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IMAGE RESTORATION USING A REACTION-DIFFUSION PROCESS

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Abstract. This study shows how partial differential equations can be employed to restore a digital image. It is in fact a generalization of the work presented by Catté [12], which modify the Perona-Malik Model by nonlinear diffusion. We give a demonstration of the consistency of the reaction-diffusion model proposed in our work.

1. Introduction

Image processing is always a challenging problem; this topic has become “hot” in recent years and a very active field of computer applications and research [14]. Various techniques have been developed in Image Processing during the last four to five decades, the use of these techniques has exploded and they are now used for all kinds of tasks in all kinds of areas: artistic effects, medical visualization, industrial inspection, human computer interfaces, etc. One of the most active topics in this field has been restoration of images, as can be ascertained from recent survey papers [4, 5]. A number of different techniques have been proposed for digital image restoration, utilizing a number of different models and assumptions. The restoration of degraded images is an important problem because it allow to recovery lost information from the observed degraded image data. Two kinds of degradations are usually encountered: spatial degradations (e.g., the loss of resolution) caused by blurring and point degradation (e.g., additive random noise), which affect only the gray levels of the individual picture points. Image is restored to its original quality by inverting the physical degradation phenomenon such as defocus, linear motion, atmospheric degradation and additive noise. Partial differential equation (PDE) methods in image processing have proven to be fundamental tools for image diffusion and restoration [4, 5, 6, 9, 24, 35]. The Perona-Malik equation [25], proposed in 1987, is one of the first attempts to derive a model that incorporates local information from an image within a PDE framework. It has stimulated a great deal of interest in image processing community [5, 34]. A nonlinear diffusion model (which they called ‘anisotropic’) was conducted by Perona and Malik in order to avoid the blurring of edges and other localization problems presented by

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linear diffusion models, they apply a diffusion process whose diffusivity is steered by derivatives of the evolving image. The model proposes a nonlinear diffusion method for avoiding the blurring and localization problems of linear diffusion filtering [24, 25] by applying a process that reduces the diffusivity in places having higher likelihood of being edges. This likelihood is measured by a function of the local gradient. Unfortunately, it was shown by Kichenassamy [19] that the basic Perona-Malik PDE model is ill-posed in the sense of Hadamard. It was shown by Kawohl and Kutev that the equation may have no global weak solutions in C^1 [17]. Zhang [36] established that the one-dimensional Perona-Malik equation admits infinitely many weak solutions. Höllig [17] constructed a forward-backward diffusion process which can have infinitely many solutions, his study has become a pessimistic results about the well-posedness of the Perona-Malik equation. In 1992, Lions and Alvarez [4, 5] offered an interesting nonlinear form of restoration equation with solving the Perona-Malik equation with a finite difference method. Although the basic model is ill-posed, its discretizations are found to be stable, this fact is sometimes referred to as the Perona-Malik paradox [19]. The explanation for these observations was given by Weickert and Benhamouda [34], who investigated the regularizing effect of a standard finite difference discretization. This observation motivated much research towards the introduction of the regularization directly into the PDE to avoid the dependence on the numerical schemes [12, 22]. A variety of spatial, spatio-temporal, and temporal regularization procedures have been proposed over the years [10, 12, 20, 28, 32, 33]. The one that has attracted much attention is the mathematically sound formulation in 1992 by Catté, Lions, Morel and Coll [12]. They suggested introducing the regularization in space and time directly into the continuous equation in order to obtain a related well-posed model which becomes more independent of the numerical implementation which causes critically dependence between the dynamics of the solution and the sort of regularization procedure. They proposed to replace the diffusivity $g(|\nabla u|^2)$by a slight variation $g(|\nabla u_\sigma|^2)$ in the Perona-Malik equation, with $u_\sigma = G_\sigma * u$, where G_σ is a smooth kernel (Gaussian of variance σ^2). Since differentiation is highly susceptible to noise. They prove existence, uniqueness and regularity for the related model and demonstrate experimentally that the related model gives similar results to the Perona-Malik equation [25]. In 2006, the study of Morfu [21] was focused on the contrast enhancement and noise filtering, he considers the Fisher equation which generally allows simulating the transport mechanisms in living cells, but also enhances the contrast and segmenting images. The model proposed by Morfu is:

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}(g(|\nabla u|)\nabla u) = f(u) \quad \text{in } Q_T, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \upsilon} = 0 \quad \text{on } \Sigma_T,\end{gathered} \tag{1.1}$$

where Ω is the domain of the image, $T > 0$, u_0 is the original image to be processed and $f(s) = s(s-a)(1-s)$ with $0 < a < 1$. The Major defects of this model are: (1) Sensitivity to noise; If we increase slightly the noise, the method gives unsatisfactory results because the image noise causes severe oscillations of the gradient and the model keeps the noise that considers edges. (2) No results of existence and consistency. To overcome this problem, we propose an improved algorithm which will be able to resist to noise and which can improve the contrast and noisy images.

The aim of our work is to modify the model of Morfu [21] by applying a Gaussian filter on the gradient of the noisy image during the calculation of the coefficient of anisotropic diffusion. The proposed model is as follows:

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}(g(|\nabla u_\sigma|)\nabla u) = f(t,x,u) \quad \text{in } Q_T, \\ u(0,x) = u_0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \upsilon} = 0 \quad \text{on } \Sigma_T\,.\end{gathered} \tag{1.2}$$

Here $\Omega =]0,1[\times]0,1[$ denotes picture domain with boundary $\partial\Omega$, with Neumann boundary conditions. Where $u(t,x)$ is the solution of this PDE (restored image) we are searching for, this solution is depended on two parameters; the scale parameter denoted by t and the spatial coordinate x. υ is an outward Normal to domain Ω and u_0 is the original image to be processed. $Q_T =]0,T[\times\Omega$ and $\Sigma_T =]0,T[\times\partial\Omega$ where T is a fixed reel number ($T > 0$). Let $\sigma > 0$, G_σ is the Gaussian filter where:

$$G_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(\frac{|x|^2}{4\sigma})}, \quad x \in \mathbb{R}^2. \tag{1.3}$$

We consider the gradient norm of w as:

$$|\nabla w| = \Big(\sum_{i=1}^{i=2} (\frac{\partial w}{\partial x_i})^2\Big)^{1/2},$$

∇w_σ is the smoothed version of gradient norm where w: $\nabla w_\sigma := \nabla(w * G_\sigma) = w * \nabla G_\sigma$. The Diffusivity g is a smooth decreasing function defined by $g : [0,+\infty[\to [0,+\infty[$ where $g(0) = 1$, and $\lim_{s\to\infty} g(s) = 0$, one of the diffusivities Perona and Malik proposed is

$$g(s) = \frac{d}{\sqrt{1 + \eta(\frac{s}{\lambda})^2}}\,, \tag{1.4}$$

where $\eta \geq 0$, $d > 0$ and λ is a threshold (contrast) parameter that separates forward and backward diffusion [33]. The nonlinearity f has no limitation of increasing. We assume that the initial data satisfy $0 \leq u_0(x)$, and for f we introduce the following assumptions:

$$f : Q_T \to \mathbb{R} \text{ is measurable and } f(t,x,.) : \mathbb{R} \to \mathbb{R} \text{ is continuous.} \tag{1.5}$$

In addition, we give here the following main properties of f:

- the positivity of the solution u of (1.1) is preserved over time, which is ensured by:

$$\text{for almost } (t,x) \in Q_T,\ f(t,x,0) \geq 0; \tag{1.6}$$

- the total mass is controlled in function of time:

$$\text{for all } u \in \mathbb{R} \text{ and for almost } (t,x) \in Q_T,\ uf(t,x,u) \leq 0. \tag{1.7}$$

The special case $f = 0$ was treated by Catté [12], where they considered the problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}(g(|\nabla u_\sigma|)\nabla u) = 0 \quad \text{in } Q_T, \\ u(0,x) = u_0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \upsilon} = 0 \quad \text{on } \Sigma_T\,.\end{gathered} \tag{1.8}$$

They established the existence, uniqueness and regularity of a solution for $\sigma > 0$ and $u_0 \in L^2(\Omega)$. This study is devoted to a generalization of their work in the case where f is nonzero. Note that if the diffusion coefficient is constant $g(s) = d$ (which corresponds to the situation where $\eta = 0$), the existence of positive global solutions have been obtained by several authors [13, 16, 29]. When $u_0 \in L^1(\Omega)$, only Pierre [26] proves the existence of global weak positive solutions. In all these works, the hypothesis (1.7) plays an important role in study of this diffusion-reaction equation. Indeed, if (1.6) is not satisfied, [23] proved the explosion in finite time of the solutions.

This work began with an introduction where we describe briefly the nonlinear diffusion model proposed by Catté [12] applied in image processing for restoration and which serves as background for our proposed model generalization. This is followed by a concept definition of solution used here and we present the main results of this work. The next section describes the global existence of our reaction diffusion equation; this is done in three steps: the first step is to truncate the equation and shows that the problem obtained has a solution. In the second step we establish appropriate estimates on the approximate solutions. In the last step, we show the convergence of the approximate system. We use a new technique recently introduced by Pierre [27] for study of semi-linear isotropic systems. Our results are a generalization of these results in the case of anisotopique reaction diffusion equation firstly introduced by [12] in the case of the equations without reaction term.

Now we will recall some functional spaces that will be used throughout this paper. For all $k \in \mathbb{N}$, $H^k(\Omega)$ is the set of functions u defined in Ω such as u and its order $D^s u$ derivatives where $|s| = \sum_{j=1}^n s_j \le k$ are in $L^2(\Omega)$. $H^k(\Omega)$ is a Hilbert space for the norm

$$\|u\|_{H^k(\Omega)} = \Big(\sum_{|s|\le k} \int_\Omega |D^s u|^2 dx \Big)^{1/2}. \tag{1.9}$$

We denote by $(H^1(\Omega))'$ the dual of $H^1(\Omega)$.

$L^p(0,T,H^k(\Omega))$ is the set of functions u such that, for all every $t \in (0,T)$, $u(t)$ belongs to $H^k(\Omega)$ with the norm

$$\|u\|_{L^p(0,T;H^k(\Omega))} = \Big(\int_0^T \|u(t)\|^p_{H^k(\Omega)} dt \Big)^{1/p}, \quad 1 < p < \infty,\ k \in \mathbb{N}. \tag{1.10}$$

$L^\infty(0,T;L^2(\Omega))$ is the set of functions u such that, for all every $t \in (0,T))$, $u(t)$ belongs to $L^2(\Omega)$ with the norm

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} = \Big(\sup_{0<t<T} \|u(t)\|^2_{L^2(\Omega)} \Big)^{1/2}. \tag{1.11}$$

$L^\infty(0,T;\mathcal{C}^\infty(\Omega))$ is the set of functions u such that, for all every $t \in (0,T)$, $u(t)$ belongs to $\mathcal{C}^\infty(\Omega)$ with the norm

$$\|u\|_{L^\infty(0,T;\mathcal{C}^\infty(\Omega))} = \inf \big\{ c,\ \|u(t)\|_{\mathcal{C}^\infty(\Omega)} \le c \text{ sur } (0,T) \big\}. \tag{1.12}$$

2. Consistency of the model: Existence and uniqueness results

2.1. **Assumptions.** Firstly, it must be specified the direction in which we want to solve the problem (1.1).

Definition 2.1. A function u is a weak solution of (1.1) if

$$\begin{gathered} u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad f(t,x,u) \in L^1(Q_T), \\ \text{for all } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0, \\ \int_{Q_T} -u\frac{\partial \varphi}{\partial t} + g(|\nabla u_\sigma|)\nabla u \nabla\varphi = \int_{Q_T} f(t,x,u(t))\varphi + \int_\Omega u_0\varphi(0,x) \end{gathered} \tag{2.1}$$

If moreover $u \in C^1(Q_T)$ then we say that u is a classical solution of (1.1).

2.2. **Main result.** Our main result in this paper is the following existence theorem.

Theorem 2.2. *Assume that* (1.5)–(1.7) *and that for all* $R \geq 0$*,*

$$\sup_{|u|\leq R} (|f(t,x,u)|) \in L^1(Q_T)\,. \tag{2.2}$$

Then for all fixed $T > 0$ *and* $\sigma > 0$ *and for any* $0 \leq u_0 \in L^2(\Omega)$ *such as* $u_0 \geq 0$*, problem* (2.1) *admits a weak positive solution.*

If moreover for all $r \geq 1$ $f(t,x,r) \leq 0$ *and* $u_0(x) \leq 1$*, we have* $0 \leq u(t,x) \leq 1$ *in* Q_T*.*

Remark 2.3. A typical example when the result of this paper can be applied is the Ficher equation outcome the population dynamics

$$f(t,x,u) = -\beta u(u-a)^{2\alpha}(1-u) \tag{2.3}$$

where $\alpha, \beta > 0$ and $0 < a < 1$.

The proof of Theorem (2.2) is done in four steps:

Step 1: Positivity of the solutions: Consider the function

$$\operatorname{sign}^-(r) = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r \geq 0 \end{cases} \tag{2.4}$$

We build a sequence of convex functions $j_\varepsilon(r)$ such as $j'_\varepsilon(r)$ is bounded and for all $r \in \mathbb{R}$, $j'_\varepsilon(r) \to \operatorname{sign}^-(r)$ when $\varepsilon \to 0$.

Let u be a solution of (2.1), we multiply both sides of the first equation by $j'_\varepsilon(u)$ and by integrating on $Q_t =]0,t[\times\Omega$ for $t \in [0,T[$, we obtain

$$\int_{Q_t} j'_\varepsilon(u)\frac{\partial u}{\partial t}\,dx\,dt + \int_{Q_t} A\nabla u.\nabla j'_\varepsilon(u)\,dx\,dt = \int_{Q_t} f(s,x,u)j'_\varepsilon(u)\,dx\,ds \tag{2.5}$$

where $A(t,x) = g(|\nabla u_\sigma|) \in L^\infty(0,T;\mathcal{C}^\infty(\Omega))$ because $u \in L^\infty(0,T;L^2(\Omega))$ and g, G_σ are C^∞ and we can show the existence of a C_0 depends only on $G_\sigma, \|u_0\|_{L^2(\Omega)}$ such as:

$$\|\nabla u_\sigma\|_{L^\infty(Q_T)} \leq C_0\,. \tag{2.6}$$

Moreover as g is decreasing, then there $a = g(C_0) > 0$ which depends only on σ and on $\|u_0\|_{L^2(\Omega)}$ such as:

$$A(t,x) \geq a \quad \forall (t,x) \in Q_T\,. \tag{2.7}$$

Consequently,

$$\int_\Omega [j_\varepsilon(u)(t) - j_\varepsilon(u)(0)]dx + a\int_{Q_t} |\nabla u|^2 j''_\varepsilon(u)\,ds\,dx \leq \int_{Q_t} f(s,x,u)j'_\varepsilon(u)\,dx\,ds\,. \tag{2.8}$$

Since $\int_\Omega j_\varepsilon(u)(0)dx = 0$ and $\int_{Q_t} |\nabla u|^2 j''_\varepsilon(u)\,dx\,ds \geq 0$ then we have

$$\begin{aligned}\int_\Omega j_\varepsilon(u)(t)dx &\leq \int_{Q_T} f(s,x,u)j'_\varepsilon(u)\,dx\,ds \\ &\leq \int_{[u<0]} f(s,x,u)j'_\varepsilon(u)\,dx\,ds + \int_{[u\geq 0]} f(s,x,u)j'_\varepsilon(u)\,dx\,ds\end{aligned}$$

On the set where $u \geq 0$ we have $j'_\varepsilon(u) = 0$ and $\int_{[u\geq 0]} f(s,x,u)j'_\varepsilon(u)\,dx\,ds = 0$; therefore

$$\int_\Omega j_\varepsilon(u)(t)dx \leq \int_{[u<0]} f(s,x,u)j'_\varepsilon(u)\,dx\,ds\,. \tag{2.9}$$

When $\varepsilon \to 0$, we obtain

$$\int_\Omega (u)^-(t)dx \leq -\int_{[u\leq 0]} f(s,x,u)\,dx\,ds\,. \tag{2.10}$$

Using (1.7) and the fact that $(u)^-(t) \geq 0$, we obtain $(u)^-(t) = 0$ on Ω; therefore $u \geq 0$ in Q_T.

Step 2: An existence result when f is bounded:

Theorem 2.4. *Assume* (1.6)–(1.5)*, and that there exists $M \geq 0$ such as for almost $(t,x) \in Q_T$ and all $r \in \mathbb{R}$,*

$$|f(t,x,r)| \leq M\,. \tag{2.11}$$

Then for all $u_0 \in L^2(\Omega)$, problem (2.1) *admits a weak solution. Moreover, there exists $C = C(M,a,T,\|u_0\|_{L^2(\Omega)})$ such that*

$$\sup_{0<t<T} \|u(t)\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H^1(\Omega))} \leq C\,. \tag{2.12}$$

Proof. We will show the existence of a weak solution by the classical Schauder fixed point theorem. Firstly we introduce the space

$$\mathcal{W}(0,T) = \left\{v \in L^2(0,T;H^1(\Omega)) : \frac{dv}{dt} \in L^2(0,T;(H^1(\Omega))')\right\} \tag{2.13}$$

which is a Hilbert space for the graph norm. Let $v \in \mathcal{W}(0,T) \cap L^\infty(0,T;L^2(\Omega))$ and we consider $u(v)$ the solution of the linear problem

$$\begin{gathered} u(v) \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\ \text{for all } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0, \\ \int_{Q_T} -u(v)\frac{\partial\varphi}{\partial t} + g(|\nabla v_\sigma|)\nabla u(v)\nabla\varphi = \int_{Q_T} f(t,x,v(t))\varphi + \int_\Omega u_0\varphi(0,x) \end{gathered} \tag{2.14}$$

According to the classical theory [7, 11], equation (2.14) admits a unique solution $u(v) \in \mathcal{W}(0,T)$ moreover by applying a classic bootstrap argument, we have $u(v)(t) \in H^1(\Omega)$ for all $t > 0$; since $f(t,x,v(t)) \in L^\infty(Q_T)$, then $u(v)(t) \in H^1(\Omega)$ for all $t > 0$. Therefore by iteration and by application the general classical theory another time [36], we deduce that $u(v)$ is a classical solution and $u(v) \in C^\infty(]0,T[\times\Omega)$. We take $\varphi = u(v)$ in (2.14), and deduce that for all $0 < t < T$,

$$\frac{1}{2}\int_\Omega u(v)^2(t) + \int_{Q_t} g(|\nabla v_\sigma|)|\nabla u(v)|^2 = \int_{Q_t} f(t,x,v(t))u(v) + \frac{1}{2}\int_\Omega u_0^2 dx \tag{2.15}$$

Using (2.7) and the assumption (2.11) on f, we obtain

$$\frac{1}{2}\int_\Omega u(v)^2(t) + a\int_{Q_t} |\nabla u(v)|^2 \le M(1+\int_{Q_t} u(v)^2) + \frac{1}{2}\int_\Omega u_0^2 dx\,. \tag{2.16}$$

Now by Gronwall's lemma, we obtain the estimation (2.12). These estimates lead us to introduce the space

$$\begin{aligned}\mathcal{W}_0(0,T) = \Big\{& v \in \mathcal{W}(0,T)\cap L^\infty(0,T;L^2(\Omega)) : v(0) = u_0 \text{and} \\ &\sup_{0<t<T} \|u(t)\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H^1(\Omega))} \le C\,,\end{aligned}$$

where $C = C(M, a, T, \|u_0\|_{L^2(\Omega)})$ is the constant obtained in (2.12). □

We can easily verify that $\mathcal{W}_0(0,T)$ is a nonempty closed convex in $\mathcal{W}(0,T)$, moreover it injects with a compact way in $L^2(0,T;L^2(\Omega))$. Then we define the application:

$$\begin{gathered} F : \mathcal{W}_0(0,T) \to \mathcal{W}_0(0,T) \\ v \mapsto F(v) = u(v), \text{ where } u \text{ is a solution of (2.14)}\,. \end{gathered} \tag{2.17}$$

Estimate (2.11) shows that F is well defined. To apply the Schauder fixed point theorem, we show that F is weakly continuous from $\mathcal{W}_0(0,T)$ in $\mathcal{W}_0(0,T)$.

Then consider a sequence (v_n) in $\mathcal{W}_0(0,T)$, such as $v_n \rightharpoonup v$ in $\mathcal{W}_0(0,T)$, and let $u_n = F(v_n)$. According to the classical results of compactness, we can extract from the sequence (u_n) a subsequence yet denoted (u_n) such that

- $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1(\Omega))$
- $u_n \rightharpoonup u$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T
- $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(0,T;L^2(\Omega))$
- $v_n \rightharpoonup v$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T
- $\nabla G_\sigma * v_n \rightharpoonup \nabla G_\sigma * v$ strongly in $L^2(0,T;L^2(\Omega))$ and almost everywhere in Q_T
- $g(|\nabla G_\sigma * v_n|) \rightharpoonup g(|\nabla G_\sigma * v|)$ strongly in $L^2(0,T;L^2(\Omega))$
- $f(t,x,v_n) \to f(t,x,v)$ strongly in $L^1(Q_T)$

The latter is obtained by applying the dominated convergence theorem. We can then pass to the limit in (2.14), with v_n instead of v, and obtain that $u = F(v)$. By uniqueness of the solution of (2.14), then the sequence $u_n = F(v_n)$ converges weakly to $u = F(v)$ in $\mathcal{W}_0(0,T)$. We deduce the existence of $u \in \mathcal{W}_0(0,T)$ such as $u = F(u)$, and thus the existence of $u \in \mathcal{W}(0,T)$ such us $u = U$.

Step 3: Approximate problem and a priori estimates Consider the truncation function $\Psi_n \in \mathcal{C}_0^\infty(\mathbb{R})$ defined by

$$\Psi_n(r) = \begin{cases} 1 & \text{if } |r| \le n, \\ 0 & \text{if } |r| \ge n+1\,. \end{cases} \tag{2.18}$$

We truncate the nonlinearity f by Ψ_n,

$$f_n(t,x,u) = \Psi_n(|u|) f(t,x,u). \tag{2.19}$$

Thus, we can easily check that f_n satisfies (1.6), (1.5), (1.7) with $M = M(n)$ and for almost $(t,x) \in Q_T$, for all $r \in \mathbb{R}$ $f_n(t,x,u) \to f(t,x,r)$.

Since $u_0 \in L^2(\Omega)$ and $|f_n(t,x,r)| \leq M_n$, theorem (2.11) is applied, then we can deduce the existence of a weak solution of the problem

$$\begin{gathered}\frac{\partial u_n}{\partial t} - \operatorname{div}(g(|\nabla(u_n)_\sigma|)\nabla u_n) = f_n(t,x,u_n) \quad \text{in } Q_T, \\ u_n(0,x) = u_0 \quad \text{on } \Omega, \\ \frac{\partial u_n}{\partial \upsilon} = 0 \quad \text{on } \Sigma_T\,.\end{gathered} \tag{2.20}$$

Remark 2.5. Since $u_0 \geq 0$ on Ω, the (i) assures that $u_n \geq 0$ is in Q_T. Moreover, under the assumption (1.7) we have also $f_n(t,x,u_n) \leq 0$ in Q_T.

Now we will show that a subsequence u_n converges to the weak solution u of problem (1.1). For this we need to prove the following result:

Lemma 2.6. *Let (u_n) the sequence of weak solutions defined by (2.12), then we have:*

(i) $\int_{Q_T} |f_n(t,x,u_n)| \leq \int_\Omega |u_0| dx$,

(ii) (u_n) *is bounded in* $L^2(0,T;H^1(\Omega))$ *and*

$$\int_{Q_T} |u_n f_n(t,x,u_n)|\, dx\, dt \leq \frac{1}{2}\int_\Omega u_0^2 dx,$$

(iii) (u_n) *is relatively compact in* $L^2(Q_T)$.

Proof. (i) By Remark 2.5, $|f_n(t,x,u_n)| = -f_n(t,x,u_n)$. Thus by integrating the equation satisfied by u_n in Q_T we obtain

$$\int_\Omega u_n(T) dx - \int_{Q_T} f_n(t,x,u_n)\, dx\, dt = \int_\Omega u_0\, dx\,; \tag{2.21}$$

therefore

$$\int_{Q_T} |f_n(t,x,u_n)|\, dx\, dt \leq \int_\Omega |u_0| dx\,. \tag{2.22}$$

(ii) Firstly we show that u_n is bounded in $L^2(Q_T)$, for this we consider $\varphi = u_n$ as a function test in (2.20), we then deduce that

$$\frac{1}{2}\int_\Omega u_n^2(t) + \int_{Q_t} g(|\nabla(u_n)_\sigma|)|\nabla u_n|^2 = \int_{Q_t} f(t,x,u_n)u_n + \frac{1}{2}\int_\Omega u_0^2 dx\,. \tag{2.23}$$

Then we use (2.7) and the hypothesis (2.8) on f to obtain

$$\frac{1}{2}\int_\Omega u_n^2(t) + a\int_{Q_t} |\nabla u_n|^2 \leq \frac{1}{2}\int_\Omega u_0^2 dx\,. \tag{2.24}$$

We have also

$$\int_{Q_T} u_n |f_n(t,x,u_n)|\, dx\, dt \leq \frac{1}{2}\int_\Omega u_0^2\, dx\,, \tag{2.25}$$

where we have

$$\sup_{0<t<T} \|u_n(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)},$$

$$\|u_n\|_{L^2(0,T;H^1(\Omega))} \leq (1 + \frac{1}{2a})\|u_0\|_{L^2(\Omega)}$$

(iii) Since $\frac{\partial u_n}{\partial t} = \operatorname{div}(A_n \nabla u_n) + f_n(t,x,u_n)$ is bounded in $L^1(0,T;(H^1(\Omega))') + L^1(\Omega)$. Since u_n is also bounded in $L^2(0,T;H^1(\Omega))$ and that the injection of $H^1(\Omega)$ in $L^2(\Omega)$ is compact, it follows that (u_n) is relatively compact in $L^2(Q_T)$ [31]. $\square$

Step 4: Convergence According to (iii), the sequence (u_n) is relatively compact in $L^2(Q_T)$, so we can extract a subsequence still denoted (u_n) such that

- $u_n \rightharpoonup u$ strongly in $L^2(Q_T)$ and almost everywhere in Q_T,
- $\nabla G_\sigma * u_n \rightharpoonup \nabla G_\sigma * u$ strongly in $L^2(Q_T)$ and almost everywhere in Q_T.
- $g(|\nabla G_\sigma * u_n|) \rightharpoonup g(|\nabla G_\sigma * u|)$ strongly in $L^2(Q_T)$
- $f_n(t,x,u_n) \to \ f(t,x,u)$ for almost everywhere in Q_T

To prove that u is a weak solution of (1.1), it suffices to prove that $f_n(t,x,u_n) \to f(t,x,u)$ in $L^1(Q_T)$. Since $f_n(t,x,u_n) \to f(t,x,u)$ almost everywhere in Q_T, we will demonstrate that $(f_n(t,x,u_n))$ is uniformly integrable in $L^1(Q_T)$. For this we show that: for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $E \subset Q_T$ measurable with $|E| < \delta$, we have

$$\int_E |f_n(t,x,u_n)| dx \le \varepsilon\,. \tag{2.26}$$

Then for all $k \ge 0$,

$$\int_E |f_n(t,x,u_n)| dx \le \int_{E\cap[u_n\le k]} |f_n(t,x,u_n)| dx + \int_{E\cap[u_n>k]} |f_n(t,x,u_n)| dx\,. \tag{2.27}$$

For the first term on the right-hand side, we have

$$\int_{E\cap[u_n\le k]} |f_n(t,x,u_n)| dx \le \int_E \sup_{|r|\le k} (|f(t,x,r)| dx\,. \tag{2.28}$$

According to (2.2), we have $\sup_{|u|\le k}(|f(t,x,u)| \in L^1(Q_T)$ is uniformly integrable in $L^1(Q_T)$, therefore for each $\varepsilon > 0$ there exist $\delta > 0$ such that if $|E| < \delta$ then

$$\int_E \sup_{|u|\le k} (|f(t,x,u)| dx \le \frac{\varepsilon}{2}\,. \tag{2.29}$$

For the second term we have

$$\int_{E\cap[u_n>k]} |f_n(t,x,u_n)| dx \le \frac{1}{k} \int_{Q_T} u_n |f_n(t,x,u_n)| dx\,. \tag{2.30}$$

Then, using (2.25) we obtain

$$\int_{E\cap[u_n>k]} |u_n f_n(t,x,u_n)| dx \le \frac{1}{2k} \|u_0\|^2_{L^2(\Omega)}\,.$$

Now if we choose $k \ge \|u_0\|^2_{L^2(\Omega)}/\varepsilon$, then we have

$$\int_{E\cap[u_n>k]} |f_n(t,x,u_n)| dx \le \frac{\varepsilon}{2}; \tag{2.31}$$

consequently, (2.26) follows from (2.29) and (2.31).

Using the following lemma, we complete the proof of Theorem 2.2.

Lemma 2.7. *Let u be a weak solution of* (2.1)*, and assume that* $0 \le u_0 \le 1$ *in* Ω*. Then* $0 \le u \le 1$ *in* Q_T*.*

Proof. We have already obtained the positivity of weak solutions if the initial data is positive. So, we assume that $u_0 \le 1$ and proof that $u \le 1$. For this, we take

$\bar{u} = 1 - u$, then we have $\nabla\bar{u} = \nabla u$, we can verify that $\bar{u}$ satisfies

$$
\begin{gathered}
\bar{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad f(t,x,1-\bar{u}) \in L^1(Q_T), \\
\text{for all } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0, \\
\int_{Q_T} -\bar{u}\frac{\partial\varphi}{\partial t} + g(|\nabla\bar{u}_\sigma|)\nabla\bar{u}\nabla\varphi = \int_{Q_T} f(t,x,1-\bar{u}(t))\varphi - \int_\Omega u_0\varphi(0,x)\,.
\end{gathered}
\tag{2.32}
$$

Then we consider the sequence of convex functions $j_\varepsilon(r)$ such as $j'_\varepsilon(r)$ is bounded and for all $r \in \mathbb{R}$, $j'_\varepsilon(r) \to \operatorname{sign}^-(r)$ when $\varepsilon \to 0$. We take $\varphi = j'_\varepsilon(\bar{u})$ as a test function in (2.32) and integrating with respect to $t \in]0,T[$, we obtain

$$
-\int_\Omega j_\varepsilon(\bar{u})(t,x)dx \le \int_0^t \int_\Omega f(t,x,1-\bar{u})j'_\varepsilon(\bar{u})\,dx\,dt\,. \tag{2.33}
$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$
-\int_\Omega (\bar{u})^-(t,x)dx \le \int_0^t \int_{[u\ge 1]} f(t,x,u)\,dx\,dt\,. \tag{2.34}
$$

Using that for all $r \ge 1$, $f(t,x,r) \le 0$, we deduce

$$
\int_\Omega (\bar{u})^-(t,x)dx \ge 0; \tag{2.35}
$$

Therefore $\bar{u}(t) \ge 0$ which implies $u = 1 - \bar{u} \le 1$. $\square$

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