
i.e.,
\[
\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left( \int_{0}^{1} \sup_{\theta \in [t-r, t]} \|f_2(\theta + \sigma)\|^p d\sigma \right) dt \to 0, \quad T \to \infty;
\]
which means that
\[
\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \sup_{\theta \in [t-r, t]} \left( \int_{0}^{1} \|f_2(\theta + \sigma)\|^p d\sigma \right) dt \to 0, \quad T \to \infty;
\]
i.e.,
\[
\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \sup_{\theta \in [t-r, t]} \left( h_2(\theta) \right) dt \to 0, \quad T \to \infty,
\]
so $h_2 \in WPP_2(\mathbb{R}, X^+, r, \rho_1, \rho_2)$, where
\[
h_2(t) = \int_{0}^{1} \|f_2(t + \sigma)\|^p d\sigma, \quad t \in \mathbb{R}.
\]

By Lemma 2.10 $h_2^{1/p} \in WPP_2(\mathbb{R}, X^+, r, \rho_1, \rho_2);$ i.e.,
\[
\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \sup_{\theta \in [t-r, t]} \left( \int_{0}^{1} \|f_2(\theta + \sigma)\|^p d\sigma \right)^{1/p} dt \to 0, \quad T \to \infty,
\]
which means that $f_2 \in \mathbb{S}^pWPP_2(\mathbb{R}, X, r, \rho_1, \rho_2)$, then $f \in \mathbb{S}^pWPP_2(\mathbb{R}, X, r, \rho_1, \rho_2)$. The proof is complete. \hfill \Box

Theorem 2.20. Assume $\rho_1, \rho_2 \in U_\infty$, $f = f_1 + f_2 \in \mathbb{S}^pWPP_2(\mathbb{R} \times Y, X, r, \rho_1, \rho_2)$ with $f_2 \in \mathbb{S}^pWPP_2(\mathbb{R} \times Y, X, r, \rho_1, \rho_2)$, $f_1(t + \omega, u) - f_1(t, u) = 0$ a.e. $t \in \mathbb{R}$, $u \in X$, and there exists a function $L_f : \mathbb{R} \to [0, +\infty)$ satisfying:

(A1') \left( \int_{t}^{t+1} \|f(s, u) - f(s, v)\|^p ds \right)^{1/p} \leq L_f(t) \|u - v\|, \text{ for all } t \in \mathbb{R}, u, v \in Y;
\]
(A2') $\limsup_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} L_f(\theta) \right) \rho_2(t) dt < \infty$;
(A3') $\lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} L_f(\theta) \right) \xi(t) \rho_2(t) dt = 0$ for each function $\xi^b \in WPP_2(\mathbb{R}, \mathbb{L}^p([0, 1], \mathbb{R}), \rho_1, \rho_2)$;
(A4') $f_1$ is uniform continuous on bounded set $K' \subset Y$ for any $t \in \mathbb{R}$.

Then $f(\cdot, h(\cdot)) \in \mathbb{S}^pWPP_2(\mathbb{R}, X, r, \rho_1, \rho_2)$ if $h \in \mathbb{S}^pWPP_2(\mathbb{R}, Y, r, \rho_1, \rho_2).$
Neutral functional differential equations of the form

\[ \frac{d}{dt}[u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \quad t \in \mathbb{R}, \]

where \( A \) is the infinitesimal generator of a semigroup of linear operators on \( X \), \( u_t \in C \) is defined by \( u_t(\theta) = u(t + \theta) \) for \( \theta \in [-r, 0] \), where \( r \) is a nonnegative constant.

First, we recall the definition of the so called exponential dichotomy of a semigroup.

**Definition 3.1.** [16] A semigroup \( (T(t))_{t \geq 0} \) is said to be exponential dichotomy if there exist projection \( P \) and constants \( M, \delta > 0 \) such that each \( T(t) \) commutes with \( P \), \( \text{Ker}P \) is invariant with respect to \( T(t), T(t) : \text{Im}Q \rightarrow \text{Im}Q \) is invertible and

\[
\|T(t)Px\| \leq Me^{-\delta t}\|x\| \quad \text{for} \ t \geq 0, \tag{3.2}
\]

\[
\|T(t)Qx\| \leq Me^{\delta t}\|x\| \quad \text{for} \ t \leq 0. \tag{3.3}
\]

where \( Q := I - P \) and \( T(t) := (T(-t))^{-1} \) for \( t \leq 0 \).

To study (3.1), we make the following assumptions:

(H1) The operator \( A : D(A) \subset X \rightarrow X \) is the infinitesimal generator of a semigroup \( (T(t))_{t \geq 0} \) which has an exponential dichotomy.

(H2) \( f \in \text{WPP}_p(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \), \( f \) is \( D(A) \)-valued, there exists a positive constant \( L_f \) such that

\[
|f(t, \psi_1) - f(t, \psi_2)|_{[D(A)]} \leq L_f\|\psi_1 - \psi_2\|_C, \quad \text{for all} \ t \in \mathbb{R}, \ \psi_1, \psi_2 \in C, \ i = 1, 2.
\]

(H3) \( g \in \text{WPP}_\infty(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \) and there exists a continuous function \( L_g : \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
\|g(t, \psi_1) - g(t, \psi_2)\| \leq L_g(t)\|\psi_1 - \psi_2\|_C, \quad \text{for all} \ t \in \mathbb{R}, \ \psi_1, \psi_2 \in C, \ i = 1, 2.
\]

(H3') \( g = g_1 + g_2 \in \text{WPP}_p(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \) with \( g_2 \in \text{WPP}_p(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \), \( g_1(t + \omega, \psi) - g(t, \psi) = 0 \) a.e. \( t \in \mathbb{R}, \ \psi \in C \), satisfying

(i) \( g_1 \) is uniform continuous on bounded set \( K \subset C \) for any \( t \in \mathbb{R} \).

(ii) there exists a positive constant \( L_g \) such that

\[
\left( \int_{t}^{t+1} \|g(t, \psi_1) - g(t, \psi_2)\|^p ds \right)^{1/p} \leq L_g\|\psi_1 - \psi_2\|_C, \quad \text{for all} \ t \in \mathbb{R}, \ \psi_1, \psi_2 \in C, \ i = 1, 2.
\]

(H4) \( \rho_1, \rho_2 \in U_T, \ \inf_{T > 0} \mu(T, \rho_2) / \mu(T, \rho_1) = \delta > 0 \) and \( \sup_{T > 0} \mu(T, \rho_2) / \mu(T, \rho_1) < \infty \).
Definition 3.2 ([6]). A continuous function \( u \) is said to be a mild solution of \([3.1]\) provided that the function \( s \to AT(t-s)Pf(s,u_s) \) is integrable on \( (-\infty,t) \), \( s \to AT(t-s)Qf(s,u_s) \) is integrable on \( (t,\infty) \) for \( t \in \mathbb{R} \), and
\[
u(t) = -f(t,u_t) - \int_{-\infty}^{t} AT(t-s)Pf(s,u_s)ds + \int_{t}^{\infty} AT(t-s)Qf(s,u_s)ds
+ \int_{-\infty}^{t} T(t-s)Pg(s,u_s)ds - \int_{t}^{\infty} T(t-s)Qg(s,u_s)ds, \quad t \in \mathbb{R}.
\]

Lemma 3.3. Assume that (H1), (H2), (H4) hold, if \( u \in WPP_{\omega} (\mathbb{R}, X, r, \rho_1, \rho_2) \), then
\[
\begin{align*}
(\Lambda_1 f)(t) &= \int_{-\infty}^{t} AT(t-s)Pf(s,u_s)ds \in WPP_{\omega} (\mathbb{R}, X, r, \rho_1, \rho_2), \\
(\Lambda_2 f)(t) &= \int_{t}^{\infty} AT(t-s)Qf(s,u_s)ds \in WPP_{\omega} (\mathbb{R}, X, r, \rho_1, \rho_2).
\end{align*}
\]

Proof. By Lemma 2.14 and Theorem 2.15, \( f(s, u_s) := h(s) \in WPP_{\omega} (\mathbb{R}, X, r, \rho_1, \rho_2) \) and \( h \) is \( D(A) \)-valued. Let \( h(s) = h_1(s) + h_2(s) \) where \( h_1 \in P_{\omega} (\mathbb{R}, X) \) and \( h_2 \in WPP_0 (\mathbb{R}, X, r, \rho_1, \rho_2) \). Then
\[
\begin{align*}
(\Lambda_1 f)(t) &= \int_{-\infty}^{t} AT(t-s)Ph_1(s)ds + \int_{-\infty}^{t} AT(t-s)Ph_2(s)ds \\
&\quad := (\Lambda_{11} h_1)(t) + (\Lambda_{12} h_2)(t),
\end{align*}
\]
where
\[
\begin{align*}
(\Lambda_{11} h_1)(t) &= \int_{-\infty}^{t} AT(t-s)Ph_1(s)ds, \quad (\Lambda_{12} h_2)(t) = \int_{-\infty}^{t} AT(t-s)Ph_2(s)ds,
\end{align*}
\]
for \( t \in \mathbb{R} \). From \( h_1 \in P_{\omega} (\mathbb{R}, X) \),
\[
(\Lambda_{11} h_1)(t + \omega) = \int_{-\infty}^{t+\omega} AT(t+\omega-s)Ph_1(s)ds = (\Lambda_{11} h_1)(t);
\]
then \( \Lambda_{11} h_1 \in P_{\omega} (\mathbb{R}, X) \).

Next, we show that \( \Lambda_{12} h_2 \in WPP_0 (\mathbb{R}, X, r, \rho_1, \rho_2) \); that is,
\[
\lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| (\Lambda_{12} h_2)(\theta) \| \right) \rho_2(t)dt = 0.
\]
In fact, for \( T > 0 \), one has
\[
\begin{align*}
&\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| (\Lambda_{12} h_2)(\theta) \| \right) \rho_2(t)dt \\
= &\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| \int_{-\infty}^{\theta} AT(\theta-s)Ph_2(s)ds \| \right) \rho_2(t)dt \\
= &\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \| \int_{-\infty}^{\theta} AT(s)Ph_2(\theta-s)ds \| \right) \rho_2(t)dt \\
\leq &\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \int_{0}^{\infty} Me^{-\delta s} \| Ah_2(\theta-s) \| ds \right) \rho_2(t)dt \\
\leq &\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \left( \sup_{\theta \in [t-r, t]} \int_{0}^{\infty} Me^{-\delta s} \| h_2(\theta-s) \| ds \right) \rho_2(t)dt
\end{align*}
\]
\[
\leq \int_0^\infty Me^{-\delta s} \Phi_T(s) ds,
\]
where
\[
\Phi_T(s) = \frac{1}{\mu(T, \rho_1)} \int_{-T}^T \left( \sup_{\theta \in [t-r,t]} ||h_2(\theta - s)||_{\Omega(A)} \right) \rho_2(t) dt.
\]
Since \( \rho_1, \rho_2 \in U_T \), by Lemma 2.12, we have \( h_2(\cdot - s) \in WPP_0(\mathbb{R}, [D(A)], r, \rho_1, \rho_2) \) for each \( s \in \mathbb{R} \); hence \( \lim_{t \to -\infty} \Phi_T(s) = 0 \) for all \( s \in \mathbb{R} \). Then \( \Lambda_2 h_2 \) belongs to \( WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \) by using the Lebesgue dominated convergence theorem, so \( \Lambda_1 f \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \).

The proof of \( \Lambda_2 f \) is similar to that of \( \Lambda_1 f \), one makes use of (3.3) rather than (3.2). This completes the proof. \( \square \)

**Lemma 3.4.** Assume that (H1), (H4) hold, if \( \phi \in S^p WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \), then
\[
\begin{align*}
(\Gamma_1 \phi)(t) &= \int_{-\infty}^t (t-s)P\phi(s) ds \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2), \\
(\Gamma_2 \phi)(t) &= \int_{-\infty}^t (t-s)Q\phi(s) ds \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2).
\end{align*}
\]

**Proof.** By \( \phi \in S^p WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \), we let \( \phi(s) = \phi_1(s) + \phi_2(s) \), where \( \phi_2 \in S^p WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \) and \( \phi_1(t + \omega) - \phi_1(t) = 0 \) a.e. \( t \in \mathbb{R} \), then
\[
(\Gamma_1 \phi)(t) = \int_{-\infty}^t (t-s)P\phi_1(s) ds + \int_{-\infty}^t (t-s)P\phi_2(s) ds := (\Gamma_1 \phi_1)(t) + (\Gamma_1 \phi_2)(t).
\]
First, we show that \( \Gamma_1 \phi_2 \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) \). Consider the integrals
\[
Y_n(t) = \int_{t-n}^{t-n+1} (t-s)P\phi_2(s) ds.
\]
Fix \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \), we have
\[
\|Y_n(t + h) - Y_n(t)\| \leq \int_{n-1}^n \|T(s)P(\phi_2(t + h - s) - \phi_2(t - s))\| ds
\]
\[
\leq M \int_{t-n}^{t-n+1} \|\phi_2(s + h) - \phi_2(s)\| ds
\]
\[
\leq M \left( \int_{t-n}^{t-n+1} \|\phi_2(s + h) - \phi_2(s)\|^p ds \right)^{1/p}.
\]
In view of \( \phi_2 \in L^p_{loc}(\mathbb{R}, X) \), we get
\[
\lim_{h \to 0} \int_{t-n}^{t-n+1} \|\phi_2(s + h) - \phi_2(s)\|^p ds = 0,
\]
which yields \( \lim_{h \to 0} \|Y_n(t + h) - Y_n(t)\| = 0 \). This means that \( Y_n(t) \) is continuous.

By Hölder’s inequality, one has
\[
\|Y_n(t)\| \leq \int_{n-1}^n \|T(s)P\phi_2(t - s)\| ds
\]
\[
\leq \int_{n-1}^n Me^{-\delta s}\|\phi_2(t - s)\| ds
\]
\[
\leq Me^{-\delta(n-1)} \int_{n-1}^n \|\phi_2(t - s)\| ds.
\]
In fact, by Hölder inequality, it follows that

\[ \sum_{n=1}^{\infty} M e^{-\delta(n-1)} \int_{t-n}^{t-n+1} \| \phi_2(s) \| ds \]
\[ \leq M e^{-\delta(n-1)} \left( \int_{t-n}^{t-n+1} \| \phi_2(s) \|^p ds \right)^{1/p} \]
\[ \leq M e^{-\delta(n-1)} \| \phi_2 \|_{S^p}. \]

Since

\[ \sum_{n=1}^{\infty} M e^{-\delta(n-1)} \| \phi_2 \|_{S^p} \leq \frac{M}{1 - e^{-\delta}} \| \phi_2 \|_{S^p} < +\infty, \]

it follows that \( \sum_{n=1}^{\infty} Y_n(t) \) converges uniformly on \( \mathbb{R} \). Let \( Y(t) = \sum_{n=1}^{\infty} Y_n(t) \) for \( t \in \mathbb{R} \). Then

\[ Y(t) = (\Gamma_1 \phi_2)(t) = \int_{-\infty}^{t} T(t-s)P \phi_2(s) ds, \quad t \in \mathbb{R}. \]

It is obvious that \( Y(t) \in BC(\mathbb{R}, X) \). So, we only need to show that

\[ \lim_{T \to \infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left( \sup_{\theta \in [t-r, t]} \| Y(\theta) \| \right) dt = 0. \quad (3.4) \]

In fact, by Hölder inequality,

\[ \| Y_n(t) \| \leq \int_{t-n}^{t} M e^{-\alpha \delta} \| \phi_2(t-s) \| ds \]
\[ \leq \tilde{M} \int_{t-n}^{t-n+1} \| \phi_2(s) \| ds \]
\[ \leq \tilde{M} \left( \int_{t-n}^{t} \| \phi_2(s) \|^p ds \right)^{1/p}, \]

for some constant \( \tilde{M} > 0 \); then

\[ \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left( \sup_{\theta \in [t-r, t]} \| Y_n(\theta) \| \right) dt \]
\[ \leq \frac{\tilde{M}}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t) \left( \sup_{\theta \in [t-r, t]} \left( \int_{\theta-n}^{\theta+1} \| \phi_2(s) \|^p ds \right)^{1/p} \right) dt, \]

and hence \( Y_n \in WP_{P_0}(\mathbb{R}, X, r, \rho_1, \rho_2) \) since \( \phi_2 \in S^p WP_{P_0}(\mathbb{R}, X, r, \rho_1, \rho_2) \). By Lemma \( 2.11 \) \( (3.4) \) holds, whence \( \Gamma_1 \phi_2 \in WP_{P_0}(\mathbb{R}, X, r, \rho_1, \rho_2) \).

From \( \phi_1(t + \omega) - \phi_1(t) = 0 \) a.e. \( t \in \mathbb{R} \), one has

\[ (\Gamma_{11} \phi_1)(t + \omega) = \int_{-\infty}^{t+\omega} T(t + \omega - s)P \phi_1(s) ds = (\Gamma_{11} \phi_1)(t), \quad a.e. \ t \in \mathbb{R}. \]

Hence \( \Gamma_1 \phi \in WP_{p,\omega}(\mathbb{R}, X, r, \rho_1, \rho_2) \).

The proof of \( \Gamma_2 \phi \) is similar to that of \( \Gamma_1 \phi \), one uses \( (3.3) \) rather than \( (3.2) \). This completes the proof. \( \square \)

**Theorem 3.5.** Assume that \( (H1)-(H4) \) hold and \( g \) satisfy the conditions \( (A2)-(A3) \), if

\[ \vartheta := \left( L \frac{2ML}{\delta} + M \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) ds + M \sup_{t \in \mathbb{R}} \int_{t}^{\infty} e^{\delta(t-s)} L_g(s) ds \right) < 1, \]
Theorem 3.6. Assume that (A2)–(A3) and \( \Theta := \left( L_f + \frac{2ML_f}{\delta} + \frac{2ML_g}{1 - e^{-\delta}} \right) < 1 \), then (3.1) has a unique mild solution of \( WPP_\omega \) type.

Proof. Define \( \mathcal{F} : WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \to WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \) as

\[
(Fu)(t) = -f(t, u_t) - \int_{-\infty}^{t} A(t-s)Pf(s, u_s)ds + \int_{t}^{\infty} A(t-s)Qf(s, u_s)ds
\]

\[
+ \int_{-\infty}^{t} T(t-s)Pg(s, u_s)ds - \int_{t}^{\infty} T(t-s)Qg(s, u_s)ds, \quad t \in \mathbb{R}.
\]

If \( u \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \), then \( u \in WPP_\omega(\mathbb{R}, C, r, \rho_1, \rho_2) \) by Lemma 2.14. Therefore, by Theorem 2.15,

\[
g(s, u_s) \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \subset SPWPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2).
\]

By Lemmas 3.3 and 3.4, it is not difficult to see that \( \mathcal{F} \) is well defined.

For any \( u, v \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \), we have

\[
\|(Fu)(t) - (Fv)(t)\|
\]

\[
\leq \|f(t, u_t) - f(t, v_t)\| + \int_{-\infty}^{t} \|A(t-s)Pf(s, u_s) - f(s, v_s)\||ds
\]

\[
+ \int_{t}^{\infty} \|A(t-s)Qf(s, u_s) - f(s, v_s)\||ds
\]

\[
+ \int_{-\infty}^{t} \|T(t-s)Pg(s, u_s) - g(s, v_s)\||ds
\]

\[
+ \int_{t}^{\infty} \|T(t-s)Qg(s, u_s) - g(s, v_s)\||ds
\]

\[
\leq L_f \|u_t - u_t\|_C + ML_f \int_{-\infty}^{t} e^{-\delta(t-s)} \|u_s - u_s\|_C ds
\]

\[
+ ML_f \int_{t}^{\infty} e^{\delta(t-s)} \|u_s - u_s\|_C ds + M \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) \|u_s - u_s\|_C ds
\]

\[
+ M \int_{t}^{\infty} e^{\delta(t-s)} L_g(s) \|u_s - u_s\|_C ds
\]

\[
\leq \left( L_f + \frac{2ML_f}{\delta} + M \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) ds \right) \|u - v\|
\]

\[
\leq \|u - v\|,
\]

then \( \mathcal{F} \) is a contraction since \( \vartheta < 1 \). By the Banach contraction mapping principle, \( \mathcal{F} \) has a unique fixed point in \( WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \), which is the unique \( WPP_\omega \) solution to (3.1). \( \square \)

In (H3), if \( L_g(t) \equiv L_g \), it is not difficult to see that \( g \) satisfies the conditions (A2)–(A3) and \( \vartheta = L_f + 2M(L_f + L_g)/\delta \).

Theorem 3.6. Assume that (H1), (H2), (H3'), (H4) hold. If

\[
\Theta := \left( L_f + \frac{2ML_f}{\delta} + \frac{2ML_g}{1 - e^{-\delta}} \right) < 1,
\]

then (3.1) has a unique mild solution of \( WPP_\omega \) type.
Proof. Define the operator \( F \) as in (3.3). Let \( u \in WPP(\mathbb{R}, X, r, \rho_1, \rho_2) \), then it is not difficult to see that \( g(s, u_s) \in S^2WPP(\mathbb{R}, X, r, \rho_1, \rho_2) \) by Theorem 2.20 so \( \Gamma \) is well defined by Lemma 3.3 and Lemma 3.4.

Let \( u, v \in WPP(\mathbb{R}, X, r, \rho_1, \rho_2) \), one has

\[
\| (F u)(t) - (F v)(t) \|
\leq L_f \| u_t - v_t \|_C + ML_f \int_{-\infty}^t e^{-\delta (t-s)} \| u_s - v_s \|_C ds
\]

\[
+ ML_f \int_{-\infty}^t \| u_s - v_s \|_C ds + M \int_{-\infty}^t \| g(s, u_s) - g(s, v_s) \| ds
\]

\[
+ M \int_{-\infty}^t \| g(s, u_s) - g(s, v_s) \| ds
\]

\[
\leq L_f \| u_t - v_t \|_C + ML_f \int_{-\infty}^t e^{-\delta (t-s)} \| u_s - v_s \|_C ds
\]

\[
+ ML_f \int_{-\infty}^t e^{-\delta (t-s)} \| u_s - v_s \|_C ds + M \int_{0}^{\infty} e^{-\delta s} \| g(t - s, u_{t-s}) - g(t - s, v_{t-s}) \| ds
\]

\[
+ M \int_{-\infty}^{0} e^{\delta s} \| g(t - s, u_{t-s}) - g(t - s, v_{t-s}) \| ds
\]

\[
\leq \left( L_f + \frac{2ML_f}{\delta} \right) \| u - v \| + M \sum_{k=0}^{\infty} \int_{k}^{k+1} e^{-\delta s} \| g(t - s, u_{t-s}) - g(t - s, v_{t-s}) \| ds
\]

\[
+ M \sum_{k=-\infty}^{0} \int_{k}^{k+1} e^{\delta s} \| g(t - s, u_{t-s}) - g(t - s, v_{t-s}) \| ds
\]

\[
\leq \left( L_f + \frac{2ML_f}{\delta} \right) \| u - v \| + M \sum_{k=0}^{\infty} e^{-\delta k} \left( \int_{t-k}^{t-k+1} \| g(s, u_s) - g(s, v_s) \|^p ds \right)^{1/p}
\]

\[
+ M \sum_{k=-\infty}^{0} e^{\delta k} \left( \int_{t-k}^{t-k+1} \| g(s, u_s) - g(s, v_s) \|^p ds \right)^{1/p}
\]

\[
\leq \left( L_f + \frac{2ML_f}{\delta} + ML \sum_{k=0}^{\infty} e^{-\delta k} + M \sum_{k=-\infty}^{0} e^{\delta k} \right) \| u - v \| \leq \Theta \| u - v \|
\]

By the Banach contraction mapping principle, \( F \) has a unique fixed point in \( WPP(\mathbb{R}, X, r, \rho_1, \rho_2) \), which is the unique \( WPP \) solution to (3.1).

Remark 3.7. It is easy to see that similar results of Theorem 3.5 and Theorem 3.6 hold for \( WPP_{wap}(\mathbb{R}, X, r, \rho_1, \rho_2) \) mild solution, that is \( (3.1) \) has a unique \( WPP_{wap} \) solution, in this case, \( f \in WPP_{wap}(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \) in (H2), \( g \in WPP_{wap}(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \) in (H3), \( g \in S^2WPP_{wap}(\mathbb{R} \times C, X, r, \rho_1, \rho_2) \) in (H3')

4. Examples

In this section, we provide some examples to illustrate our main results.
Example 4.1. Consider the partial differential equation

\[
\frac{\partial}{\partial t} \left[ u(t, \xi) + \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \eta, \xi) u(t + s, \eta) \, d\eta \, ds \right] = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(t) u(t, \xi) + \int_{-\infty}^{0} a_1(s) u(t + s, \xi) \, ds, \quad (t, \xi) \in \mathbb{R} \times [0, \pi],
\]

\[u(t, 0) = u(t, \pi) = 0,\]

where \( a_0 \in WPP_\omega(\mathbb{R}, \mathbb{R}, r, \rho_1, \rho_2), \rho_1 = e^t, \rho_2 = 1 + t^2 \) and the following conditions hold:

The functions \( b(\cdot), \frac{\partial}{\partial s} b(\tau, \eta, \zeta), i = 1, 2 \) are (Lebesgue) measurable, \( b(\tau, \eta, 0) = b(\tau, \eta, \pi) = 0 \) for every \( (\tau, \eta) \) and

\[N_1 := \max \left\{ \int_{0}^{\pi} \int_{-\infty}^{0} \int_{0}^{\pi} \left( \frac{\partial}{\partial \xi^i} b(\tau, \eta, \zeta) \right)^2 \, d\eta \, d\zeta, : i = 0, 1, 2 \right\} < \infty.\]

Under these conditions, let \( X = (L^2([0, \pi], \mathbb{R}), \| \cdot \|_{L^2}) \) and define the operator \( A \) on \( X \) by \( Au = u'' \) with

\[D(A) = \{ u \in X : u'' \in X, u(0) = u(\pi) = 0 \}.\]

It is well known that \( A \) is the infinitesimal generator of a semigroup \((T(t))_{t \geq 0}\) on \( X \) such that \( \|T(t)\| \leq e^{-t} \) for every \( t \geq 0 \).

Define the functions \( f, g : \mathbb{R} \times \mathcal{C} \to X \) by

\[f(t, \psi)(\xi) := \int_{-\infty}^{0} \int_{0}^{\pi} b(s, \eta, \xi) \psi(s, \eta) \, d\eta \, ds,
\]

\[g(t, \psi)(\xi) := a_0(t) \psi(0, \xi) + \int_{-\infty}^{0} a_1(s) \psi(s, \xi) \, ds,
\]

then (4.1) can be rewritten as an abstract system of the form (3.1), where \( u(t) = u(t, \cdot) \). By a straightforward estimation that uses (i), one can show that \( f \) is \( D(A) \)-valued, and the following hold:

\[\|Af(t, \cdot)\| \leq (N_1 r)^{1/2}, \quad t \in \mathbb{R},\]

\[\|g(t, \cdot)\| \leq \|a_0\| + r^{1/2} \left( \int_{-\infty}^{0} a_1^2(s) \, ds \right)^{1/2}, \quad t \in \mathbb{R}.
\]

See [3, 5] for more details. The next result is a consequence of Theorem 3.5.

Theorem 4.2. Under the previous assumptions, (4.1) as a unique weighted pseudo periodic solution whenever

\[3\sqrt{N_1 r} + 2\|a_0\| + 2r^{1/2} \left( \int_{-\infty}^{0} a_1^2(s) \, ds \right)^{1/2} < 1.
\]

Example 4.3. Consider the following scalar reaction-diffusion equation with delay

\[
\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + g(t, u(t - r, x)),
\]

\[u(t, 0) = u(t, \pi) = 0,
\]

\[u(\tau, x) = \varphi(\tau, x), \quad \tau \in [-r, 0], \ x \in [0, \pi],\]

where \( g = g_1 + g_2 \in SP^p WPP_\omega(\mathbb{R} \times \mathcal{C}, \mathbb{R}, r, \rho_1, \rho_2) \) with \( g_1 \in P_\omega(\mathbb{R} \times \mathcal{C}, \mathbb{R}), g_2 \in SP^p WPP_0(\mathbb{R} \times \mathcal{C}, \mathbb{R}, r, \rho_1, \rho_2), \rho_1 = e^t, \rho_2 = 1 + t^2.\)
By Theorem 3.6, one has the following result.

**Theorem 4.4.** Assume that there exists a positive constant $L_g$ such that for $i = 1, 2$,

$$\left( \int_t^{t+1} \|g(t, \psi_1) - g(t, \psi_2)\|^p ds \right)^{1/p} \leq L_g \|\psi_1 - \psi_2\|_C, \quad \text{for all } t \in \mathbb{R}, \psi_i \in C.$$ 

Then there exists a unique $WPP_\omega$ solution of $\{1.2\}$ if $2L_g < 1 - e^{-1}$.

**Acknowledgements.** This research was supported by Zhejiang Provincial Natural Science Foundation of China under Grant No. LQ13A010015.

**References**


Zhinan Xia
Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou, Zhejiang, 310023, China
E-mail address: xiazn299@zjut.edu.cn