REMARKS ON REGULARITY CRITERIA FOR THE 3D NAVIER-STOKES EQUATIONS

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Abstract. In this article, we study the regularity criteria for the 3D Navier-Stokes equations involving derivatives of the partial components of the velocity. It is proved that if $\nabla_h u$ belongs to Triebel-Lizorkin space, $\nabla u$ or $u_3$ belongs to Morrey-Campanato space, then the solution remains smooth on $[0, T]$.

1. Introduction

This article is devoted to the Cauchy problem for the following incompressible 3D Navier-Stokes equation:

$$u_t + (u \cdot \nabla)u + \nabla p = \Delta u, \quad x \in \mathbb{R}^3, \ t > 0$$
$$\text{div } u = 0, \quad x \in \mathbb{R}^3, \ t > 0$$

with initial data

$$u(x, 0) = u_0, \quad x \in \mathbb{R}^3,$$

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and the unknown scalar pressure, respectively. In the last century, Leray [11] and Hopf [8] proved the global existence of a weak solution $u(x, t) \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; H^1(\mathbb{R}^3))$ to (1.1)-(1.3) for any given initial datum $u_0(x) \in L^2(\mathbb{R}^3)$. However, whether or not such a weak solution is regular and unique is still a challenging open problem. From that time on, different criteria for regularity of the weak solutions has been proposed.

The classical Prodi-Serrin conditions (see [16, 18, 19]) say that if

$$u \in L^1(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 1, \quad 3 \leq s \leq \infty,$$

then the solution is smooth. Similar results is showed by Beirão da Veiga [1] involving the velocity gradient growth condition:

$$\nabla u \in L^1(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leq \infty.$$

Actually, whether the weak solution is smooth when a part of the velocity components is involved. As for this direction, later on, criteria just for one velocity

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component appeared. The first result in this direction is due to Neustupa et al.\[15\] (see also Zhou\[21\]), where the authors showed that if
\[ u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = \frac{1}{2}, \quad s \in (6, \infty), \]
then the solution is smooth. A similar result, for the gradient of one velocity component, is independently due to Zhou\[22\] and Pokorný\[17\]. In\[22\], Zhou proved that if
\[ \nabla u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = \frac{3}{2}, ~ 3 \leq s < \infty, \]
then the solution is smooth on\[0, T\]. This result is extended by Zhou and Pokorný\[26\]; that is,
\[ \nabla u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = \frac{23}{12}, ~ 2 \leq s \leq 3. \]
Further criteria, including several components of the velocity gradient, pressure or other quantities, can be found, here we just list some. Zhou and Pokorný\[25\] proved the regularity condition
\[ u_3 \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = \frac{3}{4} + \frac{1}{2}, \quad s > \frac{10}{3}. \]
And in\[10\], Jia and Zhou proved that if a weak solution\(u\) satisfies one of the following two conditions:
\[ u_3 \in L^\infty(0, T; L^{\frac{10}{3}}(\mathbb{R}^3)); \quad \nabla u_3 \in L^\infty(0, T; L^{\frac{30}{19}}(\mathbb{R}^3)), \]
then\(u\) is regular on\(0, T\). Dong and Zhang\[5\] proved that if the horizontal derivatives of the two velocity components
\[ \int_0^T \|\nabla_h \tilde{u}(s)\|_{\dot{B}^{0, \infty}_{q, 2}} ds < \infty, \]
then the solution keeps smoothness up to time\(T\), where\(\tilde{u} = (u_1, u_2, 0)\), and\(\nabla_h \tilde{u} = (\partial_1 \tilde{u}, \partial_2 \tilde{u}, 0)\). For other kinds of regularity criteria, see\[2, 6, 7, 9, 23, 24, 28, 29, 30\] and the references cited therein.

Throughout this paper\(C\) will denote a generic positive constant which can vary from line to line. For simplicity, we shall use\(\int f(x) dx\) to denote\(\int_{\mathbb{R}^3} f(x) dx\), use\(\|\cdot\|_p\) to denote\(\|\cdot\|_{L^p}\).

The purpose of this article is to improve and extend above known regularity criterion of weak solution for the equations (1.1), (1.2) to the Triebel-Lizorkin space and Morrey-Campanato spaces. The main results of this paper read:

**Theorem 1.1.** Assume that\(u_0 \in H^1(\mathbb{R}^3)\) with\(\text{div} \, u_0 = 0\).\(u(x,t)\) is the corresponding weak solution to (1.1) and (1.2) on\([0, T]\). If additionally
\[ \int_0^T \|\nabla_h \tilde{u}(\cdot, t)\|^p_{L_q^p(\mathbb{R}^3)} dt < \infty, \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq \infty, \quad (1.3) \]
then the solution remains smooth on\([0, T]\).

**Theorem 1.2.** Assume that\(u_0 \in H^1(\mathbb{R}^3)\) with\(\text{div} \, u_0 = 0\).\(u(x,t)\) is the corresponding weak solution to (1.1) and (1.2) on\([0, T]\). If additionally
\[ \int_0^T \|\nabla u_3(\cdot, t)\|_{\dot{B}^{\frac{8}{3} - \frac{1}{r}, \frac{4}{r}}_{m, \frac{1}{2}}} dt < \infty, \quad \text{with} \quad 0 < r \leq 1, \quad 2 \leq p \leq \frac{3}{r}, \quad (1.4) \]
then the solution remains smooth on \([0, T]\).

**Theorem 1.3.** Assume that \(u_0 \in H^1(\mathbb{R}^3)\) with \(\text{div} \ u_0 = 0\). \(u(x,t)\) is the corresponding weak solution to (1.1) and (1.2) on \([0, T]\). If additionally
\[
\int_0^T \|u_3(\cdot, t)\|_{L^{\frac{8}{3-r}}(\mathbb{R}^3)}^{\frac{8}{3-r}} dt < \infty, \quad 0 < r < \frac{3}{4}, \quad 2 \leq p \leq \frac{3}{r}, \tag{1.5}
\]
then the solution remains smooth on \([0, T]\).

**Remark 1.4.** Noticing that the classical Riesz transformation is bounded in \(\dot{B}^{0, \infty}_{\infty, \infty}\), if we take \(q = \infty\) in Theorem 1.1, then the classical Beal-Kato-Majda criterion for the Navier-Stokes equations is obtained; that is, if
\[
\int_0^T \|\nabla h \tilde{u}(\cdot, t)\|_{\dot{B}^{0, \infty}_{\infty, \infty}} dt < \infty,
\]
then the solution remains smooth on \([0, T]\).

**Remark 1.5.** Since (it is proved in [12, 13])
\[
L^q(\mathbb{R}^3) = \dot{M}^{p, q}(\mathbb{R}^3) \subset \dot{M}_r(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{M}^{p, \frac{3}{r}}(\mathbb{R}^3), \quad 1 < p \leq q < \infty,
\]
the result of Theorem 1.2 is an improved version of [27, Theorem 2]. Also we obtain, if
\[
\int_0^T \|u_3(\cdot, t)\|_{\dot{X}_r}^{\frac{8}{3-r}} dt < \infty, \quad 0 < r < \frac{3}{4},
\]
then the solution remains smooth on \([0, T]\).

### 2. Preliminaries

In this section, we shall introduce the Littlewood-Paley decomposition theory, and then give some definitions of the homogeneous Besov space, homogeneous Triebel-Lizorkin space, Morrey-Campanato space and multiplier space as well as some relate spaces used throughout this paper. Before this, let us first recall the weak solutions of (1.1)-(1.3):

Let \(u_0 \in L^2(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = 0\), a measurable \(\mathbb{R}^3\)-valued vector \(u\) is said to be a weak solution of (1.1)-(1.3) if the following conditions hold:

1. \(u(\cdot, t) \in L^\infty(0, \infty; \dot{L}^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3))\);
2. \(u\) solves (1.1) in the sense of distributions;
3. the energy inequality holds; i.e,
\[
\|u\|_2^2 + 2 \int_0^t \|\Delta u(\cdot, \tau)\|_2^2 d\tau \leq \|u_0\|_2^2, \quad 0 \leq t \leq T.
\]

Let us choose a nonnegative radial function \(\varphi \in C^\infty(\mathbb{R}^3)\) be supported in the annulus \(\{\xi \in \mathbb{R}^3 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\), such that \(\sum_{l=-\infty}^{\infty} \varphi(2^{-l} \xi) = 1, \forall \xi \neq 0\). For \(f \in S'(\mathbb{R}^3)\), the frequency projection operators \(\Delta_l\) is defined as
\[
\Delta_l f = \mathcal{F}^{-1}(\varphi(2^{-l} \cdot)) * f,
\]
where \(\mathcal{F}^{-1}(g)\) is the inverse Fourier transform of \(g\). The formal decomposition
\[
f = \sum_{l=-\infty}^{\infty} \Delta_l f. \tag{2.1}
\]
is called the homogeneous Littlewood-Paley decomposition. Noticing

$$\triangle_l f = \sum_{j=|l|-1}^{l+1} \triangle_j(\triangle_l f)$$

and using the Young inequality, we have the following class Bernstein inequality:

**Lemma 2.1 (3).** Let $\alpha \in \mathbb{N}$, then for all $1 \leq p \leq q \leq \infty$, $\sup_{|\alpha|=k} ||\partial^\alpha \triangle_l f||_q \leq C 2^{lk \alpha + \alpha \cdot \frac{k}{2}} \|\triangle_l f\|_p$, and $C$ is a constant independent of $f, l$.

For $s \in \mathbb{R}$ and $(p, q) \in [1, \infty] \times [1, \infty]$, the homogeneous Besov space $\dot{B}_{p,q}^s$ is defined by

$$\dot{S}_{p,q}^s = \{ f \in \dot{Z}'(\mathbb{R}^3) : \|f\|_{\dot{S}_{p,q}^s} < \infty \},$$

where

$$\|f\|_{\dot{S}_{p,q}^s} = \left\{ \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f(\cdot)\|_p^q \right)^{1/q} \right\}_{1 \leq q < \infty} \cup \left\{ \sup_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f(\cdot)\|_p \right\}_{q = \infty} \in L^\infty(\mathbb{R}^3).$$

and $Z'(\mathbb{R}^3)$ denote the dual space of $Z'(\mathbb{R}^3) = \{ f \in S(\mathbb{R}^3) : D^\alpha \hat{f}(0) = 0, \forall \alpha \in \mathbb{N}^3 \}$. On the other hand, for $s \in \mathbb{R}, (p, q) \in [1, \infty] \times [1, \infty]$, and for $s \in \mathbb{R}, p = q = \infty$, the homogenous Triebel-Lizorkin space is defined as

$$\dot{F}_{p,q}^s = \{ f \in \dot{Z}'(\mathbb{R}^3) : \|f\|_{\dot{F}_{p,q}^s} < \infty \},$$

where

$$\|f\|_{\dot{F}_{p,q}^s} = \left\{ \left( \sum_{j \in \mathbb{Z}} (2^{jsq} \|\triangle_j f(\cdot)\|_p^q)^{1/q} \right)_{1 \leq q < \infty} \cup \right\} \left\{ \sup_{j \in \mathbb{Z}} (2^{jsq} \|\triangle_j f(\cdot)\|_p^q)_{q = \infty} \right\}_{q = \infty} \in L^\infty(\mathbb{R}^3).$$

Notice that by Minkowski inequality, we have the following two imbedding relations:

$$\dot{B}_{p,q}^s \subset \dot{F}_{p,q}^s, \quad q \leq p;$$

$$\dot{F}_{p,q}^s \subset \dot{B}_{p,q}^s, \quad p \leq q,$$

and the following two inclusions:

$$\dot{H}^s = \dot{B}_{2,2}^s = \dot{F}_{2,2}^s, \quad L^\infty \subset \dot{F}^0_{\infty,\infty} = \dot{B}^0_{\infty,\infty}.$$ 

We refer to [20] for more properties.

For $1 < q \leq p < \infty$, the homogeneous Morrey-Campanato space in $\mathbb{R}^3$ is

$$M^{p,q} = \{ f \in L^p_{loc}(\mathbb{R}^3) : \|f\|_{M^{p,q}} = \sup_{x \in \mathbb{R}^3} \sup_{R > 0} R^{3 \frac{p}{q} - \frac{3}{q}} \|f||_{q(B(x,R))} < \infty \},$$

For $1 \leq p' \leq q' < \infty$, we define the homogeneous space

$$N^{p',q'} = \{ f \in L^{q'} = \sum_{k \in \mathbb{N}} g_k, \text{ where } g_k \in L^{q'}_{comp}(\mathbb{R}^3) \text{ and } \sum_{k \in \mathbb{N}} d^{-\frac{3}{p'} + \frac{3}{q'}} \|g_k\|_{q'} < \infty, \text{ where } d_k = \text{diam}(\text{supp } g_k) < \infty \}.$$

For $0 < \alpha < 3/2$, we say that a function belongs to the multiplier spaces $M(\dot{H}^\alpha, L^2)$ if it maps, by pointwise multiplication, $\dot{H}^\alpha$ to $L^2$:

$$\dot{X}_\alpha := M(\dot{H}^\alpha, L^2) := \{ f \in S' : \|f \cdot g\|_{L^2} \leq C \|g\|_{\dot{H}^\alpha}, \forall g \in \dot{H}^\alpha \}.$$
Thus, the above inequality implies
\[ \|f\|_{L_2} = \left( \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{f}(\xi)|^2 \right)^{1/2} < \infty. \]
where \( L^p(1 \leq p \leq \infty) \) is the Lebesgue space endowed with norm \( \| \cdot \|_p \).

**Lemma 2.2** \([4, 12]\). Let \( 1 \leq p' \leq q' < \infty \), and \( p, q \) such that \( \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1 \). Then, \( \dot{M}^{p,q} \) is the dual space of \( \dot{N}^{p',q'} \).

**Lemma 2.3** \([2, 7, 12]\). Let \( 1 < p' \leq q' < 2, m \geq 2, \text{ and } \frac{1}{p} + \frac{1}{p'} = 1 \). Denote \( \alpha = -\frac{n}{2} + \frac{n}{p} + \frac{n}{m} \in (0, 1) \) Then there exists a constant \( C > 0 \), such that for any \( u \in L^m(\mathbb{R}^n), v \in \dot{H}^\alpha(\mathbb{R}^n) \),
\[ \|u \cdot v\|_{\dot{N}_{p',q'}} \leq C\|u\|_{L^m(\mathbb{R}^n)}\|v\|_{\dot{H}^\alpha(\mathbb{R}^n)}. \]

**Lemma 2.4** \([13]\). For \( 0 \leq r \leq \frac{3}{2} \), let the space \( \mathcal{M}(\dot{B}_2^{r,1} \rightarrow L^2) \) be the space of functions which are locally square integrable on \( \mathbb{R}^3 \) and such that pointwise multiplication with these functions maps boundedly the Besov space \( \dot{B}_2^{r,1}(\mathbb{R}^3) \) to \( L^2(\mathbb{R}^3) \). The norm in \( \mathcal{M}(\dot{B}_2^{r,1} \rightarrow L^2) \) is given by the operator norm of pointwise multiplication:
\[ \|f\|_{\mathcal{M}(\dot{B}_2^{r,1} \rightarrow L^2)} = \sup\{\|fg\|_2 : \|g\|_{\dot{B}_2^{r,1}} \leq 1 \}. \]
Then, \( f \) belongs to \( \mathcal{M}(\dot{B}_2^{r,1} \rightarrow L^2) \) if and only if \( f \) belongs to \( \dot{M}^{2-r, \frac{3}{2}} \) (with equivalence of norms).

3. The proof of main results

**Proof of Theorem 1.1**. Multiplying \([1.1]\) by \(-\Delta u\), integrating by parts, noting that \( \nabla \cdot u = 0 \), we have
\[ \frac{1}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \int |\Delta u|^2 dx = \int [(u \cdot \nabla)u] \cdot \Delta u \, dx =: I. \quad (3.1) \]
Next we estimate the right-hand side of \((3.1)\), with the help of integration by parts and \(-\partial u_3 = \partial_1 u_1 + \partial_2 u_2 \), one shows that
\[ I = - \sum_{i,j,k=1}^{3} \int \partial_k u_i \partial_1 u_j \partial_2 u_j \, dx \]
\[ = - \sum_{i,j=1}^{2} \sum_{k=1}^{3} \int \partial_k u_i \partial_1 u_j \partial_2 u_j \, dx - \sum_{i,k=1}^{3} \int \partial_k u_i \partial_3 u_3 \partial_2 u_j \, dx \]
\[ - \sum_{j,k=1}^{3} \int \partial_k u_3 \partial_3 u_j \partial_2 u_j \, dx - \sum_{k=1}^{3} \int \partial_k u_j \partial_1 u_3 \partial_2 u_j \, dx \]
\[ - \sum_{i,j=1}^{2} \int \partial_3 u_i \partial_3 u_j \partial_2 u_j \, dx - \sum_{j=1}^{3} \int \partial_j u_3 \partial_1 u_3 \partial_2 u_j \, dx \]
\[ \leq C \int |\nabla_h \tilde{u}| |\nabla u|^2 dx. \]
Thus, the above inequality implies
\[ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \|\Delta u\|_2^2 \leq C \int |\nabla_h \tilde{u}| |\nabla u|^2 dx. \quad (3.2) \]
Using the Littlewood-Paley decomposition (2.1), \( \nabla_h \tilde{u} \) can be written as

\[
\nabla_h \tilde{u} = \sum_{j<-N} \Delta_j(\nabla_h \tilde{u}) + \sum_{j=-N}^N \Delta_j(\nabla_h \tilde{u}) + \sum_{j>N} \Delta_j(\nabla_h \tilde{u}).
\]

where \( N \) is a positive integer to be chosen later. Substituting this into (3.2), one obtains

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \|\Delta u\|_2^2 \\
\leq C \sum_{j<-N} \int |\Delta_j(\nabla_h \tilde{u})| |\nabla u|^2 \, dx + C \sum_{j=-N}^N \int |\Delta_j(\nabla_h \tilde{u})| |\nabla u|^2 \, dx \\
+ C \sum_{j>N} \int |\Delta_j(\nabla_h \tilde{u})| |\nabla u|^2 \, dx
\]

(3.3)

For \( K_i \) \((i = 1, 2, 3)\), we now give the estimates one by one. For \( K_1 \), using the Hölder inequality, the Young inequality and Lemma 2.1, it follows that

\[
K_1 \leq C \sum_{j<-N} \|\Delta_j(\nabla_h \tilde{u})\|_\infty \|\nabla u\|_2^2
\]

\[
\leq C \sum_{j<-N} 2^{3j/2} \|\Delta_j(\nabla_h \tilde{u})\|_2 \|\nabla u\|_2^2
\]

(3.4)

\[
\leq C \left( \sum_{j<-N} 2^{3j} \right)^{1/2} \left( \sum_{j<-N} \|\Delta_j(\nabla_h \tilde{u})\|_2^2 \right)^{1/2} \|\nabla u\|_2^2
\]

\[
\leq C 2^{-3N/2} \|\nabla u\|_2^3.
\]

Where in the last inequality, we use the fact that for all \( s \in \mathbb{R}, \dot{H}^s = \dot{B}^s_{2,2}. \)

For \( K_2 \), by the Hölder inequality and the Young inequality, one has

\[
K_2 = C \int \sum_{j=-N}^N |\Delta_j(\nabla_h \tilde{u})| |\nabla u|^2 \, dx
\]

\[
\leq CN \frac{2^q-3}{2^q} \int \left( \sum_{j=-N}^N |\Delta_j(\nabla_h \tilde{u})|^{2q/3} \right)^{3/(2q)} |\nabla u|^2 \, dx
\]

(3.5)

\[
\leq CN \frac{2^q-3}{2^q} \|\nabla_h \tilde{u}\|_{\dot{F}_{q}^{2q-3}} \|\nabla u\|_{2^{q-3}}^{2q/3} \|\nabla u\|_{2}^2
\]

\[
\leq CN \frac{2^q-3}{2^q} \|\nabla_h \tilde{u}\|_{\dot{F}_{q}^{2q-3}} \|\nabla u\|_{2}^2 \|\Delta u\|_{2}^{3/q}
\]

\[
\leq \frac{1}{2} \|\Delta u\|_2^2 + CN \|\nabla_h \tilde{u}\|_{\dot{F}_{q}^{2q-3}} \|\nabla u\|_2^2.
\]

where we used the interpolation inequality

\[
\|u\|_s \leq C \|u\|_2^{\frac{s-2}{2}} \|u\|_{\dot{H}^1}^{\frac{s-2}{2}},
\]

for \( 2 \leq s \leq 6 \).
Finally, using H"older inequality and Lemma 2.1, $K_3$ can be estimated as

$$K_3 = C \sum_{j>N} \int |\triangle_j(\nabla h \tilde{u})| |\nabla u|^2 \, dx$$

$$\leq C \sum_{j>N} \|\triangle_j(\nabla h \tilde{u})\|_3 \|\nabla u\|_3^2$$

$$\leq C \sum_{j>N} 2^j \|\triangle_j(\nabla h \tilde{u})\|_2 \|\nabla u\|_3^3$$

$$\leq C \sum_{j>N} 2^{-j/2} \sum_{j>N} 2^{2j} \|\triangle_j(\nabla h \tilde{u})\|_3^{3/2} \|\nabla u\|_2 \|\Delta u\|_2$$

$$\leq C 2^{-N/2} \|\nabla u\|_2 \|\Delta u\|_2.$$  \hfill (3.6)

Substituting (3.4), (3.5) and (3.6) in (3.3), we obtain

$$\frac{d}{dt} \|\nabla u\|_2^2 + \|\Delta u\|_2^2$$

$$\leq C 2^{-N/2} \|\nabla u\|_2^3 + CN \|\nabla h \tilde{u}\|_{L^p_{\frac{p}{q}, \frac{3q}{2}}}^2 \|\nabla u\|_2^2 + C 2^{-N/2} \|\nabla u\|_2 \|\Delta u\|_2^2.$$ \hfill (3.7)

Now we choose $N$ such that $C 2^{-N/2} \|\nabla u\|_2 \leq \frac{1}{2}$; that is

$$N \geq \ln(\|\nabla u\|_2^2 + e) + \ln C \ln 2 + 2.$$ \hfill (3.7)

Thus (3.7) implies

$$\frac{d}{dt} \|\nabla u\|_2^2 \leq C + C \|\nabla h \tilde{u}\|_{L^p_{\frac{p}{q}, \frac{3q}{2}}}^{p} \ln(\|\nabla u\|_2^2 + e) \|\nabla u\|_2^2.$$ \hfill (3.9)

Taking the Gronwall inequality into consideration, we obtain

$$\ln(\|\nabla u\|_2^2 + e) \leq C \left[1 + \int_0^T \|\nabla h \tilde{u}\|_{L^p_{\frac{p}{q}, \frac{3q}{2}}}^{p} \cdot \tau \cdot e^{f_{\frac{p}{q}, \frac{3q}{2}}^T \|\nabla h \tilde{u}\|_{L^p_{\frac{p}{q}, \frac{3q}{2}}}^{p} \cdot \tau} \right].$$

The proof of Theorem 1.1 is complete under the condition \hfill (3.3). \hfill \square

Proof of Theorem 1.2: Multiplying (1.1)$_1$ by $-\Delta_h u$, integrating by parts, noting that $\nabla \cdot u = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \int |\nabla h u|^2 \, dx + \int |\nabla \nabla_h u|^2 \, dx = \int [(u \cdot \nabla) u] \cdot \Delta_h u \, dx =: J.$$ \hfill (3.8)

Next we estimate the right-hand side of (3.8), with the help of integration by parts and $-\partial_3 u_3 = \partial_1 u_1 + \partial_2 u_2$, one shows that

$$J = -\sum_{i,j=1}^3 \sum_{k=1}^2 \int \partial_k u_i \partial_i u_j \partial_k u_j \, dx$$

$$= -\sum_{i,j,k=1}^2 \int \partial_k u_i \partial_i u_j \partial_k u_j \, dx - \sum_{i,k=1}^2 \int \partial_k u_i \partial_i u_3 \partial_k u_3 \, dx$$

$$-\sum_{j=1}^2 \sum_{k=1}^2 \int \partial_k u_3 \partial_3 u_j \partial_k u_j \, dx$$

$$=: J_1 + J_2 + J_3.$$ \hfill (3.9)
For $J_2$ and $J_3$, we obtain
\begin{equation}
|J_2 + J_3| \leq C \int |\nabla u_3||\nabla u| dx. \tag{3.10}
\end{equation}

$J_1$ is a sum of eight terms, using the fact $-\partial_3 u_3 = \partial_1 u_1 + \partial_2 u_2$, we can estimate it as
\begin{align*}
J_1 &= - \int (\partial_1 u_1 + \partial_2 u_2)((\partial_1 u_1)^2 - \partial_1 u_1 \partial_2 u_2 + (\partial_2 u_2)^2) dx \\
&\quad - \int (\partial_1 u_1 + \partial_2 u_2)((\partial_2 u_1)^2 + \partial_1 u_2 \partial_2 u_1 + (\partial_1 u_2)^2) dx \\
&\quad + \int \partial_3 u_3((\partial_1 u_1)^2 - \partial_1 u_1 \partial_2 u_2 + (\partial_2 u_2)^2 + (\partial_2 u_1)^2 + \partial_1 u_2 \partial_2 u_1 + (\partial_1 u_2)^2) dx \\
&\leq C \int |\nabla u_3||\nabla u| dx. \tag{3.11}
\end{align*}

Substituting the estimates (3.9)-(3.11) in (3.8), we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} ||\nabla u||_2^2 + ||\nabla u||_2^2 \leq C \int |\nabla u_3||\nabla u| dx =: L. \tag{3.12}
\end{equation}

when $2 < p \leq \frac{3}{2}$, using Lemmas 2.2 and 2.3 and the Young inequality, we obtain
\begin{equation}
L \leq C||\nabla u_3||_{M^p_\frac{3}{2}} ||\nabla u||_{L^p_\frac{3}{2}} ||\nabla u||_{L^p_\frac{3}{2}} \\
\leq C||\nabla u_3||_{M^p_\frac{3}{2}} ||\nabla u||_{H^r} ||\nabla u||_{2} \\
\leq C||\nabla u_3||_{M^p_\frac{3}{2}} ||\nabla u||_{L^2} ||\nabla u||_{2}^{1-r} ||\nabla u||_{H^r}^{r} \\
\leq \frac{1}{2} ||\nabla u||_{2}^2 + C||\nabla u_3||_{M^p_\frac{3}{2}}^{2} ||\nabla u||_{2}. \tag{3.13}
\end{equation}

where we used the inequality
\begin{equation}
||f||_{H^r} = ||\xi^r \hat{f}||_{2} = (\int |\xi|^{2r} |\hat{f}|^{2r} |\hat{f}|^{2-2r} d\xi)^{1/2} \leq ||f||_{2}^{1-r} ||\nabla f||_{2},
\end{equation}

with $0 < r \leq 1$.

In the case $p = 2$, using Hölder’s inequality, Lemma 2.4 and the Young inequality, we can estimate $L$ as
\begin{equation}
L \leq C||\nabla u_3|| \cdot |\nabla u||_{2} ||\nabla u||_{2} \\
\leq C||\nabla u||_{M^2_\frac{3}{2}} ||\nabla u||_{L^2} ||\nabla u||_{2} \\
\leq C||\nabla u_3||_{M^2_\frac{3}{2}} ||\nabla u||_{L^2} ||\nabla u||_{2}^{1-r} ||\nabla u||_{L^2}^{r} \\
\leq \frac{1}{2} ||\nabla u||_{2}^2 + C||\nabla u_3||_{M^2_\frac{3}{2}}^{2} ||\nabla u||_{2}. \tag{3.14}
\end{equation}

where we used the following interpolation inequality [14]: for $0 \leq r \leq 1$, $||f||_{L^2}^{1-r} \leq ||f||_{L^2}^{1/2} ||\nabla f||_{L^2}^{1/2}$.

Now, gathering (3.13) and (3.14) together and substituting into (3.12), we obtain
\begin{equation}
\frac{d}{dt} ||\nabla u||_{2}^2 + ||\nabla u||_{2}^2 \leq C||\nabla u_3||_{M^p_\frac{3}{2}}^{2} ||\nabla u||_{2}. \tag{3.15}
\end{equation}
Multiplying (1.1) by \(-\Delta u\), integrating by parts, noting that \(\nabla \cdot u = 0\), we have (see [25])

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \|\Delta u\|_2^2 = \int [(u \cdot \nabla) u] \cdot \Delta u \, dx \\
\leq C \int \|\nabla_h u\| \|\nabla u\|^2 \, dx \\
\leq C\|\nabla_h u\|_2 \|\nabla u\|_4^2 \\
\leq C\|\nabla_h u\|_2 \|\nabla u\|_2^{1/2} \|\Delta u\|_2^{1/2} \\
\leq C\|\nabla_h u\|_2 \|\nabla u\|_2^{1/2} \|\nabla_h u\|_2 \|\Delta u\|_2^{1/2}.
\]

Integrating, with respect to \(t\), yields

\[
\frac{1}{2} \|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \\
\leq \frac{1}{2} \|\nabla u_0\|_2^2 + C \sup_{0 \leq \tau \leq t} \|\nabla_h u(\tau)\|_2 (\int_0^t \|\nabla u(\tau)\|_2^2 \, d\tau)^{1/4} \\
\times \left( \int_0^t \|\nabla \nabla_h u(\tau)\|_2^2 \, d\tau \right)^{1/2} \left( \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \right)^{1/2}. \tag{3.16}
\]

Substituting (3.15) in (3.16), using Hölder's inequality and the Young inequality, we obtain

\[
\frac{1}{2} \|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \\
\leq \frac{1}{2} \|\nabla u_0\|_2^2 + (C + C \int_0^t \|\nabla u_3\|_{2^*}^2 \|\nabla u(\tau)\|_2^2 \, d\tau) \left( \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \right)^{1/4} \\
\leq C + C \left( \int_0^t \|\nabla u_3\|_{2^*}^2 \|\nabla u(\tau)\|_2 \|\nabla u(\tau)\|_2^{1/2} \, d\tau \right)^{4/3} \\
+ \frac{1}{2} \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \tag{3.17}
\]

Absorbing the last term into the left hand side, applying the Gronwall inequality and combining with the standard continuation argument, we conclude that the solutions \(u\) can be extended beyond \(t = T\) provided that \(\nabla u_3 \in L^{\frac{8}{9-2^{-1}}} (0, T; M_0^{\frac{3}{2}}(\mathbb{R}^3))\). This completes the proof of Theorem 1.2. \qed

**Proof of Theorem 1.3.** We start from (3.9), we can estimate \(J_2\) and \(J_3\) as

\[
|J_2 + J_3| \leq C \int |u_3| |\nabla u| |\nabla \nabla_h u| \, dx. \tag{3.18}
\]
From (3.11), we find that
\[ J_1 \leq C \int |u_3||\nabla u||\nabla u| \, dx. \] (3.19)

Combining (3.9), (3.17), (3.18) with (3.8), we obtain
\[ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \|\nabla u\|_2^2 \leq C \int |u_3||\nabla u||\nabla u| \, dx =: V. \] (3.20)

When \( 2 < p \leq \frac{3}{2} \), similarly as in the proof of \( L \) in (3.13), we obtain
\[ V \leq C \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}} \|\nabla u\| \cdot \|\nabla u\|_{\tilde{X}^{p,\frac{3}{2}}} \]
\[ \leq C \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}} \|\nabla u\|_{\tilde{B}^{2,\frac{3}{2}}} \|\nabla u\|_2 \]
\[ \leq C \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2}^\frac{1}{2-r} \|\Delta u\|_2 \]
\[ \leq \frac{1}{2} \|\nabla u\|_2^2 + C \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}}^2 \|\nabla u\|_{L^2}^2 (1-r) \|\Delta u\|_2^2. \] (3.21)

When \( p = 2 \), similarly as in the proof of \( L \) in (3.14), we have
\[ V \leq C \|u_3\|_{\tilde{X}^{2,2}} \|\nabla u\|_2 \]
\[ \leq C \|u_3\|_{\tilde{X}^{2,2}} \|\nabla u\|_{\tilde{B}^{2,2}} \|\nabla u\|_2 \]
\[ \leq C \|u_3\|_{\tilde{X}^{2,2}} \|\nabla u\|_{L^2} \|\nabla u\|_2 \]
\[ \leq \frac{1}{2} \|\nabla u\|_2^2 + C \|u_3\|_{\tilde{X}^{2,2}}^2 \|\nabla u\|_{L^2}^2 (1-r) \|\Delta u\|_2^2. \] (3.22)

Substituting (3.21) and (3.22) in (3.20), we find that
\[ \frac{d}{dt} \|\nabla u\|_2^2 + \|\nabla u\|_2^2 \leq C \|u_3\|_{\tilde{X}^{2,2}}^2 \|\nabla u\|_{L^2}^2 (1-r) \|\Delta u\|_2^2. \] (3.23)

Substituting (3.23) in (3.16), we obtain
\[ \frac{1}{2} \|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \]
\[ \leq \frac{1}{2} \|\nabla u_0\|_2^2 + \left( C + C \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}}^2 \|\nabla u(\tau)\|_{L^2}^2 (1-r) \|\Delta u(\tau)\|_2^2 \, d\tau \right) \]
\[ \times \left( \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \right)^{1/4} \]
\[ \leq C + \frac{1}{4} \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau + C \left( \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}}^2 \|\nabla u(\tau)\|_{L^2}^2 (1-r) \|\Delta u(\tau)\|_2^2 \, d\tau \right)^{4/3} \]
\[ \leq C + \frac{1}{4} \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau + C \left( \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}}^2 \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \right)^{4r/3} \]
\[ \times \left( \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}} \|\nabla u(\tau)\|_2^2 \, d\tau \frac{4(1-r)}{4r} \right) \]
\[ \leq C + \frac{1}{2} \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau + C \left( \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}}^2 \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{4(1-r)}{2(1+r)}} \]
\[ \leq C + \frac{1}{2} \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau + C \left( \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}}^2 \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \right)^{\frac{4(1-r)}{2(1+r)}} \]
\[ \leq C + \frac{1}{2} \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau + C \int_0^t \|u_3\|_{\tilde{X}^{p,\frac{3}{2}}} \|\nabla u(\tau)\|_2^2 \, d\tau. \]
By a similar argument as in the proof of Theorem 1.2, provided that $u_3 \in L^{\infty}(0, T; M^{p-\frac{4}{3}}(\mathbb{R}^3))$, we complete the proof of Theorem 1.3. □

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