LOWER BOUNDS FOR THE BLOWUP TIME OF SOLUTIONS TO A NONLINEAR PARABOLIC PROBLEM

HAIXIA LI, WENJIE GAO, YUZHU HAN

Abstract. In this short article, we study the blow-up properties of solutions to a parabolic problem with a gradient nonlinearity under homogeneous Dirichlet boundary conditions. By constructing an auxiliary function and by modifying the first order differential inequality technique introduced by Payne et al., we obtain a lower bound for the blow-up time of solutions in a bounded domain \( \Omega \subset \mathbb{R}^n \) for any \( n \geq 3 \). This article generalizes a result in [16].

1. Introduction

When dealing with a parabolic problem there are several interesting features to analyze, one of which is the so called finite time blow-up. The question of blow-up of solutions to nonlinear parabolic equations and systems has received considerable attention since the elegant work of Fujita [6]. We refer to the interested readers the survey papers [2, 7, 10] and the book [17].

In practical situations, one would like to know, among other things, whether the solutions blow up, and if so, at what time \( T \) blow-up occurs. However, when the solution does blow up at some finite \( T \), this time can seldom be determined explicitly, and much effort has been devoted to the calculation of bounds for \( T \). Most of the methods used until recently can only yield upper bounds for \( T \), which are of little value in particular situations when blow-up has to be avoided. By using the first-order differential inequality technique, lower bounds for the blow-up time of solutions to semilinear heat equations under different boundary conditions and suitable constraint on the data were obtained by Payne et al. [12, 13, 14, 16]. Thereafter, the differential inequality technique was successfully employed to derive lower bounds for the blow-up time of solutions to other parabolic problems, see [1, 3, 5, 11, 15].

In this article, we shall study a parabolic problem with a gradient nonlinearity of the following form

\[
\begin{align*}
u_t &= \Delta u + u^p - |\nabla u|^q, \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x) \geq 0, \quad x \in \Omega,
\end{align*}
\]

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where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta$ and $\nabla$ are the Laplace and gradient operator with respect to $x$, respectively, $T$ is the possible blow-up time and $p, q > 1$ are fixed (finite) parameters. In [4, 9], conditions on $p, q$ and $u_0(x)$ were given for which the solutions to (1.1) would blow up in finite time. In fact the restrictions on $p$ and $q$ were

$$1 < p < \frac{n+2}{n-2}, \quad 1 < q < \frac{2p}{p+1}, \quad \text{for } n \geq 2,$$

or

$$p \text{ is large enough and } q = \frac{2p}{p+1}, \quad \text{for } n = 1.$$

In a recent paper Payne et al. [16] obtained lower bounds of the blow-up time of solutions to (1.1) when $n = 3$. Naturally, we hope to obtain the lower bounds for blow-up time of solutions to (1.1) with any smooth bounds $\Omega \subset \mathbb{R}^n$ and any $n \geq 3$. That is what we will do in this article.

As indicated in [18] it is well known that if $p \leq q$ the solution will not blow up in finite time. Also it is well known that if the initial data are small enough the solution will actually decay exponentially as $t \to \infty$ (see e.g. [14, 19]). Since we are interested in a lower bound for the blow-up time $T$, only the case $p > q$ is considered.

2. A LOWER BOUND FOR THE BLOW-UP TIME

In this section we seek a lower bound for the blow-up time $T$ of solutions to (1.1) in some appropriate measure. The idea of the proof of the following theorem is inspired by that in [1].

**Theorem 2.1.** Let $u(x, t)$ be the nonnegative classical solution of problem (1.1) for $p > q > 1$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Define

$$\varphi(t) = \int_{\Omega} u^k \, dx,$$

where $k$ is a parameter restricted by the condition

$$k > \max \left\{ 1, \frac{(7n - 16)(p - 1)}{2}, (q - 1)(3n - 8) \right\}. \quad (2.1)$$

If $u(x, t)$ blows up in the measure $\varphi$ at the finite time $T$, then $T$ is bounded from below as

$$T \geq \frac{1}{\int_{\Omega} x_0 \, dx} \int_{x_0(0)}^{+\infty} \frac{1}{C_1 + C_2 \xi^{\frac{n}{n-2}}} \, d\xi, \quad (2.2)$$

where $C_1$ and $C_2$ are positive constants which will be determined in the proof.
Proof. Applying the divergence theorem to the first equation in (1.1), we have

\[
\frac{d\varphi}{dt} = k \int_{\Omega} u^{k-1} u_t \, dx \\
= k \int_{\Omega} u^{k-1} (\Delta u + u^p - |\nabla u|^q) \, dx \\
= k \int_{\Omega} u^{k-1} \Delta u \, dx + k \int_{\Omega} u^{k+p-1} \, dx - k \int_{\Omega} u^{k-1} |\nabla u|^2 \, dx \\
= -\frac{4(k-1)}{k} \int_{\Omega} |\nabla u|^{k/2} \, dx + k \int_{\Omega} u^{k+p-1} \, dx - \frac{kq^q}{(k + q - 1)^q} \int_{\Omega} \nabla u^{\frac{k+q-1}{q}} \, dx.
\]  
(2.3)

Moreover, from [12, (2.10)] it follows that

\[
\int_{\Omega} |\nabla u|^{\frac{k+q-1}{q}} \, dx \geq \left(\frac{2\sqrt{\lambda}}{q}\right)^q \int_{\Omega} u^{k+q-1} \, dx,
\]  
(2.4)

where the positive constant \(\lambda\) is the first eigenvalue of the problem

\[
\Delta w + \lambda w = 0 \text{ in } \Omega, \\
w = 0 \text{ on } \partial\Omega.
\]  
(2.5)

Thus by combining (2.3) with (2.4) we obtain

\[
\frac{d\varphi}{dt} \leq -\frac{4(k-1)}{k} \int_{\Omega} |\nabla u|^{k/2} \, dx + k \int_{\Omega} u^{k+p-1} \, dx - \frac{k(2\sqrt{\lambda})^q}{(k + q - 1)^q} \int_{\Omega} u^{k+q-1} \, dx.
\]  
(2.6)

Noticing (2.1), we can apply first Hölder’s inequality and then Young’s inequality to the second term on the right hand side of (2.3) to obtain

\[
\int_{\Omega} u^{k+p-1} \, dx \leq |\Omega|m_1 \left( \int_{\Omega} u^{\frac{k(7n-14)}{7n-16}} \, dx \right)^{m_2} \\
\leq m_1 |\Omega| + m_2 \int_{\Omega} u^{\frac{k(7n-14)}{7n-16}} \, dx,
\]  
(2.7)

where

\[
m_1 = 1 - \frac{(k + p - 1)(7n - 16)}{k(7n - 14)} \in (0, 1), \quad m_2 = \frac{(k + p - 1)(7n - 16)}{k(7n - 14)} \in (0, 1).
\]

Combining (2.7) and (2.6) yields

\[
\frac{d\varphi}{dt} \leq -\frac{4(k-1)}{k} \int_{\Omega} |\nabla u|^{k/2} \, dx + k m_1 |\Omega| + k m_2 \int_{\Omega} u^{\frac{k(7n-14)}{7n-16}} \, dx \\
- \frac{k(2\sqrt{\lambda})^q}{(k + q - 1)^q} \int_{\Omega} u^{k+q-1} \, dx.
\]  
(2.8)

We now use Hölder’s inequality in the third term on the right hand side of (2.8):

\[
\int_{\Omega} u^{\frac{k(7n-14)}{7n-16}} \, dx \leq \left( \int_{\Omega} u^k \, dx \right)^{\frac{7n-14}{7n-16}} \left( \int_{\Omega} u^{\frac{2n}{7n-16}} \, dx \right)^{1-\frac{7n-14}{7n-16}},
\]  
(2.9)
where $0 < \alpha = \frac{2(3n-7)}{7n-14} < 1$. Next, using the Sobolev inequality for $W_{0}^{1,2} \hookrightarrow L^{\frac{2n}{n-2}} (n \geq 3)$ [20], we obtain
\[
\|u^{k/2}\|_{L^{\frac{2n}{n-2}}} \leq C_{s} \|\nabla u^{k/2}\|_{L^{\frac{2n}{n-2}}},
\] (2.10)
where $C_{s} = (\frac{1}{n(n-2)^{2}})^{1/2} (\frac{n+1}{2n^2+1})^{1/n}$ is the best imbedding constant (see [8, Chap. 7]). By substituting (2.10) into (2.9), we arrive at
\[
\int_{\Omega} u^{k(7n-14)(\frac{4}{7n-14}) - 1} \leq C_{s} \left( \int_{\Omega} u^{k} \right)^{\alpha \left( \int_{\Omega} \|\nabla u^{k/2}\|^{2} \right)^{\frac{n(1-\alpha)}{n-2}}},
\] (2.11)
which, with the help of Young’s inequality, gives
\[
\int_{\Omega} u^{k(7n-14)(\frac{4}{7n-14}) - 1} \leq C_{s} \frac{6n(6n-16)}{(7n-16)^{\frac{2n-8}{3n-7}} \varepsilon_{1} \frac{2n-8}{3n-7}} \left( \int_{\Omega} u^{k} \right)^{\frac{3n-7}{n-2} - \frac{n(1-\alpha)}{n-2}} \frac{1}{n-2} \int_{\Omega} \|\nabla u^{k/2}\|^{2} dx.
\] (2.12)
Here $\varepsilon_{1}$ is a positive constant to be determined later. By Hölder’s inequality, we have
\[
\int_{\Omega} u^{q+k-1} dx \geq |\Omega|^{-\frac{q}{q+1}} \left( \int_{\Omega} u^{k} dx \right)^{1+\frac{2}{q+1}}.
\] (2.13)
Combining (2.12) and (2.13) with (2.8) gives
\[
\frac{d\varphi}{dt} \leq km_{1}|\Omega| + \left[ \frac{n(1-\alpha)}{n-2} \varepsilon_{1} km_{2} - \frac{4(k-1)}{k} \right] \int_{\Omega} \|\nabla u^{k/2}\|^{2} dx + km_{2} C_{s}^{\frac{n}{3n-7}} \frac{(6n-16)}{(7n-16)^{\frac{2n-8}{3n-7}} \varepsilon_{1} \frac{2n-8}{3n-7}} \varphi^{\frac{3n-7}{m_{3}}} - \frac{k(2\sqrt{X})^{q}}{(k+q-1)^{q}} |\Omega|^{-\frac{q}{q+1}} \varphi^{1+\frac{q}{q+1}}.
\] (2.14)
Next, we apply Young’s inequality to the third term on the right-hand side of (2.14) to conclude that
\[
\varphi^{\frac{3n-7}{m_{3}}} \leq \varepsilon_{2} \varphi^{1+\frac{q}{q+1}} + \frac{1}{m_{4}} \varphi^{\frac{m_{3}}{m_{4}}},
\] (2.15)
where
\[
m_{3} = \frac{2k-(q-1)(3n-8)}{k}, \quad m_{4} = \frac{2k-(q-1)(3n-8)}{k-(q-1)(3n-8)},
\]
and $\varepsilon_{2}$ is a positive constant to be fixed. Combining (2.15) and (2.14), we obtain
\[
\frac{d\varphi}{dt} \leq C_{1} + \left[ \frac{n(1-\alpha)}{n-2} \varepsilon_{1} km_{2} - \frac{4(k-1)}{k} \right] \int_{\Omega} \|\nabla u^{k/2}\|^{2} dx + C_{2} \varphi^{\frac{m_{3}}{m_{4}} - \frac{m_{3}}{m_{4}}}
\] + \[ \frac{\varepsilon_{2} km_{2} C_{s}^{\frac{n}{3n-7}} (6n-16)}{(7n-16)^{\frac{2n-8}{3n-7}} \varepsilon_{1} \frac{2n-8}{3n-7} m_{3}} - \frac{k(2\sqrt{X})^{q}}{(k+q-1)^{q}} \frac{\varphi^{1+\frac{q}{q+1}}}{\frac{m_{3}}{m_{4}}} \right],
\] (2.16)
where
\[
C_{1} = km_{1}|\Omega|, \quad C_{2} = \frac{km_{2} C_{s}^{\frac{n}{3n-7}} (6n-16) \varepsilon_{2} \frac{m_{3}}{m_{4}}}{(7n-16)^{\frac{2n-8}{3n-7}} \varepsilon_{1} \frac{2n-8}{3n-7} m_{4}}.
\]
Therefore, by choosing
\[
\varepsilon_{1} = \frac{4(k-1)(n-2)}{nk^{2}m_{2}(1-\alpha)}
\]
first and
\[ \varepsilon_2 = \frac{(7n - 16)m^2 k (2 \sqrt{\lambda})^q |\Omega|^{-\frac{1}{n-2}} \varepsilon_1^{\frac{n}{n-2} - q} \sqrt{\lambda}^q}{km^2 (6n - 16) C^{\frac{n}{n-2}}_n (k + q - 1)^q} \]
next, we obtain the differential inequality
\[ \frac{d \varphi}{dt} \leq C_1 + C_2 \varphi^{\frac{3n-6}{3n-8}}, \tag{2.17} \]
or equivalently
\[ \frac{d \varphi}{C_1 + C_2 \varphi^{\frac{3n-6}{3n-8}}} \leq dt. \tag{2.18} \]
Integrating of the differential inequality (2.18) from 0 to \( t \) leads to
\[ \int_{\varphi(0)}^{\varphi(t)} \frac{1}{C_1 + C_2 \xi^{\frac{3n-6}{3n-8}}} d\xi \leq t. \tag{2.19} \]
Passing to the limit as \( t \to T^- \), we obtain
\[ \int_{\varphi(0)}^{+\infty} \frac{1}{C_1 + C_2 \xi^{\frac{3n-6}{3n-8}}} d\xi \leq T. \tag{2.20} \]
Thus, the proof is complete. \( \square \)

**Remark 2.2.** It is easy to see that when \( n = 3 \), the lower bound for the blow-up time derived here is consistent with the one obtained by Payne et al. \[16\].

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