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BLOW-UP OF SOLUTIONS TO SYSTEMS OF NONLINEAR INEQUALITIES WITH SINGULARITIES ON UNBOUNDED SETS

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Abstract. We establish conditions for the blow-up of solutions to several systems of nonlinear differential inequalities, with singularities on unbounded sets.

1. Introduction

In recent years, many mathematicians study global solvability of nonlinear partial differential equations and inequalities with singular coefficients. Here nonlinear terms can depend both on the values of the unknown function and on its derivatives. This problem is not only interesting in its own right but also has important mathematical and physical applications. Thus, Liouville type theorems on nonexistence of positive solutions to nonlinear equations in the whole space or in the half-space can be used to obtain a priori estimates of solutions to respective problems in bounded domains [1, 2]. On the other hand, the class under consideration includes, in particular, Hamilton–Jacobi and Korteweg–de Vries equations that play an important role in contemporary mathematical physics [11, 12].

In [3]–[14] and their references, sharp necessary conditions for existence of global solutions to different classes of second-order elliptic equations with gradient terms were obtained. The proofs are based either on ODE techniques for radially symmetric solutions or on the nonlinear capacity method, which was suggested in [10] and extensively developed in [9].

In this article we obtain sufficient conditions for nonexistence of solutions for several classes of systems of inequalities that have singular coefficients on unbounded sets, such as straight lines and planes, as well as smooth curves and surfaces in $\mathbb{R}^N$. Here we obtain nonexistence results in natural functional classes, that is, assuming only minimal local integrability properties, unlike our previous paper [7] where lower bounds for local integrals were required. Some further related results for scalar inequalities were included into [8].

The main results of the paper are formulated in section 2. In section 3, a nonexistence theorem is proven for a system of higher-order elliptic inequalities with a nonlinearity dependent only on the values of the unknown function u, and in section 4, a similar result is obtained for a system of inequalities with a nonlinearity

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containing $|Du|$. In sections 5 and 6, the respective statements are extended to systems of second-order elliptic inequalities with a nonlinear principal part.

2. Main results

We assume that the set S satisfies a geometrical condition which is based on an idea from our paper [7], and modified according to our problem setting and functional classes under consideration. To formulate it, we need some extra notation.

Let $\varepsilon > 0$ and $x \in \mathbb{R}^N$. Denote $\rho(x) = \operatorname{dist}(x, S)$,

$$B_\varepsilon(0) = \{x \in \mathbb{R}^N : |x| \le \varepsilon\},$$

and

$$S_\varepsilon = \{x \in \mathbb{R}^N : \rho(x) < \varepsilon\}.$$

For $R > 0$, introduce the set

$$S^R = \overline{S_R \setminus S_{1/R}} \cap \overline{B_R(0)}.$$

Now we can formulate our assumption on S.

(H1) Suppose that there exists a family of functions $\xi_R \in C_0^{2k}(\mathbb{R}^N \setminus S; [0,1])$ such that

$$\xi_R(x) = \begin{cases} 0 & (x \in S_{1/(2R)} \cup (\mathbb{R}^N \setminus S_{2R})), \\ 1 & (x \in S_R \setminus S_{1/R}) \end{cases} \tag{2.1}$$

and there exists a constant $c > 0$ such that

$$|D^\alpha \xi_R(x)| \le c\rho^{-|\alpha|} \quad (x \in \mathbb{R}^N). \tag{2.2}$$

Example 2.1. We can consider as the set S a hyperplane $S = \Pi_n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^N : x_n = 0\}$. In that case we can choose $\xi_R(x) = \tilde{\xi}_R(x_n)$, where

$$\tilde{\xi}_R(x_n) = \begin{cases} 0 & (|x_n| \le \frac{1}{2R} \text{ or } |x_n| \ge 2R), \\ 1 & (\frac{1}{R} \le |x_n| \le R) \end{cases}$$

See Figure 1.

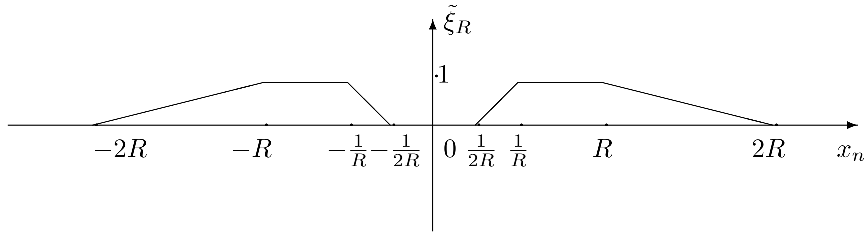

Figure 1. The function $\tilde{\xi}_R(x_n)$

Further we assume that the set S satisfies assumption (H1). We formulate our first result for a system of nonlinear elliptic inequalities

$$\begin{gathered} -\Delta_p u \ge a(x) v^{q_1} \quad (x \in \mathbb{R}^N \setminus S), \\ -\Delta_q v \ge b(x) u^{p_1} \quad (x \in \mathbb{R}^N \setminus S), \\ u(x),\, v(x) \ge 0 \quad (x \in \mathbb{R}^N \setminus S), \end{gathered} \tag{2.3}$$

where $p, q, p_1, q_1 > 1$, $p-1 < p_1$, $q-1 < q_1$, $a, b \in C(\mathbb{R}^N \setminus S)$ are nonnegative functions such that $a(x) \geq a_0 \rho^{-\alpha}|x|^{\beta}$, $b(x) \geq b_0 \rho^{-\gamma}|x|^{\delta}$ for $x \in \mathbb{R}^N \setminus S$, $a_0, b_0 > 0$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\rho(x) = \operatorname{dist}(x, S)$. Introduce the quantities

$$\begin{aligned} \sigma_1 &= \frac{(|\alpha|-\beta)(q-1) + (|\gamma|-\delta-q)q_1)(p-1) - pp_1q_1}{p_1q_1 - (p-1)(q-1)}, \\ \sigma_2 &= \frac{(|\gamma|-\delta)(p-1) + (|\alpha|-\beta-p)p_1)(q-1) - qp_1q_1}{p_1q_1 - (p-1)(q-1)}. \end{aligned} \tag{2.4}$$

Theorem 2.2. *Let $N + \min\{\sigma_1, \sigma_2\} \leq 0$. Then system* (2.3) *has no nontrivial (nonzero) solution.*

Example 2.3. For a system of inequalities (2.3) with $p = q = 2$:

$$\begin{gathered} -\Delta u \geq a(x) v^{q_1} \quad (x \in \mathbb{R}^N \setminus S), \\ -\Delta v \geq b(x) u^{p_1} \quad (x \in \mathbb{R}^N \setminus S), \\ u(x),\, v(x) \geq 0 \quad (x \in \mathbb{R}^N \setminus S), \end{gathered} \tag{2.5}$$

the quantities defined in (2.4) take the form

$$\begin{aligned} \sigma_1 &= \frac{(\alpha-\beta+(\gamma-\delta-2)q_1) - 2p_1q_1}{p_1q_1 - 1}, \\ \sigma_2 &= \frac{(\gamma-\delta+(\alpha-\beta-2)p_1) - 2p_1q_1}{p_1q_1 - 1}. \end{aligned} \tag{2.6}$$

Further we consider the system

$$\begin{aligned} -\Delta_p u &\geq a(x)|Dv|^{q_1} \quad (x \in \mathbb{R}^N \setminus S), \\ -\Delta_q v &\geq b(x)|Du|^{p_1} \quad (x \in \mathbb{R}^N \setminus S). \end{aligned} \tag{2.7}$$

For this system one has the following result.

Theorem 2.4. *Let*

$$\begin{aligned} &\max\{(p-1)(\alpha(q-1)+q_1(\beta+1)), (q-1)(\beta(p-1)+p_1(\alpha+1))\} \\ &\geq N(p_1q_1 - (p-1)(q-1)) - p_1q_1. \end{aligned}$$

Then system (2.7) *has no nontrivial (non-constant) solution.*

We also consider the systems of higher-order differential inequalities

$$\begin{aligned} (-\Delta)^k u &\geq a(x)|Dv|^q \quad (x \in \mathbb{R}^N \setminus S), \\ (-\Delta)^l v &\geq b(x)|Du|^p \quad (x \in \mathbb{R}^N \setminus S) \end{aligned} \tag{2.8}$$

and

$$\begin{gathered} (-\Delta)^k u \geq a(x) v^q \quad (x \in \mathbb{R}^N \setminus S), \\ (-\Delta)^l v \geq b(x) u^p \quad (x \in \mathbb{R}^N \setminus S), \\ u \geq 0, \quad v \geq 0 \quad (x \in \mathbb{R}^N), \end{gathered} \tag{2.9}$$

where $k, l \in \mathbb{N}$, $p, q > 1$.

For system (2.8), there holds the following theorem.

Theorem 2.5. *Let*

$$\max\Big\{|\alpha| + \beta + (|\gamma| + \delta + 2l - 1 + (2k-1)p)q,$$

$$|\gamma|+\delta+(|\alpha|+\beta+2k-1+(2l-1)q)p\Big\}\geq N(pq-1).$$

Then system (2.8) *has no nontrivial (non-constant) solution.*

For system (2.9), one has

Theorem 2.6. *Let*

$$\max\Big\{|\alpha|+\beta+(|\gamma|+\delta+2l+2kp)q,|\gamma|+\delta+(|\alpha|+\beta+2k+2lq)p\Big\}\geq N(pq-1).$$

Then system (2.9) *has no nontrivial (nonzero) solution.*

3. Proof of Theorem 2.2

Suppose that there exists (u,v) – a nontrivial solution of system (2.3). Let $\varphi_R\in C_0^\infty(\overline{\mathbb{R}^N};\mathbb{R}_+)$ be a family of test functions to be specified below.

Multiplying the first inequality (2.3) by $u_\varepsilon^\lambda\varphi_R$ and the second one by $v_\varepsilon^\lambda\varphi_R$, where $u_\varepsilon=u+\varepsilon$, $v_\varepsilon=v+\varepsilon$, $\varepsilon>0$ and $\max\{1-p,1-q\}<\lambda<0$, we obtain

$$\int a(x)v^{q_1}u_\varepsilon^\lambda\varphi_R\,dx\leq c\lambda\int|Du|^pu_\varepsilon^{\lambda-1}\varphi_R\,dx+\int|Du|^{p-1}|D\varphi_R|u_\varepsilon^\lambda\,dx, \tag{3.1}$$

$$\int b(x)u^{p_1}v_\varepsilon^\lambda\varphi_R\,dx\leq c\lambda\int|Dv|^qv_\varepsilon^{\lambda-1}\varphi_R\,dx\ +\int|Dv|^{q-1}|D\varphi_R|v_\varepsilon^\lambda\,dx. \tag{3.2}$$

Application of Young's inequality to the first terms on the right-hand sides of the obtained relations results in

$$\int a(x)v^{q_1}u_\varepsilon^\lambda\varphi_R\,dx+\frac{c|\lambda|}{2}\int|Du|^pu_\varepsilon^{\lambda-1}\varphi_R\,dx\leq c_\lambda\int\frac{|D\varphi_R|^p}{\varphi_R^{p-1}}u_\varepsilon^{\lambda+p-1}\,dx, \tag{3.3}$$

$$\int b(x)u^{p_1}v_\varepsilon^\lambda\varphi_R\,dx+\frac{c|\lambda|}{2}\int|Dv|^qv_\varepsilon^{\lambda-1}\varphi_R\,dx\leq d_\lambda\int\frac{|D\varphi_R|^q}{\varphi_R^{q-1}}v_\varepsilon^{\lambda+q-1}\,dx, \tag{3.4}$$

where the constants c_λ and d_λ depend only on p,q, and λ. Further, multiplying each differential inequality (2.3) by φ_R and integrating by parts, we arrive at

$$\int a(x)v^{q_1}\varphi_R\,dx\leq\Big(\int|Du|^pu_\varepsilon^{\lambda-1}\varphi_R\,dx\Big)^{\frac{p-1}{p}}\Big(\int\frac{|D\varphi_R|^p}{\varphi_R^{p-1}}u_\varepsilon^{(1-\lambda)(p-1)}\,dx\Big)^{1/p}, \tag{3.5}$$

$$\int b(x)u^{p_1}\varphi_R\,dx\leq\Big(\int|Dv|^qv_\varepsilon^{\lambda-1}\varphi_R\,dx\Big)^{\frac{q-1}{q}}\Big(\int\frac{|D\varphi_R|^q}{\varphi_R^{q-1}}v_\varepsilon^{(1-\lambda)(q-1)}\,dx\Big)^{1/q}. \tag{3.6}$$

Combining (3.3)–(3.6) and taking $\varepsilon\to0$, we obtain a priori estimates

$$\int a(x)v^{q_1}\varphi_R\,dx\leq D_\lambda\Big(\int\frac{|D\varphi_R|^p}{\varphi_R^{p-1}}u^{\lambda+p-1}\,dx\Big)^{\frac{p-1}{p}}\Big(\int\frac{|D\varphi_R|^p}{\varphi_R^{p-1}}u^{(1-\lambda)(p-1)}\,dx\Big)^{1/p}, \tag{3.7}$$

$$\int b(x)u^{p_1}\varphi_R\,dx\leq E_\lambda\Big(\int\frac{|D\varphi_R|^q}{\varphi_R^{q-1}}v^{\lambda+q-1}\,dx\Big)^{\frac{q-1}{q}}\Big(\int\frac{|D\varphi_R|^q}{\varphi_R^{q-1}}v^{(1-\lambda)(q-1)}\,dx\Big)^{1/q}, \tag{3.8}$$

where D_λ, $E_\lambda>0$ depend only on p, q, and λ.

Applying the Hölder inequality with exponent r to the first integral on the right-hand side of (3.7), we obtain

$$\begin{aligned} &\Big(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}} u^{\lambda+p-1}\,dx\Big)^{\frac{p-1}{p}} \\ &\le \Big(\int b(x) u^{(\lambda+p-1)r}\varphi_R\,dx\Big)^{\frac{p-1}{pr}} \Big(\int b^{-\frac{r'}{r}}(x)\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\Big)^{\frac{p-1}{pr'}}, \end{aligned} \tag{3.9}$$

where $\frac{1}{r}+\frac{1}{r'}=1$.

Choosing the exponent r so that $(\lambda+p-1)r=p_1$, from (3.7) and (3.9) we have

$$\begin{aligned} \int a(x)v^{q_1}\varphi_R\,dx &\le D_\lambda\Big(\int b(x)u^{p_1}\varphi_R\,dx\Big)^{\frac{p-1}{pr}}\Big(\int b^{-\frac{r'}{r}}(x)\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\Big)^{\frac{p-1}{pr'}} \\ &\quad\times\Big(\int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}}u^{(1-\lambda)(p-1)}\,dx\Big)^{1/p}. \end{aligned} \tag{3.10}$$

Applying the Hölder inequality with exponent $y>1$ to the last integral on the right-hand side of (3.10), we obtain

$$\begin{aligned} \int \frac{|D\varphi_R|^p}{\varphi_R^{p-1}}u^{(1-\lambda)(p-1)}\,dx &\le \Big(\int b(x)u^{(1-\lambda)(p-1)y}\varphi_R\,dx\Big)^{1/y} \\ &\quad\times\Big(\int b^{-\frac{y'}{y}}(x)\frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\Big)^{1/y'}, \end{aligned} \tag{3.11}$$

where $\frac{1}{y}+\frac{1}{y'}=1$.

Choosing y in (3.11) so that $(1-\lambda)(p-1)y=p_1$ and taking into account (3.10), we reach the estimate

$$\begin{aligned} \int a(x)v^{q_1}\varphi_R\,dx &\le D_\lambda\Big(\int b(x)u^{p_1}\varphi_R\,dx\Big)^{\frac{p-1}{pr}}\Big(\int b^{-\frac{r'}{r}}(x)\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\Big)^{\frac{p-1}{pr'}} \\ &\quad\times\Big(\int b(x)u^{p_1}\varphi_R\,dx\Big)^{\frac{1}{py}}\Big(\int b^{-\frac{y'}{y}}(x)\frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\Big)^{\frac{1}{py'}}; \end{aligned}$$

i.e.,

$$\begin{aligned} &\int a(x)v^{q_1}\varphi_R\,dx \\ &\le D_\lambda\Big(\int b(x)u^{p_1}\varphi_R\,dx\Big)^{\frac{p-1}{pr}+\frac{1}{py}}\Big(\int b^{-\frac{r'}{r}}(x)\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\Big)^{\frac{p-1}{pr'}} \\ &\quad\times\Big(\int b^{-\frac{y'}{y}}(x)\frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\Big)^{\frac{1}{py'}}, \end{aligned} \tag{3.12}$$

where the exponents r and y are chosen so that

$$\begin{gathered} \frac{1}{y}+\frac{1}{y'}=1,\quad (1-\lambda)(p-1)y=p_1, \\ \frac{1}{r}+\frac{1}{r'}=1,\quad (\lambda+p-1)r=p_1. \end{gathered} \tag{3.13}$$

Note that such choice of r and y is possible due to our hypotheses on p and p_1 provided that $\lambda<0$ is small enough in absolute value. Similarly, choosing s and z

such that

$$\begin{gathered}
\frac{1}{z}+\frac{1}{z'}=1, \quad (1-\lambda)(q-1)z=q_1, \\
\frac{1}{s}+\frac{1}{s'}=1, \quad (\lambda+q-1)s=q_1,
\end{gathered} \tag{3.14}$$

and estimating the right-hand side of (3.8) by the Hölder inequality, we obtain

$$\begin{aligned}
&\int b(x)u^{p_1}\varphi_R\,dx \\
&\le E_\lambda\Big(\int a(x)v^{q_1}\varphi_R\,dx\Big)^{\frac{q-1}{qs}+\frac{1}{qz}}\Big(\int a^{-\frac{s'}{s}}(x)\frac{|D\varphi_R|^{qs'}}{\varphi_R^{qs'-1}}\,dx\Big)^{\frac{q-1}{qs'}} \\
&\quad\times\Big(\int a^{-\frac{z'}{z}}(x)\frac{|D\varphi_R|^{qz'}}{\varphi_R^{qz'-1}}\,dx\Big)^{\frac{1}{qz'}}.
\end{aligned} \tag{3.15}$$

Combining (3.12) and (3.15), we finally arrive at

$$\begin{aligned}
&\Big(\int a(x)v^{q_1}\varphi_R\,dx\Big)^{1-mn} \\
&\le D_\lambda E_\lambda^N\Big(\int a^{-\frac{s'}{s}}(x)\frac{|D\varphi_R|^{qs'}}{\varphi_R^{qs'-1}}\,dx\Big)^{\frac{n(q-1)}{qs'}}\Big(\int a^{-\frac{z'}{z}}(x)\frac{|D\varphi_R|^{qz'}}{\varphi_R^{qz'-1}}\,dx\Big)^{\frac{n}{qz'}} \\
&\quad\times\Big(\int b^{-\frac{r'}{r}}(x)\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\Big)^{\frac{p-1}{pr'}}\Big(\int b^{-\frac{y'}{y}}(x)\frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\Big)^{\frac{1}{py'}}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\Big(\int b(x)u^{p_1}\varphi_R\,dx\Big)^{1-mn} \\
&\le E_\lambda D_\lambda^m\Big(\int b^{-\frac{r'}{r}}(x)\frac{|D\varphi_R|^{pr'}}{\varphi_R^{pr'-1}}\,dx\Big)^{\frac{m(p-1)}{pr'}}\Big(\int b^{-\frac{y'}{y}}(x)\frac{|D\varphi_R|^{py'}}{\varphi_R^{py'-1}}\,dx\Big)^{\frac{m}{py'}} \\
&\quad\times\Big(\int a^{-\frac{s'}{s}}(x)\frac{|D\varphi_R|^{qs'}}{\varphi_R^{qs'-1}}\,dx\Big)^{\frac{q-1}{qs'}}\Big(\int a^{-\frac{z'}{z}}(x)\frac{|D\varphi_R|^{qz'}}{\varphi_R^{qz'-1}}\,dx\Big)^{\frac{1}{qz'}},
\end{aligned} \tag{3.17}$$

$$n:=\frac{p-1}{pr}+\frac{1}{py}, \quad m:=\frac{q-1}{qs}+\frac{1}{qz}. \tag{3.18}$$

Simple calculations taking into account (3.13) and (3.14) give explicit values of m and n, namely,

$$m=\frac{q-1}{q_1}, \quad n=\frac{p-1}{p_1}. \tag{3.19}$$

Our assumptions imply that the exponent on the left-hand side of (3.16), (3.17) is such that

$$1-mn=\frac{p_1q_1-(p-1)(q-1)}{p_1q_1}>0.$$

Thus from (3.17) and our assumptions on a and b we have

$$\begin{gathered}
\int_{(S^R\setminus S^{1/R})\cap B_R(0)} a(x)v^{q_1}\,dx\le CR^{N+\sigma_1}, \\
\int_{(S^R\setminus S^{1/R})\cap B_R(0)} b(x)u^{p_1}\,dx\le CR^{N+\sigma_2}.
\end{gathered} \tag{3.20}$$

Taking $R \to \infty$ in (3.20), under condition $N + \min\{\sigma_1, \sigma_2\} \le 0$ we come to a contradiction, which completes the proof of Theorem 2.2.

4. Proof of Theorem 2.4

Multiplying inequalities (2.7) by the test function $\varphi_R \in C_0^1(\mathbb{R}^N; [0,1])$ and integrating by parts, we obtain

$$\begin{gathered}
\int_{\mathbb{R}^N} a(x)|Dv|^{q_1}\varphi_R(x)\,dx \le \int_{\mathbb{R}^N} (|Du|^{p-2}Du, D\varphi_R)\,dx,\\
\int_{\mathbb{R}^N} b(x)|Du|^{p_1}\varphi_R(x)\,dx \le \int_{\mathbb{R}^N} (|Dv|^{q-2}Dv, D\varphi_R)\,dx,
\end{gathered}$$

which because of relations

$$(|Du|^{p-2}Du, D\varphi_R) \le |Du|^{p-1}|D\varphi_R|, \quad (|Dv|^{q-2}Dv, D\varphi_R) \le |Dv|^{q-1}|D\varphi_R|$$

and the Hölder inequality, results in

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|Dv|^{q_1}\varphi_R(x)\,dx\\
&\le \Big(\int_{\mathbb{R}^N} b(x)|Du|^{p_1}\varphi_R(x)\,dx\Big)^{\frac{p-1}{p_1}} \qquad (4.1)\\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{p-1}{p_1-p+1}}(x)|D\varphi_R|^{\frac{p_1}{p_1-p+1}}\varphi_R^{1-\frac{p_1}{p_1-p+1}}(x)\,dx\Big)^{\frac{p_1-p+1}{p_1}},\\
&\int_{\mathbb{R}^N} b(x)|Du|^{p_1}\varphi_R(x)\,dx\\
&\le \Big(\int_{\mathbb{R}^N} a(x)|Dv|^{q_1}\varphi_R(x)\,dx\Big)^{\frac{q-1}{q_1}} \qquad (4.2)\\
&\quad\times \Big(\int_{\mathbb{R}^N} a^{-\frac{q-1}{q_1-q+1}}(x)|D\varphi_R|^{\frac{q_1}{q_1-q+1}}\varphi_R^{1-\frac{q_1}{q_1-q+1}}(x)\,dx\Big)^{\frac{q_1-q+1}{q_1}}.
\end{aligned}$$

Substituting (4.1) into (4.2) and vice versa, we obtain

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N} a(x)|Dv|^{q_1}\varphi_R(x)\,dx\Big)^{1-\frac{(p-1)(q-1)}{p_1q_1}}\\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{q-1}{q_1-q+1}}(x)|D\varphi_R|^{\frac{q_1}{q_1-q+1}}\varphi_R^{1-\frac{q_1}{q_1-q+1}}(x)\,dx\Big)^{\frac{(p-1)(q_1-q+1)}{p_1q_1}}\\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{p-1}{p_1-p+1}}(x)|D\varphi_R|^{\frac{p_1}{p_1-p+1}}\varphi_R^{1-\frac{p_1}{p_1-p+1}}(x)\,dx\Big)^{\frac{p_1-p+1}{p_1}},
\end{aligned}$$

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N} b(x)|Du|^{p_1}\varphi_R(x)\,dx\Big)^{1-\frac{(p-1)(q-1)}{p_1q_1}}\\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{q-1}{q_1-q+1}}(x)|D\varphi_R|^{\frac{q_1}{q_1-q+1}}\varphi_R^{1-\frac{q_1}{q_1-q+1}}(x)\,dx\Big)^{\frac{q_1-q+1}{q_1}}\\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{p-1}{p_1-p+1}}(x)|D\varphi_R|^{\frac{p_1}{p_1-p+1}}\varphi_R^{1-\frac{p_1}{p_1-p+1}}(x)\,dx\Big)^{\frac{(q-1)(p_1-p+1)}{p_1q_1}};
\end{aligned}$$

i. e.,

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|Dv|^{q_1}\varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{q-1}{q_1-q+1}}(x)|D\varphi_R|^{\frac{q_1}{q_1-q+1}}\varphi_R^{1-\frac{q_1}{q_1-q+1}}(x)\,dx\Big)^{\frac{(p-1)(q_1-q+1)}{p_1q_1-(p-1)(q-1)}} \\
&\quad\times\Big(\int_{\mathbb{R}^N} b^{-\frac{p-1}{p_1-p+1}}(x)|D\varphi_R|^{\frac{p_1}{p_1-p+1}}\varphi_R^{1-\frac{p_1}{p_1-p+1}}(x)\,dx\Big)^{\frac{q_1(p_1-p+1)}{p_1q_1-(p-1)(q-1)}},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N} b(x)|Du|^{p_1}\varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{q-1}{q_1-q+1}}(x)|D\varphi_R|^{\frac{q_1}{q_1-q+1}}\varphi_R^{1-\frac{q_1}{q_1-q+1}}(x)\,dx\Big)^{\frac{p_1(q_1-q+1)}{p_1q_1-(p-1)(q-1)}} \\
&\quad\times\Big(\int_{\mathbb{R}^N} b^{-\frac{p-1}{p_1-p+1}}(x)|D\varphi_R|^{\frac{p_1}{p_1-p+1}}\varphi_R^{1-\frac{p_1}{p_1-p+1}}(x)\,dx\Big)^{\frac{(q-1)(p_1-p+1)}{p_1q_1-(p-1)(q-1)}}.
\end{aligned} \tag{4.4}$$

Choosing test function $\varphi_R \in C_0^1(\mathbb{R}^N;[0,1])$ so that

$$\varphi_R(x) = \begin{cases} 1 & (|x| \le R), \\ 0 & (|x| \ge 2R), \end{cases}$$

and

$$|D\varphi_R(x)| \le cR^{-1} \quad (x \in \mathbb{R}^N), \tag{4.5}$$

we obtain

$$\begin{gathered}
\int_{B_R(0)} a(x)|Dv|^{q_1}\,dx \le cR^{N-\frac{(p-1)((|\alpha|+\beta)(q-1)+q_1(|\gamma|+\delta+1))+p_1q_1}{p_1q_1-(p-1)(q-1)}}, \\
\int_{B_R(0)} b(x)|Du|^{p_1}\,dx \le cR^{N-\frac{(q-1)((|\gamma|+\delta)(p-1)+p_1(|\alpha|+\beta+1))+p_1q_1}{p_1q_1-(p-1)(q-1)}}.
\end{gathered}$$

Taking $R \to \infty$, we complete the proof similarly to Theorem 2.2.

5. Proof of Theorem 2.5

Multiplying inequalities (2.8) by the test function $\varphi_R \in C_0^{2k-1}(\mathbb{R}^N;[0,1])$ and integrating by parts, we obtain

$$\begin{gathered}
\int_{\mathbb{R}^N} a(x)|Dv|^q\varphi_R(x)\,dx \le \int_{\mathbb{R}^N}(Du, D((-\Delta)^{k-1}\varphi_R))\,dx, \\
\int_{\mathbb{R}^N} b(x)|Du|^p\varphi_R(x)\,dx \le \int_{\mathbb{R}^N}(Dv, D((-\Delta)^{l-1}\varphi_R))\,dx,
\end{gathered}$$

which by relations

$$\begin{gathered}
(Du, D((-\Delta)^{k-1}\varphi_R)) \le |Du|\cdot|D((-\Delta)^{k-1}\varphi_R)|, \\
(Dv, D((-\Delta)^{l-1}\varphi_R)) \le |Dv|\cdot|D((-\Delta)^{l-1}\varphi_R)|,
\end{gathered}$$

and the Hölder inequality, results in

$$
\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|Dv|^q\varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} b(x)|Du|^p\varphi_R(x)\,dx\Big)^{1/p} \\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|D((-\Delta)^{k-1}\varphi_R)|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{p}},
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^N} b(x)|Du|^p\varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a(x)|Dv|^q\varphi_R(x)\,dx\Big)^{1/q} \\
&\quad\times \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|D((-\Delta)^{l-1}\varphi_R)|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{q}}.
\end{aligned}
\tag{5.2}
$$

Substituting (5.1) into (5.2) and vice versa, we obtain

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^N} a(x)|Dv|^q\varphi_R(x)\,dx\Big)^{1-\frac{1}{pq}} \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|D((-\Delta)^{l-1}\varphi_R)|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{pq}} \\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|D((-\Delta)^{k-1}\varphi_R)|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{p}},
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^N} b(x)|Du|^p\varphi_R(x)\,dx\Big)^{1-\frac{1}{pq}} \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|D((-\Delta)^{l-1}\varphi_R)|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{q}} \\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|D((-\Delta)^{k-1}\varphi_R)|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{pq}};
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|Dv|^q\varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|D((-\Delta)^{l-1}\varphi_R)|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{pq-1}} \\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|D((-\Delta)^{k-1}\varphi_R)|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{(p-1)q}{pq-1}},
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^N} b(x)|Du|^p\varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|D((-\Delta)^{l-1}\varphi_R)|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{p(q-1)}{pq-1}} \\
&\quad\times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|D((-\Delta)^{k-1}\varphi_R)|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{pq-1}}.
\end{aligned}
\tag{5.4}
$$

Denote $K=\max\{k,l\}$. Choosing the test function $\varphi_R\in C_0^{2K-1}(\mathbb{R}^N;[0,1])$ so that
$$\varphi_R(x)=\begin{cases}1 & (|x|\le R),\\ 0 & (|x|\ge 2R),\end{cases}$$
and
$$|D^\mu\varphi_R(x)|\le cR^{-1}\quad (x\in\mathbb{R}^N; 0\le|\mu|\le 2K-1), \tag{5.5}$$
we have
$$\int_{B_R(0)} a(x)|Dv|^q\,dx\le cR^{N-\frac{|\alpha|+\beta+(|\gamma|+\delta+2l-1+(2k-1)p)q}{pq-1}},$$
$$\int_{B_R(0)} b(x)|Du|^p\,dx\le cR^{N-\frac{|\gamma|+\delta+(|\alpha|+\beta+2k-1+(2l-1)q)p}{pq-1}}.$$
Taking $R\to\infty$, we complete the proof similarly to Theorems 2.2 and 2.4.

6. Proof of Theorem 2.6

Multiplying inequalities (2.9) by the test function $\varphi_R\in C_0^{2K}(\mathbb{R}^N;[0,1])$, where $K=\max\{k,l\}$, and integrating by parts, we obtain
$$\int_{\mathbb{R}^N} a(x)v^q\varphi_R(x)\,dx\le\int_{\mathbb{R}^N} u\cdot(-\Delta)^k\varphi_R\,dx,$$
$$\int_{\mathbb{R}^N} b(x)u^p\varphi_R(x)\,dx\le\int_{\mathbb{R}^N} v\cdot(-\Delta)^l\varphi_R\,dx.$$
Taking into account that
$$u\cdot(-\Delta)^k\varphi_R\le u\cdot|(-\Delta)^k\varphi_R)|,$$
$$v\cdot(-\Delta)^l\varphi_R\le v\cdot|(-\Delta)^l\varphi_R)|$$
and using the Hölder inequality, we arrive at
$$\begin{aligned}\int_{\mathbb{R}^N} a(x)v^q\varphi_R(x)\,dx&\le\Big(\int_{\mathbb{R}^N} b(x)u^p\varphi_R(x)\,dx\Big)^{1/p}\\&\quad\times\Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|(-\Delta)^k\varphi_R|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{p}},\end{aligned}\tag{6.1}$$
$$\begin{aligned}\int_{\mathbb{R}^N} b(x)u^p\varphi_R(x)\,dx&\le\Big(\int_{\mathbb{R}^N} a(x)v^q\varphi_R(x)\,dx\Big)^{1/q}\\&\quad\times\Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|(-\Delta)^l\varphi_R|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{q}}.\end{aligned}\tag{6.2}$$
Substituting (6.1) into (6.2) and vice versa, we obtain
$$\begin{aligned}\Big(\int_{\mathbb{R}^N} a(x)v^q\varphi_R(x)\,dx\Big)^{1-\frac{1}{pq}}&\le\Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|(-\Delta)^l\varphi_R|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{pq}}\\&\quad\times\Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|(-\Delta)^k\varphi_R|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{p}},\end{aligned}$$
$$\begin{aligned}\Big(\int_{\mathbb{R}^N} b(x)u^p\varphi_R(x)\,dx\Big)^{1-\frac{1}{pq}}&\le\Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x)|(-\Delta)^l\varphi_R|^{\frac{q}{q-1}}\varphi_R^{-\frac{1}{q-1}}(x)\,dx\Big)^{\frac{q-1}{q}}\\&\quad\times\Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x)|(-\Delta)^k\varphi_R|^{\frac{p}{p-1}}\varphi_R^{-\frac{1}{p-1}}(x)\,dx\Big)^{\frac{p-1}{pq}};\end{aligned}$$

i.e.,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} a(x) v^q \varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x) |(-\Delta)^l \varphi_R|^{\frac{q}{q-1}} \varphi_R^{-\frac{1}{q-1}}(x)\,dx \Big)^{\frac{q-1}{pq-1}} \qquad (6.3)\\
&\quad \times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x) |(-\Delta)^k \varphi_R|^{\frac{p}{p-1}} \varphi_R^{-\frac{1}{p-1}}(x)\,dx \Big)^{\frac{(p-1)q}{pq-1}}, \\
&\int_{\mathbb{R}^N} b(x) u^p \varphi_R(x)\,dx \\
&\le \Big(\int_{\mathbb{R}^N} a^{-\frac{1}{q-1}}(x) |(-\Delta)^l \varphi_R|^{\frac{q}{q-1}} \varphi_R^{-\frac{1}{q-1}}(x)\,dx \Big)^{\frac{p(q-1)}{pq-1}} \qquad (6.4)\\
&\quad \times \Big(\int_{\mathbb{R}^N} b^{-\frac{1}{p-1}}(x) |(-\Delta)^k \varphi_R|^{\frac{p}{p-1}} \varphi_R^{-\frac{1}{p-1}}(x)\,dx \Big)^{\frac{p-1}{pq-1}}.
\end{aligned}
$$

Choosing the test function $\varphi_R \in C_0^{2K}(\mathbb{R}^N;[0,1])$ so that

$$
\varphi_R(x) = \begin{cases} 1 & (|x| \le R), \\ 0 & (|x| \ge 2R) \end{cases}
$$

and estimate (5.5) holds, we obtain

$$
\begin{gathered}
\int_{B_R(0)} a(x)|Dv|^q\,dx \le cR^{N - \frac{|\alpha|+\beta+(|\gamma|+\delta+2l+2kp)q}{pq-1}}, \\
\int_{B_R(0)} b(x)|Du|^p\,dx \le cR^{N - \frac{|\gamma|+\delta+(|\alpha|+\beta+2k+2lq)p}{pq-1}}.
\end{gathered}
$$

Taking $R \to \infty$, we complete the proof similarly to Theorems 2.2, 2.4 and 2.5.

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