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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO MIXED TYPE DIFFERENTIAL EQUATIONS

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Abstract. This work concerns the asymptotic behavior of solutions to the differential equation

$$\dot{x}(t)+\sum_{i=1}^{m}a_i(t)x(r_i(t))+\sum_{j=1}^{n}b_j(t)x(\tau_j(t))=0,$$

where $a_j(t)$ and $b_j(t)$ are real-valued continuous functions and $r_j(t)$ and $\tau_j(t)$ are non-negative functions such that

$$\begin{aligned}
&r_i(t)\le t,\ t\ge t_0,\quad \lim_{t\to\infty} r_i(t)=\infty,\ i=1,\dots,m;\\
&\tau_j(t)\ge t,\ t\ge t_0,\quad \lim_{t\to\infty} \tau_j(t)=\infty,\ j=1,\dots,n.
\end{aligned}$$

1. Introduction

In recent years, the theory of delay differential equations with advanced and retarded arguments (mixed type) has provided a natural framework for mathematical modeling of many real world phenomena, namely optimal control problems [6], nerve conduction theory [3], the slowing down of neutrons in nuclear reactors [9], models for economic dynamics [6, 7] and the description of traveling waves in a spatial lattice [4, 5]. See Bellman and Cooke [1] for more applications of differential equations of mixed type. The concept of delay is related to the memory of systems, where past events influence the current behavior. The concept of advance is related to a potential future events which are known at the current time, and which could be useful for decision making. It is well known that the solutions of these types of equations cannot be obtained in closed form. In the absence of a closed form, a viable alternative is studying the qualitative behavior of solutions. As a first step, we need existence and uniqueness of solutions which can be a complicated issue for mixed type equations.

In this article we study the asymptotic behavior of the advanced and retarded differential equation

$$x'(t)+\sum_{i=1}^{m}a_i(t)x(r_i(t))+\sum_{j=1}^{n}b_j(t)x(\tau_j(t))=0,\quad t\ge t_0, \tag{1.1}$$

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where $a_i(t)$ and $b_j(t)$ are real continuous functions and $r_j(t)$ and $\tau_j(t)$ are non-negative functions such that

$$\begin{gathered}r_i(t) \le t,\ t \ge t_0, \quad \lim_{t\to\infty} r_i(t) = \infty,\ i = 1,\dots,m;\\ \tau_j(t) \ge t,\ t \ge t_0, \quad \lim_{t\to\infty} \tau_j(t) = \infty,\ j = 1,\dots,n.\end{gathered}$$

About fifteen years ago, a new technique of fixed points was developed for studying stability of delay differential equations [2]. In the present article we apply this technique to mixed type differential equation. It is possible to find in the literature some conditions to ensure the stability of a solution of a delay differential equation, but it is not easy to find conditions for stability for mixed type differential equations. We will establish necessary and sufficient conditions for all solutions of (1.1) to converge to zero.

In the second section we establish the main results, and in the third section we illustrate the previous results with an example.

2. Main results

Let $r_0 = \inf\{r_i(s) : s \ge t_0, i = 1,\dots,m\}$. Then $r_0 \le t_0$, and the initial condition for (1.1) is determined by a function ϕ, continuous on $[r_0,t_0]$,

$$x(t) = \phi(t) \quad \text{for } r_0 \le t \le t_0\,. \tag{2.1}$$

For short notation, we write $x_0 = x(t_0)$ and $y_0 = y(t_0)$.

By a solution to (1.1) we mean a continuous function $x : [r_0,\infty) \to \mathbb{R}$ satisfying (2.1), and differentiable on $[t_0,\infty)$ and satisfies (1.1).

Theorem 2.1. *Let $a_i(t)$ and $b_j(t)$ non-positive functions. Suppose that the inequality*

$$y(t) \ge -\sum_{i=1}^{m} a_i(t)e^{-\int_{r_i(t)}^{t} y(s)\,ds} - \sum_{j=1}^{n} b_j(t)e^{\int_t^{\tau_j(t)} y(s)\,ds} \quad \text{for } t \ge t_0 \tag{2.2}$$

has a nonnegative solution which is integrable on each interval $[t_0,b]$. Then (1.1) *has a positive solution.*

Proof. Let $y_0(t)$ be a nonnegative solution of (2.2). Define the iteration

$$y_{k+1}(t) = \begin{cases} y_k(t) & r_0 \le t \le t_0,\\ -\sum_{i=1}^{m} a_i(t)e^{-\int_{r_i(t)}^{t} y_k(s)\,ds} - \sum_{j=1}^{n} b_j(t)e^{\int_t^{\tau_j(t)} y_k(s)\,ds}, & t_0 \le t \end{cases}$$

for $k = 0,1,\dots$. Then, by (2.2), we have

$$y_1(t) = -\sum_{i=1}^{m} a_i(t)e^{-\int_{r_i(t)}^{t} y_0(s)\,ds} - \sum_{j=1}^{n} b_j(t)e^{\int_t^{\tau_j(t)} y_0(s)\,ds} \le y_0(t)\,.$$

By induction we have $0 \le y_{k+1}(t) \le y_k(t) \le \dots \le y_0(t)$. Hence, there exists a pointwise limit $y(t) = \lim_{k\to\infty} y_k(t)$. By the Lesbesgue convergence theorem, we have

$$y(t) = -\sum_{i=1}^{m} a_i(t)e^{-\int_{r_i(t)}^{t} y(s)\,ds} - \sum_{j=1}^{n} b_j(t)e^{\int_t^{\tau_j(t)} y(s)\,ds}.$$

Then, the function

$$x(t) = \begin{cases} y(t_0)e^{\int_{t_0}^t y(s)\,ds} & t \ge t_0, \\ y(t) & r_0 \le t \le t_0 \end{cases}$$

is a positive solution of (1.1). $\square$

Theorem 2.2. *Let $a_i(t)$ and $b_j(t)$ be non-positive functions. If*

$$\int_{t_0}^{\infty} \sum_{j=1}^{n} b_j(s)\,ds = -\infty$$

and $x(t)$ is a non-oscillatory solution of (1.1)*, then* $\lim_{t\to\infty} x(t) = \infty$.

Proof. Since x is non-oscillatory, it must be eventually positive or eventually negative. We consider only the positive case, because in the negative case we can consider $-x$ which is also a solution. Suppose that $x(t) > 0$ for $t \ge t_1'$. Select $t_1 \ge t_1'$ such that $t_1' \le \inf\{r_i(s) : s \ge t_1,\ i = 1, \dots, m\}$. Then $x'(t) \ge 0$ for $t \ge t_1$, and

$$\begin{aligned} x'(t) &= -\sum_{i=1}^{m} a_i(t)x(r_i(t)) - \sum_{j=1}^{n} b_j(t)x(\tau_j(t)) \\ &\ge -\sum_{j=1}^{n} b_j(t)x(\tau_j(t)) \\ &\ge -x(t_1)\sum_{j=1}^{n} b_j(t), \end{aligned}$$

which implies

$$x(t) \ge -x(t_1)\int_{t_0}^{t} \sum_{j=1}^{n} b_j(s)\,ds.$$

Thus, $\lim_{t\to\infty} x(t) = \infty$. $\square$

Theorem 2.3. *Let $a_i(t)$ and $b_j(t)$ non-negative functions. If, either*

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} a_i(s)\,ds = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \sum_{j=1}^{n} b_j(s)\,ds = \infty\,,$$

and $x(t)$ is a non-oscillatory solution of (1.1)*, then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t) > 0$ for $t \ge t_1$. Select $t_1 \ge t_1'$ such that $t_1' \le \inf\{r_i(s) : s \ge t_1,\ i = 1, \dots, m\}$. Then $x'(t) \le 0$ for $t \ge t_1$. thus, $x(t)$ is non-increasing and positive. It must have a finite limit. If $\lim_{t\to\infty} x(t) = d > 0$, then $x(t) > d$ for $t \ge t_1$, and

$$x'(t) \le -d\Big(\sum_{i=1}^{m} a_i(s) + \sum_{j=1}^{n} b_j(t)\Big)$$

which implies $\lim_{t\to\infty} x(t) = -\infty$. This contradicts to the assumption that $x(t)$ is positive, and therefore $\lim_{t\to\infty} x(t) = 0$. $\square$

Next we study the asymptotic behavior of (1.1), independently of the sign of the coefficients. In the next lemma we establish an equivalence between the differential equation (1.1) and an integral equation.

Lemma 2.4. *A function $x(t)$ is a solution of* (1.1) *and* (2.1) *if and only if $x(t)$ is a solution of*

$$
\begin{aligned}
x(t) &= x_0 e^{-\int_{t_0}^{t}(A(s)+B(s))\,ds} - \int_{t_0}^{t} e^{-\int_u^t (A(s)+B(s))\,ds} \\
&\quad \times \Big(\sum_{i=1}^{m} a_i(u)\int_{r_i(u)}^{u} E_x(s)\,ds - \sum_{j=1}^{n} b_j(u)\int_{u}^{\tau_j(u)} E_x(s)\,ds\Big)\,du
\end{aligned}
\tag{2.3}
$$

for $t \ge t_0$, and (2.1) *is satisfied. Here we use the notation*

$$
A(t) = \sum_{i=1}^{m} a_i(t), \quad B(t) = \sum_{j=1}^{m} b_j(t),
$$
$$
E_x(t) = \sum_{i=1}^{m} a_i(t)x(r_i(t)) + \sum_{j=1}^{n} b_j(t)x(\tau_j(t)).
$$

Proof. Note that by (1.1), $x'(t) = -E_x(t)$ and that

$$
x(r_i(t)) = x(t) - \int_{r_i(t)}^{t} x'(u)\,du = x(t) + \int_{r_i(t)}^{t} E_x(u)\,du,
$$
$$
x(\tau_j(t)) = x(t) + \int_{t}^{\tau_j(t)} x'(u)\,du = x(t) - \int_{t}^{\tau_j(t)} E_x(u)\,du\,.
$$

Then (1.1) can be re-written as

$$
x'(t) + \sum_{i=1}^{m} a_i(t)\Big(x(t) - \int_{r_i(t)}^{t} \dot{x}(s)\,ds\Big) + \sum_{j=1}^{n} b_j(t)\Big(x(t) - \int_{\tau_j(t)}^{t} \dot{x}(s)\,ds\Big) = 0
$$

which is equivalent to

$$
x'(t) + (A(t)+B(t))x(t) = -\sum_{i=1}^{m} a_i(t)\int_{r_i(t)}^{t} E_x(s)\,ds + \sum_{j=1}^{n} b_j(t)\int_{t}^{\tau_j(t)} E_x(s)\,ds.
\tag{2.4}
$$

Multiplying both sides by the integrating factor $\exp(\int_{t_0}^{t}(A(s)+B(s))\,ds)$, we obtain differential equation equivalent to the one above:

$$
\begin{aligned}
&\frac{d}{dt}\Big(e^{\int_{t_0}^{t}(A(s)+B(s))\,ds}x(t)\Big) \\
&= -e^{\int_{t_0}^{t}(A(s)+B(s))\,ds}\Big(\sum_{i=1}^{m} a_i(t)\int_{r_i(t)}^{t} E_x(s)\,ds - \sum_{j=1}^{n} b_j(t)\int_{t}^{\tau_j(t)} E_x(s)\,ds\Big).
\end{aligned}
$$

Integrating from t_0 to t, we obtain (2.3).

Now starting from (2.3) differentiate with respect to t, and retrace the steps above to obtain (1.1). The proof is complete. □

Theorem 2.5. *Assume that there exists a constant c such that*

$$
\begin{aligned}
&\int_{t_0}^{t} e^{-\int_u^t (A(s)+B(s))\,ds}\Big(\sum_{i=1}^{m} |a_i(u)|\int_{r_i(u)}^{u}\Big(\sum_{k=1}^{m}|a_k(s)| + \sum_{\ell=1}^{n}|b_\ell(s)|\Big)\,ds \\
&+ \sum_{j=1}^{n}|b_j(u)|\int_{u}^{\tau_j(u)}\Big(\sum_{k=1}^{m}|a_k(s)| + \sum_{\ell=1}^{n}|b_\ell(s)|\Big)\,ds\Big)\,du \le c < 1\,.
\end{aligned}
\tag{2.5}
$$

Then for each initial condition (2.1)*, there exists a unique solution to* (1.1)*. Furthermore, if*

$$\lim_{t\to\infty}\int_{t_0}^{t}(A(s)+B(s))\,ds=\infty, \tag{2.6}$$

then, every solution of (1.1) *converges to zero.*

Proof. Let $C([t_0,\infty))$ be the set of real-valued functions, continuous on $[t_0,\infty)$. Then this is a Banach space with the norm

$$\|x\|=\sup_{t\ge t_0}|x(t)|.$$

Based on (2.3), for a function $x\in C([r_0,\infty))$, we define the operator

$$(Tx)(t)=\begin{cases} x(t) \quad \text{if } r_0\le t\le t_0,\\ e^{-\int_{t_0}^{t}(A(s)+B(s))\,ds}x(t_0)-\int_{t_0}^{t}e^{-\int_{u}^{t}(A(s)+B(s))\,ds}\\ \times\Big(\sum_{i=1}^{m}a_i(u)\int_{r_i(u)}^{u}E_x(s)\,ds-\sum_{j=1}^{n}b_j(u)\int_{u}^{\tau_j(u)}E_x(s)\,ds\Big)\,du\\ \quad \text{if } t_0\le t.\end{cases}$$

It is clear that T maps $C([t_0,\infty))$ into $C([t_0,\infty))$ and preserves the values of $x(t)$ for $t\in[r_0,t_0]$. We will proof that T is a contraction.

Let x, y be two continuous function on $[t_0,\infty)$, and satisfying the same initial conditions (2.1). Then for $t\ge t_0$, we have

$$\begin{aligned} &|(Tx)(t)-(Ty)(t)|\\ &\le e^{-\int_{t_0}^{t}(A(s)+B(s))\,ds}|(x(t_0)-y(t_0))|+\int_{t_0}^{t}e^{-\int_{u}^{t}(A(s)+B(s))\,ds}\\ &\quad\times\Big(\sum_{i=1}^{m}|a_i(u)|\int_{r_i(u)}^{u}|E_x(s)-E_y(s)|\,ds\,du\\ &\quad+\sum_{j=1}^{n}|b_j(u)|\int_{u}^{\tau_j(u)}|E_x(s)-E_y(s)|\,ds\,du\Big). \end{aligned}$$

Since $x(t)=y(t)$ for $r_0\le t\le t_0$, and

$$\begin{aligned} |E_x(s)-E_y(s)| &\le |\sum_{k=1}^{m}a_k(t)x(r_k(t))-\sum_{k=1}^{m}a_k(t)y(r_k(t))|\\ &\quad+|\sum_{\ell=1}^{n}b_\ell(t)x(\tau_\ell(t))-\sum_{\ell=1}^{n}b_\ell(t)y(\tau_\ell(t))|\\ &\le\sum_{k=1}^{m}|a_k(t)||x(r_k(t))-y(r_k(t))|+\sum_{\ell=1}^{n}|b_\ell(t)||x(\tau_\ell(t))-y(\tau_\ell(t))|, \end{aligned}$$

by (2.5), we obtain

$$|(Tx)(t)-(Ty)(t)|\le c\|x(t)-y(t)\|\,.$$

Consequently, the operator T has a unique fixed point in $C([t_0,\infty))$, and this fixed point satisfies (2.1).

Let

$$L=\big\{x\in C([t_0,\infty)):\lim_{t\to\infty}x(t)=0\big\},$$

which is a closed subspace of $C([t_0,\infty))$. Now, we claim that $T(L)\subset L$ and that T preserves the initial conditions (2.1). Indeed, for $x\in L$, we have

$$\begin{aligned}|(Tx)(t)| &\le |x_0|e^{-\int_{t_0}^t (A(s)+B(s))\,ds}+\int_{t_0}^t e^{-\int_u^t (A(s)+B(s))\,ds}\\ &\quad\times\Big(\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u |E_x(s)|\,ds+\sum_{j=1}^n |b_j(u)|\int_u^{\tau_j(u)} |E_x(s)|\,ds\Big)\,du\,.\end{aligned} \tag{2.7}$$

Note that by (2.6),

$$\lim_{t\to\infty}|x_0|e^{-\int_{t_0}^t (A(s)+B(s))\,ds}=0\,.$$

On the other hand, since $x\in L$, for each $\epsilon>0$ there exists $t_1'\ge t_0$ such that $|x(t)|<\epsilon/2$ for all $t\ge t_1'$. Select $t_1\ge t_1'$ such that $t_1'\le \inf\{r_i(s): s\ge t_1,\, i=1,\dots,m\}$. Then for $t\ge t_1$,

$$\begin{aligned}&\int_{t_0}^t e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u |E_x(s)|\,ds\,du\\ &\le \int_{t_0}^{t_1} e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u |E_x(s)|\,ds\,du\\ &\quad+\frac{\epsilon}{2}\int_{t_1}^t e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_j(u)}^u \Big(\sum_{k=1}^m |a_k(s)|+\sum_{\ell=1}^n |b_\ell(s)|\Big)\,ds\,du\end{aligned}$$

We observe that the first term on the right-hand side approaches zero as $t\to\infty$. Then there exists $t_2\ge t_1$ such that

$$\int_{t_0}^{t_1} e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u |E_x(s)|\,ds\,du<\frac{\epsilon}{2}\,.$$

Then by (2.5)

$$\begin{aligned}&\int_{t_0}^t e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u |E_x(s)|\,ds\,du\\ &\le \frac{\epsilon}{2}+\frac{\epsilon}{2}\int_{t_1}^t e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u \Big(\sum_{k=1}^m |a_k(s)|+\sum_{\ell=1}^n |b_\ell(s)|\Big)\,ds\,du<\epsilon.\end{aligned}$$

Since ϵ is arbitrarily small,

$$\lim_{t\to\infty}\int_{t_0}^t e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{i=1}^m |a_i(u)|\int_{r_i(u)}^u |E_x(s)|\,ds\,du=0. \tag{2.8}$$

By a similar process, we prove that

$$\lim_{t\to\infty}\int_{t_0}^t e^{-\int_u^t (A(s)+B(s))\,ds}\sum_{j=1}^n |b_j(u)|\int_u^{\tau_j(u)} |E_x(s)|\,ds=0\,. \tag{2.9}$$

Therefore $(Tx)(t)\to 0$; i.e., the fixed point $x=Tx$, satisfies $\lim_{t\to\infty}x(t)=0$. $\square$

The above theorem provides sufficient conditions for the convergence of solutions to zero. The next theorem provides necessary conditions.

Theorem 2.6. *Assume that* (2.6) *holds, and*

$$\liminf_{t\to\infty}\int_{t_0}^{t}(A(s)+B(s))\,ds>-\infty. \tag{2.10}$$

If all solutions of (1.1) *converge to zero, then* (2.7) *holds.*

Proof. For the shake of contradiction suppose that (2.7) does not hold; i. e.,

$$\liminf_{t\to\infty}\int_{t_0}^{t_n}(A(s)+B(s))\,ds=\alpha<\infty.$$

Then by (2.10), $\alpha>-\infty$. Then there exists a sequence $\{t_n\}$ approaching ∞, such that

$$\lim_{n\to\infty}\int_{t_0}^{t_n}(A(s)+B(s))\,ds=\alpha\,.$$

Let x be a solution such that $x(t_0)=x_0\neq 0$. then

$$\lim_{n\to\infty}x_0e^{-\int_{t_0}^{t_n}(A(s)+B(s))\,ds}=x_0e^{\alpha}\neq 0\,. \tag{2.11}$$

By Lemma 2.4, $x(t_n)$ satisfies (2.3) with t_n instead of t. By (2.8) and (2.9),

$$\begin{aligned}&\lim_{n\to\infty}\int_{t_0}^{t_n}e^{-\int_u^{t_n}(A(s)+B(s))\,ds}\\&\times\Big(\sum_{i=1}^{m}a_i(u)\int_{r_i(u)}^{u}E_x(s)\,ds-\sum_{j=1}^{n}b_j(u)\int_u^{\tau_j(u)}E_x(s)\,ds\Big)\,du=0\,.\end{aligned} \tag{2.12}$$

Since all solutions approach zero, by (2.3), (2.11) and (2.12), it follows that

$$0=\lim_{n\to\infty}x(t_n)=x_0e^{\alpha}+0\neq 0\,.$$

This contradiction competes the proof. $\square$

3. An example

In this section we provide an example to illustrate our results.

Example 3.1. Consider the equation

$$\dot x(t)+\sin(t)e^{-(a-1)t/a}x(t/a)+\big(1-\sin(t)\big)e^{(b-1)t}x(bt)=0,\quad t>0 \tag{3.1}$$

where $a>b>1$ and $b-1<(a-1)/a$. Note that $r_0=t_0=0$ in this example, so the initial condition (2.1) reduces to $x(t_0)=x_0$. We want to check that all the assumptions of Theorem 2.5 are satisfied.

$$\begin{aligned}&\int_0^t(A(s)+B(s))\,ds\\&=\int_0^t\big(\sin(t)e^{-(a-1)t/a}+\big(1-\sin(t)\big)e^{(b-1)t}\big)\,ds\\&=\frac{e^{-\alpha t}}{\alpha^2+1}\big(\alpha\sin(t)-\cos(t)\big)+e^{\beta t}\big(\frac{1}{\beta}-\frac{\beta}{\beta^2+1}\sin(t)+\frac{1}{\beta^2+1}\cos(t)\big)+k,\end{aligned}$$

where $\alpha=(a-1)/a$, $\beta=b-1$ and

$$k=\frac{1}{\alpha^2+1}+\frac{1}{\beta}+\frac{1}{\beta^2+1}.$$

Since

$$\frac{e^{-\alpha t}}{\alpha^2+1}(\alpha\sin(t)-\cos(t))\to 0,\quad\text{and}$$
$$e^{\beta t}(\frac{1}{\beta}-\frac{\beta}{\beta^2+1}\sin(t)+\frac{1}{\beta^2+1}\cos(t))\to\infty,$$

as $t\to\infty$, it follows that (2.6) holds. Also note that

$$\begin{aligned}
&\int_0^t e^{-\int_u^t\left(\sin(s)e^{-\alpha s}+(1-\sin(s))e^{\beta s}\right)ds}\Big\{|\sin(u)|e^{-\alpha u}\int_{u/a}^{u}\big(|\sin(s)|e^{-\alpha s}\\
&\quad+|1-\sin(s)|e^{\beta s}\big)\,ds+|1-\sin(u)|e^{\beta u}\int_u^{bu}(|\sin(s)|e^{-\alpha s}+|1-\sin(s)|e^{\beta s})\,ds\Big\}\,du\\
&<\int_0^t e^{-\int_u^t(\sin(s)e^{-\alpha s}+(1-\sin(s))e^{\beta s})\,ds}\Big\{e^{-\alpha u}\int_{u/a}^{u}(e^{-\alpha s}\\
&\quad+2e^{\beta s})\,ds+2e^{\beta u}\int_u^{bu}(e^{-\alpha s}+2e^{\beta s})\,ds\Big\}\,du\\
&=\int_0^t e^{-\int_u^t(\sin se^{-\alpha s}+(1-\sin s)e^{\beta s})\,ds}\Big\{e^{-\alpha u}(\frac{e^{-\alpha u/a}-e^{-\alpha u}}{\alpha}+2\frac{e^{\beta u}-e^{\beta u/a}}{\beta})\\
&\quad+2e^{\beta u}(\frac{e^{-\alpha u}-e^{-\alpha bu}}{\alpha}+2\frac{e^{\beta bu}-e^{\beta u}}{\beta})\Big\}\,du\\
&<te^{-\int_0^t(\sin(s)e^{-\alpha s}+(1-\sin(s))e^{\beta s})\,ds}\Big\{e^{-\alpha t}(\frac{e^{-\alpha t/a}-e^{-\alpha t}}{\alpha}+2\frac{e^{\beta t}-e^{\beta t/a}}{\beta})\\
&\quad+2e^{\beta t}(\frac{e^{-\alpha t}-e^{-\alpha bt}}{\alpha}+2\frac{e^{\beta bt}-e^{\beta t}}{\beta})\Big\}\to 0\quad\text{as } t\to\infty\,.
\end{aligned}$$

Note that

$$te^{-\int_0^t(\sin(s)e^{-\alpha s}+(1-\sin(s))e^{\beta s})\,ds}\to 0\quad\text{as } t\to\infty,$$

and

$$e^{-\alpha t}\Big(\frac{e^{-\alpha t/a}-e^{-\alpha t}}{\alpha}+2\frac{e^{\beta t}-e^{\beta t/a}}{\beta}\Big)+2e^{\beta t}\Big(\frac{e^{-\alpha t}-e^{-\alpha bt}}{\alpha}+2\frac{e^{\beta bt}-e^{\beta t}}{\beta}\Big)\to 0$$

as $t\to\infty$, when $a>b>1$ and $b-1<(a-1)/a$. So, there exists an ϵ, $0<\epsilon<c/2$, such that

$$e^{-\int_0^t(\sin(t)e^{-(a-1)t/a}+(1-\sin(t))e^{(b-1)t})\,ds}<\epsilon,$$

and

$$\begin{aligned}
&\int_0^t e^{-\int_u^t(\sin(s)e^{-\alpha s}+(1-\sin(s))e^{\beta s})\,ds}\Big\{|\sin(u)|e^{-\alpha u}\int_{u/a}^{u}\big(|\sin(s)|e^{-\alpha s}\\
&+|1-\sin(s)|e^{\beta s}\big)\,ds+|1-\sin(u)|e^{\beta u}\int_u^{bu}\big(|\sin(s)|e^{-\alpha s}+|1-\sin(s)|e^{\beta s}\big)\,ds\Big\}\,du\\
&<\epsilon
\end{aligned}$$

Therefore, the conditions of Theorem 2.5 are satisfied, consequently all solutions of (3.1) converge to zero. In fact the function is $x(t)=x_0e^{-t}$ which converges to zero, for each initial condition x_0.

Note that condition (2.9) holds, so the conditions for Theorem 2.6 are also satisfied.

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