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# SOLUTION TO SECOND-ORDER DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDE

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Abstract. We consider an ordinary differential equation of second order with discontinuous nonlinearity relative to the phase variable. Phase trajectories are studied. We establish a theorem on the existence of a continuum set for nontrivial solutions and the theorem on the boundedness of solutions.

## 1. Introduction and statement of problem

Over a number of years differential equations with discontinuous right sides have attracted researchers' attention. Equations with discontinuous nonlinearities are of interest on both theoretical and practical grounds. The problem on existence of solutions for the Sturm-Liouville task with discontinuous nonlinearity is considered in [1]–[4]. The applications of such problems are shown in [5, 6], and other papers. Periodic solutions of second-order differential equations with discontinuous right sides are studied in [7, 8]. This paper extends this research.

We study the existence of solutions to the second-order ordinary differential equation with discontinuous nonlinearity of the form

$$-u'' = g(x, u(x)), \quad x \in \mathbb{R}, \tag{1.1}$$

$$g(x,u) = \begin{cases} m_1 & \text{for } u < f(x), \\ m_2 & \text{for } u \geq f(x). \end{cases} \tag{1.2}$$

Here the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $m_i$ $(i = 1, 2)$ are constants in $\mathbb{R}$, the function $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth and one-to-one.

Let us remark that systems of ordinary differential equations with multiple-valued discontinuous nonlinearity of this type are investigated, for instance, in [9]–[11].

From (1.1), (1.2) we receive the equations

$$-u'' = m_i \quad (i = 1, 2). \tag{1.3}$$

It follows from (1.3) that $u' = -m_i x + c_1$, $u = -\frac{m_i}{2}x^2 + c_1 x + c_2$, where $c_1$, $c_2$ are real constants. Then the phase curves on the plane $(u, u')$ are defined by the equations $2m_i(c_2 - u) = (u')^2 - c_1^2$.

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We assume that the function $u' = \psi(u)$, which graph is a curve without contact for the phase trajectories of system (1.1), (1.2), is assigned to the function $u = f(x)$ on the phase $(uOu')$-plane. In other words, the phase trajectories have the only isolated points of tangency to this curve, i.e. the points of tangency do not belong to segments of this curve.

Let, in particular, the discontinuity surface be the straight line $u = f(x) = kx+b$, where coefficients $k, b \in \mathbb{R}$. Note that such switching surfaces arise quite often in automatic control systems. On the phase $(uOu')$-plane the straight line $u' = k$ is a switching line. Its graph represents a curve without contact for the phase trajectories of system (1.1), (1.2).

Let us consider all possible relations between parameters $m_1$ and $m_2$.

## 2. Solution of the problem

*Case 1.* Suppose $m_i > 0$ $(i = 1, 2)$; then the phase trajectories consist of parabola pieces. The branches of the parabolas are directed aside opposite to the positive direction of the $Ou$-axis. From any initial point on the half-plane $u' > k$ the representative point reaches the switching line and then along the parabola on the half-plane $u' < k$ it goes to infinity $(u \to -\infty)$. If the initial point lies on the half-plane $u' < k$, then the representative point also tends to infinity $(u \to -\infty)$. The phase trajectories are called respectively trajectories of "parabola – parabola" type or "parabola" one.

*Case 2.* The similar situation happens when $m_i < 0$ $(i = 1, 2)$. In this case the parabola branches lying on both half-planes are directed towards the positive direction of the $Ou$-axis. From any initial point on the half-plane $u' > k$ the representative point moves along the parabolic trajectory to infinity $(u \to +\infty)$. If the initial point is on the half-plane $u' < k$, then the representative point comes to the switching line along the parabolic trajectory and after that it goes into infinity $(u \to +\infty)$. The types of the phase trajectories are the same as in the previous case.

Both cases considered above correspond to the condition $m_1 m_2 > 0$. Suppose $m_1$ and $m_2$ have opposite signs.

*Case 3.* To be precise, let $m_1 < 0$, $m_2 > 0$. Then from any initial point that belongs to the phase plane and does not to the line $u' = k$ the representative point comes to the straight line $u' = k$ along the parabolic trajectory. The phase trajectories are of "parabola" type.

*Case 4.* Now we assume that $m_1 > 0$, $m_2 < 0$. From any initial point on $u' > k$ the representative point extends to infinity $(u \to +\infty)$ along the parabola, if on $u' < k$, then it also tends to infinity $(u \to -\infty)$ along the parabola. We have the phase trajectories of "parabola" type.

The two latter cases correspond to the condition $m_1 m_2 < 0$.

*Case 5.* Let $m_1 < 0$, $m_2 = 0$. If the initial point belongs to the half-plane $u' \geq k$ $(k > 0)$, then the representative point moves to infinity $(u \to +\infty)$ along the straight line parallel to the $Ou$-axis. However, if the initial point is on the half-plane $u' < k$, then the representative point comes to the straight line $u' = k$ along the parabolic trajectory and also goes to infinity $(u \to +\infty)$ along the straight line $u' = k$. Note that the straight line $u' = 0$ is a set of equilibrium points. Let $k < 0$. If the initial point is on the half-plane $u' < k$, then the trajectory has "parabola – straight line" type and the representative point tends to infinity $(u \to -\infty)$ along

$u' = k$. At the same time, if the initial point is on the half-plane $u' > 0$, then the representative point goes to infinity ($u \to +\infty$) along the straight lines parallel to the $Ou$-axis. Here the $Ou$-axis is a set of equilibrium points. If the initial point is from the set $k < u' < 0$, then the representative point leaves for infinity along the lines parallel to the $Ou$-axis, so that $u \to -\infty$. We receive two types of the trajectories, namely, "straight line" or "parabola – straight line".

*Case 6.* Further, let $m_1 = 0$ and $m_2 > 0$. Obviously, the qualitative picture of splitting the phase plane into trajectories is not changed in comparison with the previous case, but pieces of the parabolic trajectories lie on the upper half-plane. The types of the phase trajectories are the same as above. Really, if the initial point belongs to the half-plane $u' > k$ ($k > 0$), then the representative point comes to the line $u' = k$ along the parabolic trajectories and goes to infinity ($u \to +\infty$) along this straight line. On the other hand, from any initial point on the set $0 < u' < k$ the representative point tends to infinity ($u \to +\infty$) along the straight line parallel to the $Ou$-axis. If $u' < 0$, then the representative point leaves for infinity ($u \to -\infty$) along the straight line parallel the $Ou$-axis. Let $k < 0$. If the initial point is on the half-plane $u' > k$, then along the parabolic trajectory the representative point comes to the line $u' = k$ and along this line it goes to infinity ($u \to -\infty$). But if the initial point is on the half-plane $u' < k$, then the representative point moves to infinity ($u \to -\infty$) along the line parallel to the $Ou$-axis. The straight line $u' = k$ is a set of equilibrium positions when $k = 0$.

*Case 7.* Let $m_1 = 0$, $m_2 < 0$. This case differs from Case 6 in motion directions along the parabolic pieces of trajectories. Indeed, let $k > 0$. If the initial point belongs to the plane $u' \geq k$, then the representative point goes to infinity ($u \to +\infty$) along the parabolic trajectory. The representative point moves to infinity ($u \to +\infty$) along the straight lines parallel to the $Ou$-axis when $0 < u' < k$. Note that the straight line $u' = 0$ is a set of equilibrium points. The representative point goes to infinity ($u \to -\infty$) along the straight lines parallel to the $Ou$-axis when $u' < 0$. Let $k < 0$. If $u' \geq k$, then the representative point goes to infinity ($u \to +\infty$) along the parabolic trajectory. If $u' < k$, then the representative point goes to infinity ($u \to -\infty$) along the straight line parallel to the $Ou$-axis. The types of the trajectories are "parabola" or "straight line".

*Case 8.* Let $m_1 > 0$, $m_2 = 0$. This case differs from Case 5 in motion directions along the parabolic trajectories. For example, let $k > 0$. If the initial point belongs to the half-plane $u' \geq k$, then the representative point moves to infinity ($u \to +\infty$) along the straight lines parallel to the $Ou$-axis. If $u' < k$, then the representative point goes to infinity ($u \to -\infty$) along the parabolic trajectories. Let $k < 0$. If $u' > 0$, then the representative point goes to infinity ($u \to +\infty$) along the straight lines parallel to the $Ou$-axis. Here the straight line $u' = 0$ is a set of equilibrium points. If $k \leq u' < 0$, then the representative point goes to infinity ($u \to -\infty$) along the straight lines parallel to the $Ou$-axis. If the initial point is taken from the set $u' < k$, then the representative point goes to infinity ($u \to -\infty$) along the parabolic trajectories. The types of the trajectories are "parabola" or "straight line".

*Case 9.* Let $m_1 = m_2 = 0$. Let $k > 0$. If the initial point belongs to either $u' \geq k$ or $0 < u' < k$, then the representative point moves to infinity ($u \to +\infty$) along the straight line parallel to the $Ou$-axis. The line $u' = 0$ is a set of equilibrium points. If the initial point is taken from $u' < 0$, then the representative point moves to infinity ($u \to -\infty$) along the straight lines parallel to the $Ou$-axis. Now let $k < 0$. If the initial point belongs to the half-plane $u' > 0$, then the representative point moves to infinity ($u \to +\infty$) along the straight lines parallel to the $Ou$-axis. The straight line $u' = 0$ is a set of equilibrium points. If the initial point is taken from $k < u' < 0$ or $u' \le k$, then the representative point moves to infinity ($u \to -\infty$). The trajectories are of "straight line" type.

So, we have considered all cases of relations between the parameters $m_1$ and $m_2$. All studied types of the trajectories take place under the conditions on $f(x)$ imposed above. The function $f$ is not only linear. Thus the following theorem on existence of solutions for problem (1.1), (1.2) is fair.

**Theorem 2.1.** *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the function $f : \mathbb{R} \to \mathbb{R}$ be piecewise smooth and one-to-one. The graph of $f$ on the phase $(uOu')$-plane is a curve without contact for the phase trajectories of system* (1.1), (1.2). *Then there is a continuum set of nontrivial solutions for problem* (1.1), (1.2) *such that the phase trajectories are the piecewise smooth curves consisting of the pieces of parabolas and straight lines.*

Notice that Theorem 2.1 agrees with the results received in [4] for one-dimensional analog of the Gol'dshtik model for separated flows of incompressible fluid.

The following corollary follows from Theorem 2.1.

**Corollary 2.2.** *Let the conditions of Theorem 2.1 hold and in addition $m_1 m_2 > 0$, $f(x) = kx + b$, $k \ne 0$. Then for each point of the switching line there exists a neighborhood such that switching of the phase trajectory pieces in it does not lead to qualitative change of the phase trajectories in the whole.*

As established above, nontrivial solutions of problem (1.1), (1.2) belong to the class of piecewise smooth functions. The phase trajectories are "sewed" on continuity on the curve $u' = \psi(u)$ to which the set $\{x \in \mathbb{R} : u(x) = f(x)\}$ is assigned.

## 3. Boundedness

Further, let the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, where $\Omega \subset \mathbb{R}$ is a bounded connected set. Then the set of points $\{(\psi(u), u) : x \in \Omega,\ u(x) = f(x)\}$ has zero measure and is closed with respect to the closed set $\Omega$. In particular, this is fair for the set $\Omega = [x_1, x_2]$ ($x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$). We note that the similar result on properties of the "separating" set is received in [12, 13] for equations of elliptic type with discontinuous nonlinearities.

We have
$$|g(x,u)| \le \max\{|m_1|, |m_2|\} = m$$
for any $x \in \Omega$ and $u \in \mathbb{R}$. It follows from the inequality above and equation (1.1) that
$$0 \le |-u''(x)| \le m,$$
where $m$ is a real non-negative number defined above. So, the estimation for the differential operator of problem (1.1), (1.2) is received.

For $\Omega = [x_1, x_2]$, we get
$$|u'(x_2) - u'(x_1)| = \Big|\int_{x_1}^{x_2} u''(x)dx\Big| \le \int_{x_1}^{x_2} |u''(x)|dx \le m(x_2 - x_1)$$
and
$$|u'(x)| \le m|x_2| + |c_1| = C_1.$$

Note that such kind of estimations are also fair for any bounded closed set $\Omega$.

From the form of solutions $u(x)$ on the set $\Omega = [x_1, x_2]$ it follows that

$$|u(x)| \le \frac{m}{2}x_2^2 + |c_1 x_2| + |c_2| = C_2,$$

which means boundedness of the solutions $u(x)$.

Thus the theorem on boundedness of solutions and their derivatives is received for problem (1.1), (1.2).

**Theorem 3.1.** *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$, where $\Omega$ is the bounded closed set in $\mathbb{R}$. Then the solutions $u(x)$ of problem* (1.1), (1.2) *are bounded on $\Omega$. Also $u'(x)$ and $u''(x)$ are bounded on the corresponding subsets of their existence of $\Omega$.*

**Remark.** Notice that solutions $u(x)$ of problem (1.1), (1.2) are bounded with respect to the norm in the corresponding functional spaces.

As an example, let us consider the norm in the Sobolev space $H^1_\circ([x_1, x_2])$:

$$\|u\| = \Big(\int_{x_1}^{x_2} |u'(x)|^2 dx\Big)^{1/2}.$$

We obtain

$$\begin{aligned}
\|u(x)\| &= \Big(\int_{x_1}^{x_2} (c_1 - m_i x)^2 dx\Big)^{1/2} \\
&= \sqrt{c_1^2(x_2 - x_1) - c_1 m_i (x_2^2 - x_1^2) + \frac{m_i^2}{3}(x_2^3 - x_1^3)} \le C_3.
\end{aligned}$$

Since the space $H^1_\circ([x_1, x_2])$ is compactly embedded in $C([x_1, x_2])$, we obtain (see, for example, [3]):

$$\|u\|_\infty \le \frac{1}{\sqrt{2\frac{2\operatorname{ess\,inf}_{[x_1,x_2]} 1}{x_2 - x_1}}}\|u\|.$$

Thus,

$$\|u(x)\|_\infty \le \frac{\sqrt{x_2 - x_1}}{2}\|u(x)\| \le \frac{\sqrt{x_2 - x_1}}{2}C_3 = C_4.$$

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