Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 21, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

IDENTIFICATION OF THE DENSITY DEPENDENT COEFFICIENT IN AN INVERSE REACTION-DIFFUSION PROBLEM FROM A SINGLE BOUNDARY DATA

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Abstract. This study is devoted to the numerical solution of an inverse coefficient problem for a density dependent nonlinear reaction-diffusion equation. The method is based on approximating the unknown coefficient by polynomials. An optimal idea for solving the inverse problem is to minimize an error functional between the output data and the additional data. For this purpose, we find a polynomial of degree n that minimizes the error functional; i.e, n^{th} degree polynomial approximation of the unknown coefficient for the desired n.

1. Introduction

Problems involving the determination of unknown coefficients in ordinary and partial differential equations by some additional conditions are well known in the mathematical literature as the inverse coefficient problems. These additional conditions may be given on the whole domain, on the boundary of the domain, or at the final time. As it is known, a direct problem aims to find a solution that satisfies given ordinary or partial differential equation with initial and boundary conditions. In some problems the main ordinary or partial differential equation and the initial and boundary conditions are not sufficient to obtain the solution, but, instead some additional conditions are required. Such problems are called the inverse problems. A problem is said to be well-posed or properly posed in the sense of Hadamard [10] if it satisfies the following three conditions: there exists a solution of the problem (existence), there is at most one solution of the problem (uniqueness), and the solution depends continuously on the data (stability). If at least one of these properties does not hold, then the equation is called ill-posed. In this context, another definition of the inverse problems can be given as follows: If one of two problems which are inverse to each other is ill-posed, we call it the inverse problem and the other one the direct problem. It is well-known that inverse problems are often ill-posed.

2000 *Mathematics Subject Classification.* 45K05, 35R30, 65M32.
Key words and phrases. Fractional derivative; fractional Laplacian; weak solution; inverse problem; Mittag-Leffler function; Cauchy problem.

Submitted November 8, 2013. Published January 10, 2014.

In this paper, we consider the one dimensional nonlinear inverse reaction-diffusion problem

$$\begin{aligned} u_t &= (a(u)u_x)_x + |u|^p, \quad (x,t) \in \Omega_T, p \geq 2, \\ u(x,0) &= 0, \quad x \in \overline{\Omega}, \\ -a(u(0,t))u_x(0,t) &= g(t), \quad t \in [0,T], \\ u_x(1,t) &= 0, \quad t \in [0,T], \\ u(0,t) &= f(t), \quad t \in [0,T], \end{aligned} \tag{1.1}$$

where Ω is an open interval in $\mathbb{R}$, $\Omega_T := \Omega \times (0,T)$ a domain in $\mathbb{R} \times \mathbb{R}^+$. The inverse problem here consists of determining the unknown coefficient $a(u)$ in the inverse problem (1.1). The last condition, i.e., $u(0,t) = f(t)$ is taken to be an additional condition. In this context, for given inputs $a(u)$, $g(t)$ and p the nonlinear problem (1.1), without the additional condition, is defined as a direct (forward) problem. Henceforth, the expression *direct problem* will mean the studied problem without the additional condition. In the problem (1.1), the compatibility condition $f(0) = 0$ is satisfied.

The density dependent nonlinear reaction-diffusion equation $u_t = (a(u)u_x)_x + R(u)$ models many transport phenomena where $a(u)$ and $R(u)$ are called the diffusion coefficient and the reaction term respectively. The applications of this equations have had wide variety involving transport in porous medium, population dynamics, plasma physics and combustion theory. The density dependent nonlinear reaction-diffusion equation becomes $u_t = (a(u)u_x)_x + |u|^p$ for the reaction term $R(u) = |u|^p$ and has been used to model many different applications. For instance, it is used to model the flow of groundwater in a homogeneous, isotropic, rigid and unsaturated porous medium [1]. If we choose the coordinate x to measure the vertical height from ground level and pointing upward, the soil is represented by the vertical column $(-L, 0)$ [2], however as noted in [2], they assumed that absorption and chemical osmotic and thermal effects are negligible and there is no source inside the material. But in our work, we consider the effect of the nonlinear source term as well. This equation can be obtained easily by combining Darcy's law and the continuity equation. First initial condition and third boundary conditions in (1.1) represent the initial moisture content and moisture content at $x = 1$ respectively. The flux of moisture and moisture content at $x = 0$ are identified by the second and last conditions in problem (1.1) respectively. In this context, source term can be interpreted as material source.

Some numerical methods have been introduced for nonlinear diffusion equations. In [3, 4] for example, the authors considered an inverse problem for the nonlinear diffusion equation

$$u_t = (a(u)u_x)_x, \quad (x,t) \in \Omega_T.$$

The inverse problem is reformulated as an auxiliary inverse problem. Also it is proved that this auxiliary inverse problem has at least one solution in a specific admissible class. Finally, the auxiliary inverse problem is approximated by an associated identification problem. In addition to these, the authors presented a numerical method to solve the inverse problem for a special class of admissible coefficients. In the method, the partial differential equation is solved directly employing finite difference approach and the optimization part is solved using the program ZXMIN of the IMSL package. Furthermore, the intersecting graph technique is defined

as a second numerical method [3, 4]. We note that there are some other numerical methods introduced for numerical solution of the nonlinear inverse diffusion problem. A numerical algorithm based on the finite difference method and the least-squares scheme was given in [5]. According to this algorithm, Taylor's expansion is employed to linearize nonlinear terms and then finite difference method is applied to discretize the problem domain. Also this approach rearranges the matrix form of the governing differential equations and estimate the unknown coefficient. In [6], the given algorithm is based on linearizing nonlinear terms by Taylor's series expansion and removing the time-dependent terms by Laplace transform. Finite difference technique is used to discretize the problem.

In this paper, we develop a numerical algorithm to solve the inverse coefficient problem (1.1). The algorithm is based on the optimization of an error functional between the output data and the additional data. The algorithm attempts to minimize the error functional by using polynomials of a predetermined degree n. In doing so, it is assumed that the error functional is differentiable with respect to the coefficients of the polynomial which enables us to use the gradient descent method. The numerical experiments show that the algorithm is effective in practical use. A detailed analysis of the factors affecting the algorithm is also given.

The remainder of this paper comprises of five sections: In the next section, some theoretical background is recalled for the inverse problem including the existence and uniqueness of the solution. Our numerical method is given in section 3. Some numerical examples are presented to show the efficiency of the method in section 4. In section 5, analysis of the results are given. The final section of the paper contains discussions and comments on planned studies.

2. Existence and uniqueness for the inverse coefficient problem

The theoretical aspect of an inverse problem, similar to (1.1) is studied in [7]. The inverse problem (1.1) differs from the inverse problem in [7] in that it imposes the condition $-a(u(0,t))u_x(0,t) = g(t)$, (see [3]); whereas in [7], the condition $-a(u(0,t))u_t(0,t) = g(t)$, (see [9]) is used instead. In this paper, the authors have proved that the inverse problem has a unique solution under certain conditions. But the existence and uniqueness theorem also holds for the inverse problem (1.1). In this section, we present the existence and uniqueness theorem for the self-containment of the paper, but first we give some preliminaries.

We define the following norms and function spaces:

$$|u|_D = \sup\{u(s), s \in D\},$$
$$H_\alpha(u) = \sup\Big\{\frac{u(p)-u(q)}{d(p,g)^\alpha} : p, q \in D, p \neq q\Big\},$$
$$|u|_\alpha = |u|_D + H_\alpha(u), \quad |u|_{1+\alpha} = |u|_\alpha + \Big|\frac{\partial u}{\partial x}\Big|_\alpha,$$
$$|u|_{2+\alpha} = |u|_\alpha + \Big|\frac{\partial u}{\partial x}\Big|_\alpha + \Big|\frac{\partial^2 u}{\partial x^2}\Big|_\alpha + \Big|\frac{\partial u}{\partial t}\Big|_\alpha,$$

where $D = \Omega_T$, $d(p,q)$ is the usual Euclidean metric for the points p and q in D and $\alpha > 0$ is a constant. The space of all functions u for which $|u|_{2+\alpha} < \alpha$ is denoted by $C_{2+\alpha}(D)$. In [8], it is proved that the space $C_{2+\alpha}(D)$ is a Banach space with the corresponding norm.

Definition 2.1. A set $\mathbb{A}$ satisfying the following conditions is called the *class of admissible coefficients* for the inverse coefficient problem (1.1)

(C1) $a \in C_{2+\alpha}(I)$ with $|a|_{2+\alpha} \le c_1$,
(C2) $\nu \le a \le \mu$ and $a'(s) > 0$, for $s \in I$,
(C3) $|a'| \le \delta$ and $|a''| \le \delta$ for $s \in I$,

where $\alpha \in (0,1)$, I is a closed interval, $a : I \to \mathbb{R}$ and c_1, ν, μ, δ are positive constants.

Inspired by [3, 4], the authors in [7] use the transformation $v(x,t) = T_a(u(x,t)) = \int_0^{u(x,t)} a(s)ds$ to transform the inverse problem (1.1) into the problem

$$\begin{aligned} v_t = a(T_a^{-1}(v))v_{xx} + a(T_a^{-1}(v))|(T_a^{-1}(v))|^p, \quad (x,t) \in \Omega_T,\ p \ge 2, \\ v(x,0) = 0, \quad x \in \overline{\Omega}, \\ -v_x(0,t) = g(t), \quad t \in [0,T], \\ v_x(1,t) = 0, \quad t \in [0,T], \\ v(0,t) = F(t), \quad t \in [0,T], \end{aligned} \tag{2.1}$$

where $F(t) = \int_0^{f(t)} a(s)ds$. In problem (2.1), the compatibility condition $F(0) = 0$ also holds. We note that $\frac{d}{du}T_a(u) \ge \nu > 0$ implies that $T_a(u(x,t))$ is invertible. So the term $a(T_a^{-1}(v))$ in (2.1) makes sense. It is clear that the unknown coefficient is not in divergence form in (2.1), that is why the inverse problem (1.1) is needed to be transformed into a new one. Moreover, determination of the unknown coefficient $a(u)$ in the problem (2.1) is equivalent to determination of the unknown coefficient $A(v) := a(T_a^{-1}(v))$ in the problem (2.1). Therefore the authors study the inverse problem (2.1) instead of (1.1). Before we state the existence and uniqueness theorem, we need the following lemmas for the functions that belong the class of admissible coefficients $\mathbb{A}$.

Lemma 2.2 ([3]). *For each $a \in \mathbb{A}$, there exists a unique function $p_a(u,v)$ defined on $I \times I$ such that $p_a(u,v)$ is a number between u and v. Moreover, the following equality holds*

$$a(u) - a(v) = a'(p_a(u,v))(u - v).$$

It is important to emphasize that the above lemma can be applied to $T_a^{-1}(v)$. Because the following equalities imply that the inverse function $T_a^{-1}(v)$ also belongs to the set $\mathbb{A}$,

$$\begin{aligned} \frac{\partial}{\partial v}(T_a^{-1}(v)) &= \frac{1}{T_a'(v)} = \frac{1}{a(v)}, \\ \frac{\partial^2}{\partial v^2}(T_a^{-1}(v)) &= \frac{\partial}{\partial v}\frac{1}{a(v)} = -\frac{a'(v)}{a(v)^2}. \end{aligned}$$

Hence we have the following lemma.

Lemma 2.3 ([3]). *There exists a unique function $q_a(\cdot,\cdot)$ such that $q_a(u,v)$ is a number between u and v. Moreover, the following equality holds,*

$$T_a^{-1}(u) - T_a^{-1}(v) = (T_a^{-1})'(q_a(u,v))(u - v).$$

Lemma 2.4 ([9]). *Suppose that $\{w_n\}$ is a bounded and monotone increasing sequence of functions in $C_{2+\alpha}(\Omega_T)$. Then, there exists a function $w \in C_{2+\alpha}(\Omega_T)$*

such that $D^\beta D_t^j w_n \to D^\beta D_t^j w, |\beta| \le 2$, $0 \le j \le 1$ *uniformly (on compact subsets of D), where* $D^\beta u(x,t) = \frac{\partial^{|\beta|}}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} u(x,t), \beta = (m_1, m_2, \dots, m_n), |\beta| = m_1 + m_2 + \cdots + m_n$ *and* $D_t^m u(x,t) = \frac{\partial^m}{\partial t^m} u(x,t)$.

Using the above lemmas, following [7] closely, it can be proved that the inverse coefficient problem (2.1) has a unique solution under certain conditions. This is stated in the following theorem. For the sake of completeness we provide a sketch of the proof.

Theorem 2.5. *Assume that* $\frac{dF}{dt}$ *and* $g(t)$ *are positive continuous functions on* $[0,T]$ *and* $C^1([0,T])$ *respectively. Then the inverse problem* (2.1) *(equivalently,* (1.1) *) has a unique solution.*

Sketch of the proof. Step 1 (Existence). Let $\hat{v}_0 = 0$ and $\hat{v}_n$, $n = 1, 2, \dots$, be solution of the problem

$$\begin{gathered}
(\hat{v}_n)_t = a(T_a^{-1}((\hat{v}_{n-1})))v_{xx} + a(T_a^{-1}((\hat{v}_{n-1})))|(T_a^{-1}((\hat{v}_{n-1})))|^p, \ (x,t) \in \Omega_T, \ p \ge 2, \\
\hat{v}_n(x,0) = 0, \quad x \in \overline{\Omega}, \\
-(\hat{v}_n)_x(0,t) = g(t), \quad t \in [0,T], \\
(\hat{v}_n)_x(1,t) = 0, \quad t \in [0,T], \\
\hat{v}_n(0,t) = F(t), \quad t \in [0,T].
\end{gathered}$$

First it is not difficult to show that the sequence $\{\hat{v}_n\}$ is monotone increasing. Also, by applying Lemma 2.4 for $\beta = 1$ and $0 \le j \le 1$, we deduce

$$D^\beta D_t^j \hat{v}_n \to D^\beta D_t^j \hat{v}.$$

Since $\hat{v}_n$ is the a solution of the problem in Step 1, we have

$$(\hat{v}_n)_t = a(T_a^{-1}((\hat{v}_{n-1})))v_{xx} + a(T_a^{-1}((\hat{v}_{n-1})))|(T_a^{-1}((\hat{v}_{n-1})))|^p. \tag{2.2}$$

Letting $n \to \infty$, we deduce that $\hat{v}$ is a solution of (2.1).

Step 2 (Uniqueness). Suppose $v(x,t)$ and $u(x,t)$ are two solutions of (2.1). Let $z(x,t) = v(x,t) - u(x,t)$. Then $z(x,t)$ must satisfy the problem

$$\begin{gathered}
z_t = a(T_a^{-1}(v))z_{xx} + C_*(x,t)z, \\
z(x,0) = 0, \quad x \in \overline{\Omega}, \\
z_x(0,t) = z_x(1,t) = 0, \quad t \in [0,T], \\
z(0,t) = 0, \quad t \in [0,T],
\end{gathered} \tag{2.3}$$

where

$$\begin{aligned}
C_*(x,t) &= C(x,t) + \frac{h'(T_a^{-1}(\bar{u}))}{a(q_a(v(x,t),u(x,t)))}, \\
C(x,t) &= \frac{a'\Big(p_a\Big(T_a^{-1}(v(x,t)), T_a^{-1}(u(x,t))\Big)\Big)}{a\Big(q_a\Big(v(x,t),u(x,t)\Big)\Big)}.
\end{aligned}$$

By using the maximum principle we conclude that $z(x,t) \equiv 0$. Therefore the solution of problem (2.1) must be unique. □

3. Overview of the method

In this section, we present our numerical method. The essence of the method is to approximate the unknown diffusion coefficient $a(u)$ by polynomials. Since the unknown diffusion coefficient $a(u)$ is continuous on a compact domain Ω_T in the problem (3), there exists a sequence of polynomials converging to $a(u)$. However, finding such a sequence which guarantees the solution of the inverse problem is difficult. It is known that the direct problem has a unique solution if $a(u)$ satisfies certain conditions [8].

Our starting point is that the correct $a(u)$ will yield the solution satisfying the condition $u(0,t) = f(t)$, hence $a(u)$ will minimize the functional

$$F(c) = \|u(c,0,t) - f(t)\|_2^2,$$

where $u(c,x,t)$ is the solution of the direct problem with the diffusion coefficient $c(u)$ and $\|\cdot\|_2$ is the L^2 norm on Ω. Hence, our strategy is to find a polynomial of degree n that minimizes $F(c)$, i.e, n^{th} degree polynomial approximation of $a(u)$ for the desired n. From now on we take $c(u) = c_0 + c_1u + \cdots + c_nu^n$ as $c = (c_0, \ldots, c_n)$ hence $F(c)$ is a function of n variables. To overcome the ill-posedness of the inverse problem, Tikhonov regularization is applied. A regularization term with a regularization parameter λ is added to $F(c)$

$$G(c) = \|u(c,0,t) - f(t)\|_2^2 + \lambda\|c\|^2,$$

where $\|c\|$ denotes the Euclidean norm of c. From now on, we fix n and λ and we leave the discussions about the regularization parameter to the next section.

The method for minimizing $G(c)$ depends on the properties of $F(c)$, e.g., convexity, differentiability etc. In our case, the convexity or differentiability of $F(c)$ is not clear due to the term $u(c,x,t)$. However, we do not envision a major drawback in assuming the differentiability of $F(c)$ in numerical implementations. For this reason, we proceed the minimization of $G(c)$ by the steepest descent method which will utilize the gradient of F.

In this method, the algorithm starts with an initial point b_0, then the point providing the minimum is approximated by the points

$$b_{i+1} = b_i + \Delta b_i,$$

where Δb_i is the feasible direction which minimizes

$$E(\Delta b) = G(b_i + \Delta b).$$

This procedure is repeated until a stop criterion is satisfied; i.e, $\|\Delta b_i\| < \epsilon$ or $|G(b_{i+1}) - G(b_i)| < \epsilon$ or a certain number of iterations. In the minimization of $E(\Delta b)$, we use the following estimate on $u(b_i + \Delta b, 0, t)$;

$$u(b_i + \Delta b, 0, t) \simeq u(b_i, 0, t) + \nabla u(b_i, 0, t) \cdot \Delta b,$$

where ∇denotes the gradient of $u(b,0,t)$ with respect to b. Hence $E(\Delta b)$ turns out to be

$$E(\Delta b) = \|\nabla u(b_i, 0, t) \cdot \Delta b + u(b_i, 0, t) - f(t)\|_2^2 + \lambda\|\Delta b\|_2^2.$$

In numerical calculations, we note that $\|\cdot\|_2$ can be discretized by using a finite number of points in $[0,T]$, i.e., for $t_1 = 0 < t_2 < \cdots < t_q = T$, hence $E(\Delta b)$ has its new form as

$$E(\Delta b) \simeq \sum_{k=1}^{q} (u(b_i, 0, t_k) + \nabla u(b_i, 0, t_k) \cdot \Delta b - f(t_k))^2 + \lambda\|\Delta b\|_2^2. \tag{3.1}$$

Now the minimization of this problem is a least squares problem whose solution leads to the following normal equation (see [11])

$$(\lambda I + A^T A)\Delta b = A^T K,$$

where

$$A = [\nabla u(b_i, 0, t_1)^T \dots \nabla u(b_i, 0, t_q)^T],$$
$$K = [u(b_i, 0, t_1) - f(t_1) \dots u(b_i, 0, t_q) - f(t_q)]^T.$$

Now the optimal direction is found by

$$\Delta b = (\lambda I + A^T A)^{-1} A^T K. \tag{3.2}$$

In forming A, the computation (or estimation) of s^{th} component of the vector $\nabla u(b_i, 0, t_k)$ can be achieved by

$$\frac{u(b_i + he_s, 0, t_k) - u(b_i + he_s, 0, t_k)}{h}, \tag{3.3}$$

where e_s is the standard unit vector whose s^{th} component is 1 and h is the differential step. Now we give the algorithm.

Step 1: Set b_0, n, λ and a stopping criterion k or ϵ (iteration number less than k or size of $\|\Delta b_i\| \le \epsilon$).

Step 2: Calculate Δb_i using (3.2) and set $b_{i+1} = b_i + \Delta b_i$.

Step 3: Stop when the criterion is achieved.

4. Numerical examples with noisy and noise-free data

In this section we examine the algorithm with three inverse problems. The computations have been carried out in MATLAB. In solving the direct problem for each value of c, MATLAB PDE solver is used. All examples are considered in the following form where $a(u)$ is to be found

$$\begin{aligned} u_t = (a(u)u_x)_x + |u|^p + H(x,t), \quad (x,t) \in (0,1) \times (0,1), \\ u(x,0) = 0, \quad x \in [0,1], \\ -a(u(0,t))u_x(0,t) = g(t), \quad t \in [0,1], \\ u_x(1,t) = 0, \quad t \in [0,1], \\ u(0,t) = f(t), \quad t \in [0,1]. \end{aligned} \tag{4.1}$$

Note that in the problem statement above, the term $H(x,t)$ has been added to the main equation. This is due to the difficulty in finding an analytical solution of $u_t = (a(u)u_x)_x + |u|^p$. In particular, it is done to check the algorithm for $a(u) = e^u$ which brings a nonzero $H(x,t)$. The algorithm has also been tested with the analytic solution in the second example where $a(u) = e^u$. However, t he numerical solution of the direct problem is used in all examples.

Due to the discretization of the problem, many variables appear in computations. These factors and their values in our computations are listed below:

(1) The degree of the polynomial $c(u)$ to approximate $a(u)$: $n = 2, 3, 4, 5$ and 6 are taken in the examples.
(2) Initial guess for the coefficients of $c(u)$: All initial guesses for the coefficients are taken to be vectors composed of 1's to get an objective observation.
(3) Differential step h in (3.3): $h = 0.1$, $h = 0.01$ are taken in the examples.

(4) Number of t points; q in (3.1): $q = 10$ and $q = 100$ are taken in the examples.
(5) Number of (x, t) points in mesh grid used in Matlab PDE solver: taken to be $q \times q$ where q is already determined in (4).
(6) Stopping criterion: $\|\Delta b_i\| < \epsilon = 0.01$ or maximum iteration number $M = 100$.
(7) Regularization parameter (λ is in (3.1)): λ is taken to be zero in the noise-free examples, but an optimal λ is searched to deal with noisy data.

In our examples, the correct $a(u)$ is already known and $u(0, t)$ is extracted from the numerical solution of the direct problem for the correct $a(u)$. In the second example $u(x, t)$ is given analytically, the results for the analytical solution is also provided. The expected solution for $a(u)$ is of nth degree Taylor polynomial for given n.

Example 4.1. $p = 2.5$, $g(t) = \sin t$, $H(x, t) = 0$, and $u(0, t) = f(t)$ is found numerically. The correct solution is $a(u) = 1 + 2u + 3u^2 + u^3$. See Table 1.

TABLE 1. Initial guesses and results for $n = 2, 3, 4, 5$ and 6

Initial guess	$h = 0.1, q = 10$
(1,1)	(0.8329, 3.7928)
(1,1,1)	(1.0094, 1.8164, 3.8355)
(1,1,1,1)	(1.0000, 2.0000, 3.0001, 0.9999)
(1,1,1,1,1)	(1.0000, 2.0000, 2.9999, 1.0004, -0.0005)
(1,1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0000, -0.0001, 0.0001)
	$h = 0.01, q = 10$
(1,1)	(0.8315, 3.7928)
(1,1,1)	(1.0093 1.8179 3.8332)
(1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0000)
(1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0000, -0.0000)
(1,1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0000, -0.0000, 0.0000)
	$h = 0.1, q = 100$
(1,1)	(0.8486, 3.6889)
(1,1,1)	(1.0077, 1.8393, 3.7754)
(1,1,1,1)	(1.0000, 2.0000, 3.0001, 0.9999)
(1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0002, -0.0002)
(1,1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0001, -0.0001, 0.0000)
	$h = 0.01, q = 100$
(1,1)	(0.8486, 3.6894)
(1,1,1)	(1.0087, 1.8293, 3.7937)
(1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0000)
(1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0000, 0.0000)
(1,1,1,1,1,1)	(1.0000, 2.0000, 3.0000, 1.0002, -0.0003, 0.0002)

Example 4.2. $p = 2$, $g(t) = t^2$, $H(x, t)$ is found accordingly and $u(x, t) = t^2(\frac{x^2}{2} - x)$, hence $u(0, t) = f(t) = 0$. The correct solution is $a(u) = e^u$. The expected coefficients are the Taylor coefficients of e^u which is $(1, 1, 0.5, \frac{1}{6}, \frac{1}{24}, \frac{1}{120})$. Both

results obtained from the analytical solution $u(0,t) = 0$ (See Table 2) and the numerical solution for $u(0,t)$ (See Table 3) are given.

TABLE 2. Initial guesses and results for $n = 2, 3, 4, 5$

Initial guess	$h = 0.1, q = 10$
(1,1)	(0.9777, 0.7918)
(1,1,1)	(0.9791, 0.8209, 0.0966)
(1,1,1,1)	(0.9746, 0.6361, -1.4027, -3.0406)
(1,1,1,1,1)	(0.9706 0.3645 -5.4992 -23.2347 -30.5746)
	$h = 0.01, q = 10$
(1,1)	(0.9778 0.7919)
(1,1,1)	(0.9792, 0.8216, 0.0988)
(1,1,1,1)	(0.9997, 0.9940, 0.4622, 0.0832)
(1,1,1,1,1)	(0.9702 0.3443 -5.7337 -24.0679 -31.3722)
	$h = 0.1, q = 100$
(1,1)	(0.9944, 0.8693))
(1,1,1)	(0.9996, 0.9894, 0.4232)
(1,1,1,1)	(0.9997, 0.9940, 0.4622, 0.0832)
(1,1,1,1,1)	(0.9996, 0.9904, 0.4082, -0.1814, -0.3939)
	$h = 0.01, q = 100$
(1,1)	(0.9943, 0.8693)
(1,1,1)	(0.9996, 0.9894, 0.4231)
(1,1,1,1)	(0.9997, 0.9934, 0.4572, 0.0729)
(1,1,1,1,1)	(0.9996, 0.9896, 0.3971, -0.2315, -0.4616)

Example 4.3. $p = 3$, $g(t) = t^2$, $H(x,t) = 0$, $u(0,t) = f(t)$ is found numerically. The correct solution is $a(u) = 2 + \sin(u)$. The expected coefficients are the Taylor coefficients of $2 + \sin(u)$ which is $(2, 1, 0, -\frac{1}{6}, 0, \frac{1}{120})$. (See Table 4).

In the examples above $u(0,t) = f(t)$ for the correct $a(u)$ is obtained analytically or numerically without any noise on $u(0,t)$ except the error resulting from the computation of $u(0,t)$ with PDE Solver. However, in the applications the additional data $u(0,t)$ is generally given with a noise; i.e., $u(0,t) + \gamma u(0,t)$ where γ is called noise level and is generally less than 0.1. The examples above are now tested with $u(0,t)$ plus some noise. The algorithm is run for the best choices of h, q and the initial guesses in the previous calculations; i.e., $h = 0.1, q = 100$ for all examples. The noise levels will be taken as $\gamma = +0.03, -0.05$. Table 5 shows the results. In these tables we also give the relative error for each example and perturbation which is defined as

$$\frac{\|u - u_a\|_\infty}{\|u\|_\infty}$$

where $\|\cdot\|_\infty$ denotes maximum norm, u and u_a are the solutions corresponding to the correct $a(u)$ and observed $a(u)$ (i.e coefficients obtained in Tables 5 and 6) respectively. Defining the relative error provides a gauge to compare the results for the noisy data for different regularization parameters. The results are given for each example in Table 5.

TABLE 3. Initial guesses and results for $n = 2, 3, 4, 5$ and 6.

Initial guess	$h = 0.1, q = 10$
(1,1)	(0.9937, 0.8628)
(1,1,1)	(0.9998, 0.9911, 0.4253)
(1,1,1,1)	(1, 0.997, 0.4948, 0.1408)
(1,1,1,1,1)	(1.0000, 1.0000, 0.5001, 0.1669, 0.0384)
(1,1,1,1,1,1)	(1,1.0004,0.5128,0.3247,0.7995,1.1887)
	$h = 0.01, q = 10$
(1,1)	(0.9937, 0.8629)
(1,1,1)	(.9998, 0.9912, 0.4254)
(1,1,1,1)	(1, 0.997, 0.4948, 0.1409)
(1,1,1,1,1)	(1.0000, 1.0000, 0.4998, 0.1648, 0.0351)
(1,1,1,1,1,1)	(1,1,0.5,0.1666,0.0412,0.007)
	$h = 0.1, q = 100$
(1,1)	(0.9945, 0.8702)
(1,1,1)	(0 .9998, 0.9919, 0.4286)
(1,1,1,1)	(1, 0.9997, 0.4953, 0.1417)
(1,1,1,1,1)	(1.0000, 1.0003, 0.5045, 0.1856, 0.0635)
(1,1,1,1,1,1)	(1,1.0031,0.5721,0.7732,2.0824,2.3360)
	$h = 0.01,\ q = 100$
(1,1)	(0.9945, 0.8702)
(1,1,1)	(0 .9998, 0.9918, 0.4288)
(1,1,1,1)	(1, 0.9996, 0.4947, 0.1407)
(1,1,1,1,1)	(1.0000, 1.0013, 0.5207, 0.2704, 0.1955)
(1,1,1,1,1,1)	(1,1.0022,0.5689,0.8506,2.5924,3.1306)

Note that the coefficients in Table 5 are far from the coefficients of the correct $a(u)$. This is due to the ill-posedness of the problem. A small perturbation in $u(0, t)$ causes the algorithm to deviate much from the correct $a(u)$. In order to overcome this problem, regularization parameter is used. The regularization parameter is now added to the algorithm, i.e., λ is nonzero. In our experiments below, the best λ value is sought. Since the problem is highly nonlinear, we seek the best regularization parameter empirically. We present the optimal regularization parameters and the corresponding relative errors in Table 6.

5. Analysis of results

The experiments have clearly indicated that the initial guess for q and n are the main factors affecting the accuracy of the solutions. The initial guesses have to be chosen close enough to the coefficients of the correct solution. However, it is hard to give a radius of the trust region around the expected coefficients. One way to overcome this problem is to start with $n = 1$ with several initial guesses then choose the best one for it (call it x_0) then make it $n = 2$, use the solution $(x_0, 1)$ as an initial guess and repeat it for the other dimensions. Although the initial guesses in the above experiments have not been determined with this procedure, that approach also has been observed to work well in all examples. It should also be noted that the initial guesses for different dimensions are intended to be same in all

TABLE 4. Initial guesses and results for $n = 2, 3, 4, 5$ and 6.

Initial guesses	$h = 0.1, q = 10$
(1,1)	(2.0039, 0.9665)
(1,1,1)	(1.9991, 1.0210, -0.1131)
(1,1,1,1)	(2.0000, 1.0002, -0.0017, -0.1621)
(1,1,1,1,1)	(2.0000, 1.0000, 0.0005, -0.1701, 0.0090)
(1,1,1,1,1,1)	(2.0000, 1.0000, -0.0000, -0.1664, -0.0007, 0.0089)
	$h = 0.01, q = 10$
(1,1)	(2.0040, 0.9665)
(1,1,1)	(1.9991, 1.0211, -0.1132)
(1,1,1,1)	(2.0000, 1.0002, -0.0017, -0.1621)
(1,1,1,1,1)	(2.0000, 1.0000, 0.0005, -0.1701, 0.0089)
(1,1,1,1,1,1)	(2.0000, 1.0000, 0.0000, -0.1667, 0.0001, 0.0081)
	$h = 0.1, q = 100$
(1,1)	(2.0032, 0.9710)
(1,1,1)	(1.9992 ,1.0191, -0.1077)
(1,1,1,1)	(2.0000, 1.0015, -0.0096, -0.1510)
(1,1,1,1,1)	(2.0001, 0.9970, 0.1128, -0.9067 ,1.1098)
(1,1,1,1,1,1)	(2.0000, 1.0012, -0.0169, -0.0727, -0.2284, 0.2087)
	$h = 0.01, q = 100$
(1,1)	(2.0033, 0.9700)
(1,1,1)	(1.9992 ,1.0191, -0.1081)
(1,1,1,1)	(2.0000, 1.0007, -0.0047, -0.1579)
(1,1,1,1,1)	(2.0001, 0.9969, 0.1138, -0.9096, 1.1129)
(1,1,1,1,1,1)	(2.0000 ,1.0007 ,-0.0094 ,-0.1157 ,-0.1197, 0.1096)

examples and they are not very far from the expected coefficients. However, using the same initial guesses aims to get an insight about the behavior of the inversion algorithm on different types of $a(u)$; i.e., a polynomial, an exponential function and a uniformly bounded function.

It is observed that q has a significant impact on the solutions. However, the way how it affects the algorithm is not very clear. It appears that in all examples $q = 10$ works better (See Tables 1, 3, 4). When the analytical solution is used in the inversion algorithm in the second example, $q = 100$ turns out to be better than $q = 10$ (See Table 2). It seems that using an additional data $u(0, t) = f(t)$ that is numerically found by PDE Solver from the correct $a(u)$ is more preferable. Indeed, in applications an analytical $u(0, t) = f(t)$ is often not given. It should be noted that when the solutions $u(c, x, t)$ and $u(0, t)$ in $F(c)$ are obtained numerically by PDE solver, they bear an error caused by q. These errors seem to cancel each other or add to the error of the main algorithm because $F(c)$ measures the difference between $u(c, x, t)$ and $u(0, t)$. This might be the fact behind the result $q = 10$ works better than $q = 100$ using the same initial guesses in all examples (See Tables 1, 3, 4). Additionally these errors brings some perturbation to the additional data $u(0, t)$ (see example 4.2). In the second example when analytical solution is used, the analytical solution $u(0, t)$ is perceived as a perturbation of the numerical $u(0, t)$ and that leads to the results in Table 3. This is due to the ill-posedness of the

TABLE 5. The results for given γ values

Example 4.1	$\gamma = +0.03$
(1,1,1,1)	(0.9439, 1.7790, 2.8595, -0.8672)
Relative error	0.0297
Example 4.1	$\gamma = -0.05$
(1,1,1,1)	(1.0895, 2.9046, 0.2247, 12.1565)
Relative error	0.0499
Example 4.2	$\gamma = +0.03$
(1,1,1,1,1,1)	(1.0011,1.0441,1.2929,6.3858,21.0167,24.7614)
Relative Error	0.0024
Example 4.2	$\gamma = -0.05$
(1,1,1,1,1,1)	(0.9981,0.9300,-0.7489,-9.5259,-32.1563,-37.5388)
Relative Error	0.0034
Example 4.3	$\gamma = +0.03$
(1,1,1,1,1)	(1.8833,0.3501,2.8646,-11.5998,22.3717, -16.4603)
Relative error	0.0299
Example 4.3	$\gamma = -0.05$
(1,1,1,1,1)	(2.2204, 2.6791, -8.6243, 40.0126, -90.2433, 76.5890)
Relative error	0.05

problem which is made clear in the examples with noisy data (Compare the fourth row for $q = 10$ with the findings in Table 5). Despite the ambiguous but significant effect of q, for practical uses $q = 10$ appears to be efficient for non-polynomial $a(u)$'s. Using a higher dimension n is more accurate in estimating $a(u)$ provided that the algorithm is started with a good initial guess. The ill-posedness of the inverse problems is observed when noisy data is used in the examples. Table 5 shows clearly that a noise level even as small as 0.03% causes a big shift in the coefficients of $a(u)$.

The effect of regularization parameter becomes apparent in noisy examples. Since the problem is highly nonlinear, we seek the best regularization parameter empirically. We present the best regularization parameter with their relative errors. When Tables 5 and 6 are compared, we see a significant enhancement in results in terms of relative errors. When the optimal regularization parameter is used, the algorithm ends at relatively better coefficients.

The changes in differential step h is observed to have a negligible effect in finding feasible directions. In our experiments $h = 0.1$ appears to be good enough for a satisfactory solution.

Concluding remarks. The presented numerical method has been successfully applied to a nonlinear inverse reaction-diffusion problem. The demonstrated numerical examples show that the method allows to reconstruct the unknown coefficient with high accuracy, even for acceptable noise levels. The assumption on the smoothness of $F(c)$ appears to be true at least in the space of polynomials. The authors of this paper plan to consider simultaneously determination of the coefficient $a(u)$ and the number p in the considered inverse problem. In this context, an existence and uniqueness theorem for the solution will be provided. Later we will study the same kind of inverse problems involving determination of the coefficient $a(u_x^2)$ and/or the

TABLE 6. Regularization parameters and relative errors for different noise levels

Example 4.1	$\gamma = +0.03$
(1,1,1,1)	(0.9418, 1.9809, 1.4719, 1.2065)
λ	0.0004
Relative error	0.0283
Example 4.1	$\gamma = -0.05$
(1,1,1,1)	(1.4532, 1.1701, 1.0707, 1.0319)
λ	0.58
Relative error	0.0287
Example 4.2	$\gamma = +0.03$
(1,1,1,1,1,1)	(1.0027, 1.0661, 0.9099, 1.0385, 0.9855, 1.0052)
λ	0.000007
Relative Error	0.000286
Example 4.2	$\gamma = -0.05$
(1,1,1,1,1,1)	(1.0011, 1.0539, 0.8900, 1.0448, 0.9826, 1.0060)
λ	0.0000071
Relative Error	0.00031
Example 4.3	$\gamma = +0.03$
(1,1,1,1,1,1)	(1.7049, 0.9091, 0.9263, 0.9644, 0.9845, 0.9934)
λ	0.012
Relative error	0.023
Example 4.3	$\gamma = -0.05$
(1,1,1,1,1,1)	(1.7880, 1.1874, 1.0551, 1.0186, 1.0068, 1.0026)
λ	0.37
Relative error	0.0106

number p related to the nonlinear equation $u_t = \left(a(u_x^2)u_x\right)_x + |u|^p$. Also we generalize the nonlinear source $|u|^p$ to be $f(u)$ and then study some inverse problems for these two equations; i.e., $u_t = \left(a(u)u_x\right)_x + f(u)$ and $u_t = \left(a(u_x^2)u_x\right)_x + f(u)$. These are subjects of the planned studies by the authors of this article.

Acknowledgments. This research was partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK), also by the Zirve University Research Fund.

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