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NULL CONTROLLABILITY OF A MODEL IN POPULATION DYNAMICS

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Abstract. In this article, we study the null controllability of a linear model with degenerate diffusion in population dynamics. We develop first a Carleman type inequality for the adjoint system of an intermediate model, and then an observability inequality. By a fixed point technique, we establish the existence of a control acting on a subset of the space domain that leads the population of a certain age to extinction in a finite time.

1. Introduction

We consider the linear population dynamics model

$$\begin{gathered}
\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-(k(x)y_x)_x+\mu(t,a,x)y=\vartheta\chi_\omega \quad \text{in } Q,\\
y(t,a,1)=y(t,a,0)=0 \quad \text{on } (0,T)\times(0,A),\\
y(0,a,x)=y_0(a,x) \quad \text{in } Q_A,\\
y(t,0,x)=\int_0^A \beta(t,a,x)y(t,a,x)da \quad \text{in } Q_T,
\end{gathered} \tag{1.1}$$

where $Q=(0,T)\times(0,A)\times(0,1)$, $Q_A=(0,A)\times(0,1)$, $Q_T=(0,T)\times(0,1)$ and we will denote $q=(0,T)\times(0,A)\times\omega$. The system (1.1) models the dispersion of a gene in a given population. In this case, x represents the gene type and $y(t,a,x)$ is the distribution of individuals of age a at time t and of gene type x of the population. The parameters $\beta(t,a,x)$, $\mu(t,a,x)$ are respectively the natural fertility and mortality rates of individuals of age a at time t and of gene type x, A is the maximal age of life of population, and k is the gene dispersion coefficient. The subset $\omega=(x_1,x_2)\Subset(0,1)$ is the region where a control ϑ is acting. This control corresponds to an external supply or to removal of individuals on the subdomain ω. Finally, $\int_0^A \beta(t,a,x)y(t,a,x)da$ is the distribution of the newborns of population that are of gene type x at time t. The variable x can also represent a space variable, as in some diffusion population models studied in the literature.

The question of null controllability is widely investigated in many papers, among them we find [1, 2, 3, 4, 17] and the references therein. In [3, 4], the authors

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proved the existence of a control that leads the population to its steady state y_s. This is equivalent to show the null controllability for the system satisfied by $y - y_s$. To reach this goal, the authors took the adjoint system as a collection of parabolic equations along characteristic lines, and used Carleman and observability inequalities for the heat equation proved in [12]. In [1, 2, 17], following the same strategy of [12], the authors showed a direct Carleman estimate for the backward adjoint system of the population model (1.1) and deduced its null controllability by showing adequate observability inequalities. Note that in [17], Traore considered a nonlinear distribution of newborns under the form $F(\int_0^A \beta(t,a,x)y(t,a,x)da)$. In this contribution and contrary to the previous works, we consider that the dispersion coefficient k in our problem depend on x and degenerate at the left boundary; i.e., $k(0) = 0$, e.g. $k(x) = x^\alpha$. In this case, we say that the model (1.1) is a degenerate population dynamics system. Genetically speaking, this assumption is naturel because it means that if each population is not of a gene type, then this gene can not be transmitted to its offspring.

In this context of degeneracy, we will study the null controllability of the degenerate model (1.1) at each fixed time $T > 0$. More exactly, we show that for all $y_0 \in L^2(Q_A)$ and any $\delta \in (0, A)$, there exists a control $\vartheta \in L^2(q)$ such that the associated solution of (1.1) verifies

$$y(T, a, x) = 0, \quad \text{a.e. in } (\delta, A) \times (0, 1). \tag{1.2}$$

Such a control does not depend only on the initial distribution y_0, but also on the parameter δ. As in [2] and [17], we prove this result by developing a new Carleman estimate. This will be obtained by following the method of the work done in [6] for degenerate heat equation.

The remainder of this article is organized as follows: in Section 3, we give the functional framework in which system (1.1) is wellposed and provide the proof of the Carleman inequality for an intermediate trivial adjoint system. With the help of this inequality, we establish the observability inequality and show the null controllability of the intermediate system. Using a generalization of the Leray-Schauder fixed point theorem, we will deduce in Section 4 the main result of null controllability of (1.1). The last section is an appendix which is devoted to the proof of a Caccioppoli's inequality which plays a crucial rule in the proof of the Carleman estimate.

2. Well-posedness result

In this article, we assume that the dispersion coefficient k satisfies the hypotheses

$$\begin{gathered} k \in C([0,1]) \cap C^1((0,1]), \quad k > 0 \text{ in } (0,1] \text{ and } k(0) = 0, \\ \exists \gamma \in [0,1) : xk'(x) \leq \gamma k(x), \quad x \in [0,1]. \end{gathered} \tag{2.1}$$

The above hypothesis on k means in the case of $k(x) = x^\alpha$ that $0 \leq \alpha < 1$. Similarly, all results of this chapter can be obtained also in the case of $1 \leq \alpha < 2$ taking, instead of Dirichlet condition, the Newmann condition $(k(x)u_x)(0) = 0$ on $x = 0$.

On the other hand, we assume that the rates μ and β satisfy

$$\begin{gathered} \mu \in L^\infty(Q), \quad \mu \geq 0 \text{ a.e. in } Q, \\ \beta \in C^2([0,T] \times [0,A] \times [0,1]), \quad \beta \geq 0 \text{ a.e. in } Q. \end{gathered} \tag{2.2}$$

To prove the well-posedness of (1.1), we introduce the following weighted Sobolev spaces

$$\begin{aligned} H_k^1(0,1) := \big\{u \in L^2(0,1) : u \text{ is abs. cont. in } [0,1], \\ \sqrt{k}u_x \in L^2(0,1), u(1) = u(0) = 0\big\}, \\ H_k^2(0,1) := \big\{u \in H_k^1(0,1) : k(x)u_x \in H^1(0,1)\big\}, \end{aligned} \tag{2.3}$$

endowed respectively with the norms

$$\begin{gathered} \|u\|^2_{H_k^1(0,1)} := \|u\|^2_{L^2(0,1)} + \|\sqrt{k}u_x\|^2_{L^2(0,1)}, \quad u \in H_k^1(0,1), \\ \|u\|^2_{H_k^2} := \|u\|^2_{H_k^1(0,1)} + \|(k(x)u_x)_x\|^2_{L^2(0,1)}, \quad u \in H_k^2(0,1). \end{gathered} \tag{2.4}$$

We recall from [10, 11] that the operator $Cu := (k(x)u_x)_x$, $u \in D(C) = H_k^2(0,1)$, is closed self-adjoint and negative with dense domain in $L^2(0,1)$.

Using properties of the operator C, one can show as in [13, 14, 19] the existence of a unique solution of the model (1.1) and that this solution is generated by a C_0-semigroup on the space $L^2((0,A) \times (0,1))$. Moreover, this solution has additional time, age and gene regularity. More precisely, the following well-posedness result holds.

Theorem 2.1. *Under the assumptions* (2.1) *and* (2.2) *and for all* $\vartheta \in L^2(Q)$ *and* $y_0 \in L^2(Q_A)$, *the system* (1.1) *admits a unique solution* y. *This solution belongs to* $E := C([0,T], L^2((0,A) \times (0,1))) \cap C([0,A], L^2((0,T) \times (0,1))) \cap L^2((0,T) \times (0,A), H_k^1(0,1))$. *Moreover, the solution of* (1.1) *satisfies the inequality*

$$\begin{aligned} &\sup_{t \in [0,T]} \|y(t)\|^2_{L^2(Q_A)} + \sup_{a \in [0,A]} \|y(a)\|^2_{L^2(Q_T)} + \int_0^1 \int_0^A \int_0^T (\sqrt{k(x)}y_x)^2 \,dt\,da\,dx \\ &\le C\Big(\int_q \vartheta^2 + \int_{Q_A} y_0^2 \,da\,dx\Big). \end{aligned}$$

The properties of operator C allow us also to define the root of the operator $B = -C$ denoted by $B^{1/2}$. On the other hand, by the definitions (2.3) and (2.4) and following the same arguments used in the proofs of [18, Propositions 3.5.1, 3.6.1] one can show that $D(B^{1/2}) = H_k^1(0,1)$. Moreover, the following result is needed in the sequel. For the proof, see [18, Corollary 3.4.6].

Proposition 2.2. *The operator* B *defined above has a unique extension*

$$B \in \mathcal{L}(H_k^1(0,1), H_k^{-1}(0,1)), \tag{2.5}$$

where $H_k^{-1}(0,1)$ *denotes the dual space of* $H_k^1(0,1)$ *with respect to the pivot space* $L^2(0,1)$.

3. Null controllability of an intermediate system

In this section, we investigate the null controllability of the system

$$\begin{gathered} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k(x)y_x)_x + \mu(t,a,x)y = \vartheta\chi_\omega \quad \text{in } Q, \\ y(t,a,1) = y(t,a,0) = 0 \quad \text{in } (0,T) \times (0,A), \\ y(0,a,x) = y_0(a,x) \quad \text{in } Q_A, \\ y(t,0,x) = b(t,x) \quad \text{in } Q_T, \end{gathered} \tag{3.1}$$

with $b \in L^2(Q_T)$. To reach this target, we show first a Carleman estimate for the adjoint system of (3.1).

3.1. **Carleman inequalities results.** Consider the adjoint system of (3.1),

$$\begin{gathered}\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+(k(x)w_x)_x-\mu(t,a,x)w=0,\\ w(t,a,1)=w(t,a,0)=0,\\ w(T,a,x)=w_T(a,x),\\ w(t,A,x)=0.\end{gathered}\tag{3.2}$$

We assume that μ satisfies (2.2), $w_T\in L^2(Q_A)$ and that the coefficient of diffusion k satisfies (2.1). Let us introduce the weight functions

$$\begin{gathered}\Theta(t,a):=\frac{1}{(t(T-t))^4a^4},\quad \psi(x):=c_1\Big(\int_0^x\frac{r}{k(r)}dr-c_2\Big),\\ \varphi(t,a,x):=\Theta(t,a)\psi(x).\end{gathered}\tag{3.3}$$

For the moment, we suppose that $c_2>\frac{1}{k(1)(2-\gamma)}$ and $c_1>0$. One can observe that $\psi(x)<0$, $x\in(0,1)$, or $\Theta(a,t)\to+\infty$ as $t\to0^+,T^-$ and $a\to0^+$. The first result of this paragraph is the following proposition.

Proposition 3.1. *Consider the two following systems with $h\in L^2(Q)$,*

$$\begin{gathered}\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+(k(x)w_x)_x=h,\\ w(a,t,1)=w(a,t,0)=0,\\ w(a,T,x)=w_T(a,x),\\ w(A,t,x)=0,\end{gathered}\tag{3.4}$$

$$\begin{gathered}\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}+(k(x)w_x)_x-\mu(t,a,x)w=h,\\ w(t,a,1)=w(t,a,0)=0,\\ w(T,a,x)=w_T(a,x),\\ w(t,A,x)=0.\end{gathered}\tag{3.5}$$

Then, there exist $C>0$ and $s_0>0$, such that every solutions of (3.4) or (3.5) satisfy, for $s\ge s_0$, the inequality

$$\begin{aligned}&s^3\int_Q\Theta^3\frac{x^2}{k(x)}w^2e^{2s\varphi}\,dt\,da\,dx+s\int_Q\Theta k(x)w_x^2e^{2s\varphi}\,dt\,da\,dx\\ &\le C\Big(\int_Q|h|^2e^{2s\varphi}\,dt\,da\,dx+sk(1)\int_0^A\int_0^T\Theta w_x^2(a,t,1)e^{2s\varphi(a,t,1)}\,dt\,da\Big).\end{aligned}\tag{3.6}$$

Proof. We establish the inequality (3.6) for every solution of system (3.4), and then deduce the result for the model (3.5). Let w be the solution of (3.4). The function $\nu(t,a,x):=e^{s\varphi(t,a,x)}w(t,a,x)$ satisfies the system

$$\begin{gathered}L_s^+\nu+L_s^-\nu=e^{s\varphi}h,\\ \nu(t,a,1)=\nu(t,a,0)=\nu(T,a,x)=\nu(0,a,x)=\nu(t,A,x)=\nu(t,0,x)=0,\end{gathered}\tag{3.7}$$

where

$$\begin{gathered}L_s^+\nu:=(k(x)\nu_x)_x-s(\varphi_a+\varphi_t)\nu+s^2\varphi_x^2k(x)\nu,\\ L_s^-\nu:=\nu_t+\nu_a-2sk(x)\varphi_x\nu_x-s(k(x)\varphi_x)_x\nu.\end{gathered}$$

Passing to the norm in (3.7), one has

$$\|L_s^+\nu\|^2_{L^2(Q)} + \|L_s^-\nu\|^2_{L^2(Q)} + 2\langle L_s^+\nu, L_s^-\nu\rangle_{L^2(Q)} = \|e^{s\varphi(a,t,x)}h\|^2_{L^2(Q)}.$$

Then, the proof of step one is based on the calculus of the inner product $\langle L_s^+\nu, L_s^-\nu\rangle$ whose a first expression is given in the following lemma.

Lemma 3.2. *The identity* $\langle L_s^+\nu, L_s^-\nu\rangle = S_1 + S_2$ *holds with*

$$\begin{aligned}
S_1 &= s\int_Q (k(x)\nu_x)^2\varphi_{xx}\,dt\,da\,dx - s^3\int_Q (k(x)\varphi_x)_x k(x)\varphi_x^2\nu^2\,dt\,da\,dx\\
&\quad + s^2\int_Q (\varphi_a+\varphi_t)(k(x)\varphi_x)_x\nu^2\,dt\,da\,dx\\
&\quad + s\int_Q k(x)\nu_x((k(x)\varphi_x)_{xx}\nu + (k(x)\varphi_x)_x\nu_x)\,dt\,da\,dx\\
&\quad + s^3\int_Q (k^2\varphi_x^3)_x\nu^2\,dt\,da\,dx - s^2\int_Q (k(x)(\varphi_a+\varphi_t)\varphi_x)_x\nu^2\,dt\,da\,dx\\
&\quad + \frac{s}{2}\int_Q (\varphi_{at}+\varphi_{tt})\nu^2\,dt\,da\,dx - \frac{s^2}{2}\int_Q (\varphi_x^2)_t k(x)\nu^2\,dt\,da\,dx\\
&\quad + \frac{s}{2}\int_Q (\varphi_{at}+\varphi_{aa})\nu^2\,dt\,da\,dx - \frac{s^2}{2}\int_Q (\varphi_x^2)_a k(x)\nu^2\,dt\,da\,dx,
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \int_0^A\int_0^T [k(x)\nu_x\nu_a]_0^1\,dt\,da + \int_0^A\int_0^T [k(x)\nu_x\nu_t]_0^1\,dt\,da\\
&\quad + s^2\int_0^A\int_0^T [k(x)\varphi_x(\varphi_a+\varphi_t)\nu^2]_0^1\,dt\,da - s^3\int_0^A\int_0^T [k^2(x)\varphi_x^3\nu^2]_0^1\,dt\,da\\
&\quad - s\int_0^A\int_0^T [k(x)\nu\nu_x(k(x)\varphi_x)_x]_0^1\,dt\,da - s\int_0^A\int_0^T [(k(x)\nu_x)^2\varphi_x]_0^1\,dt\,da.
\end{aligned}$$

Proof. We have

$$I_{11} = \int_Q (k(x)\nu_x)_x\nu_t\,dt\,da\,dx = \int_0^A\int_0^T [k(x)\nu_x\nu_t]_0^1\,dt\,da - \int_0^1\int_0^A [\frac{k(x)}{2}\nu_x^2]_0^T\,da\,dx.$$

By the definition of ν, one has

$$I_{11} = \int_0^A\int_0^T [k(x)\nu_x\nu_t]_0^1\,dt\,da,$$

$$\begin{aligned}
I_{12} &= \int_Q (k(x)\nu_x)_x\nu_a\,dt\,da\,dx\\
&= \int_0^A\int_0^T [k(x)\nu_x\nu_a]_0^1\,dt\,da - \int_Q k(x)\nu_x\nu_{xa}\,dt\,da\,dx\\
&= \int_0^A\int_0^T [k(x)\nu_x\nu_a]_0^1\,dt\,da,
\end{aligned}$$

$$
\begin{aligned}
I_{13} &= \int_Q -2sk(x)\varphi_x\nu_x(k(x)\nu_x)_x\,dt\,da\,dx\\
&= \int_Q -s\varphi_x((k(x)\nu_x)^2)_x\,dt\,da\,dx\\
&= -s\int_0^A\int_0^T[(k(x)\nu_x)^2\varphi_x]_0^1\,dt\,da + s\int_Q(k(x)\nu_x)^2\varphi_{xx}\,dt\,da\,dx.
\end{aligned}
$$

$$
\begin{aligned}
I_{14} &= \int_Q(-s(k(x)\varphi_x)_x\nu)(k(x)\nu_x)_x\,dt\,da\,dx\\
&= -s\int_0^A\int_0^T[k(x)\nu_x\nu(k(x)\varphi_x)_x]_0^1\,dt\,da\\
&\quad + s\int_Q k(x)\nu_x(\nu(k(x)\varphi_x)_{xx} + \nu_x(k(x)\varphi_x)_x)\,dt\,da\,dx.
\end{aligned}
$$

$$
\begin{aligned}
I_{21} &= -s\int_Q(\varphi_a+\varphi_t)\nu\nu_t\,dt\,da\,dx = \frac{-s}{2}\int_Q(\varphi_a+\varphi_t)(\nu^2)_t\,dt\,da\,dx\\
&= \frac{s}{2}\int_Q(\varphi_{ta}+\varphi_{tt})\nu^2\,dt\,da\,dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{22} &= -s\int_Q(\varphi_a+\varphi_t)\nu\nu_a\,dt\,da\,dx = \frac{-s}{2}\int_Q(\varphi_a+\varphi_t)(\nu^2)_a\,dt\,da\,dx\\
&= \frac{s}{2}\int_Q(\varphi_{aa}+\varphi_{ta})\nu^2\,dt\,da\,dx,
\end{aligned}
$$

$$
\begin{aligned}
I_{23} &= \int_Q(2sk(x)\varphi_x\nu_x)(s(\varphi_a+\varphi_t)\nu)\,dt\,da\,dx\\
&= -\int_Q s^2\nu^2(k(x)(\varphi_a+\varphi_t)\varphi_x)_x\,dt\,da\,dx\\
&\quad + s^2\int_0^A\int_0^T[k(x)(\varphi_a+\varphi_t)\varphi_x\nu^2]_0^1\,dt\,da.
\end{aligned}
$$

$$
\begin{aligned}
I_{24} &= \int_Q(s(\varphi_a+\varphi_t)\nu)(s(k(x)\varphi_x)_x\nu)\,dt\,da\,dx\\
&= \int_Q s^2(\varphi_a+\varphi_t)(k(x)\varphi_x)_x\nu^2\,dt\,da\,dx.
\end{aligned}
$$

$$
\begin{aligned}
I_{31} &= \int_Q s^2\varphi_x^2k(x)\nu\nu_t\,dt\,da\,dx\\
&= \int_0^1\int_0^A[\frac{s^2}{2}\varphi_x^2k(x)\nu^2]_0^T\,da\,dx - \frac{s^2}{2}\int_Q(\varphi_x^2k(x))_t\nu^2\,dt\,da\,dx\\
&= \frac{-s^2}{2}\int_Q(\varphi_x^2k(x))_t\nu^2\,dt\,da\,dx,
\end{aligned}
$$

$$
I_{32} = s^2\int_Q\varphi_x^2k(x)\nu\nu_a\,dt\,da\,dx = \frac{-s^2}{2}\int_Q(\varphi_x^2k(x))_a\nu^2\,dt\,da\,dx.
$$

$$
\begin{aligned}
I_{33} &= \int_Q (-2sk(x)\varphi_x \nu_x)(s^2\varphi_x^2 k(x)\nu)\,dt\,da\,dx \\
&= \int_Q -s^3 k^2(x)\varphi_x^3(\nu^2)_x\,dt\,da\,dx \\
&= -s^3\int_0^A\int_0^T [k^2(x)\varphi_x^3\nu^2]_0^1\,dt\,da + s^3\int_Q (k^2(x)\varphi_x^3)_x\nu^2\,dt\,da\,dx.
\end{aligned}
$$

$$
\begin{aligned}
I_{34} &= \int_Q -(s(k(x)\varphi_x)_x\nu)(s^2\varphi_x^2 k(x)\nu)\,dt\,da\,dx \\
&= -s^3\int_Q (k(x)\varphi_x)_x k(x)\varphi_x^2\nu^2\,dt\,da\,dx.
\end{aligned}
$$

By adding all these identities, the result follows. □

Back to the proof of Proposition 3.1. Now, using the definitions of φ and ψ given in (3.3), the Dirichlet boundary conditions satisfied by ν and the assumption $k(0) = 0$, the expressions of S_1 and S_2 stated in the previous lemma can be simplified follows,

$$
\begin{aligned}
S_1 = \frac{s}{2}&\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2 dx\,dt\,da + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx \\
&+ sc_1\int_Q \Theta(2k(x)-xk'(x))\nu_x^2\,dt\,da\,dx - 2s^2\int_Q \Theta c_1^2\frac{x^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx \\
&+ s^3\int_Q \Theta^3 c_1^3\Big(\frac{x}{k(x)}\Big)^2(2k(x)-xk'(x))\nu^2\,dt\,da\,dx,
\end{aligned}
$$

and

$$
S_2 = -sc_1k(1)\int_0^A\int_0^T \Theta\nu_x^2(a,t,1)\,dt\,da + 2s^3\int_0^A\int_0^T \Theta^3c_1^3\Big[\frac{x^3}{k(x)}\nu^2\Big]_{x=0}\,dt\,da.
$$

From the third condition in assumptions (2.1), we deduce that the function $x\mapsto \frac{x^3}{k(x)}$ is nondecreasing in $(0,1]$, and then, $0<\frac{x^3}{k(x)}\le\frac{1}{k(1)}$, $0\le\frac{x^3}{k(x)}\nu^2\le\frac{1}{k(1)}\nu^2$, $x\in(0,1]$. Hence, $\lim_{x\to0^+}\frac{x^3}{k(x)}\nu^2=0$. Accordingly,

$$
\begin{aligned}
&\langle L_s^+\nu, L_s^-\nu\rangle \\
&= \frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx \\
&\quad + sc_1\int_Q \Theta(2k(x)-xk'(x))\nu_x^2\,dt\,da\,dx - 2s^2\int_Q \Theta c_1^2\frac{x^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx \\
&\quad + s^3\int_Q \Theta^3c_1^3\Big(\frac{x}{k(x)}\Big)^2(2k(x)-xk'(x))\nu^2\,dt\,da\,dx \\
&\quad - sc_1k(1)\int_0^A\int_0^T \Theta\nu_x^2(a,t,1)\,dt\,da.
\end{aligned}
$$

Thanks to the third assumption in (2.1), we have

$$
\begin{aligned}
S_1 \geq{}& \frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx\\
&+ sc_1\int_Q \Theta k(x)\nu_x^2\,dt\,da\,dx - 2s^2\int_Q \Theta c_1^2\frac{x^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx\\
&+ s^3\int_Q \Theta^3 c_1^3\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx.
\end{aligned}
\tag{3.8}
$$

Now, using the $|\Theta(\Theta_a+\Theta_t)|\leq c\Theta^3$, we infer for s quite large that

$$
\begin{aligned}
&|-2s^2\int_Q \Theta c_1^2\frac{x^2}{k(x)}(\Theta_a+\Theta_t)\nu^2\,dt\,da\,dx|\\
&\leq 2s^2c_1^2c\int_Q \frac{x^2}{k(x)}\Theta^3\nu^2\,dt\,da\,dx \leq \frac{c_1^3}{4}s^3\int_Q \frac{x^2}{k(x)}\Theta^3\nu^2\,dt\,da\,dx.
\end{aligned}
\tag{3.9}
$$

On the other hand, we have

$$
|\psi(x)| = |c_1 l(x) - c_1c_2| \leq c_1|\int_0^x \frac{r}{k(r)}dr| + c_1c_2 \leq \frac{c_1}{(2-\gamma)k(1)} + c_1c_2, \tag{3.10}
$$

and this yields

$$
\begin{aligned}
&|\frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx|\\
&\leq s\Big(\frac{c_1}{(2-\gamma)k(1)}+c_1c_2\Big)\int_Q \Big(\frac{\Theta_{aa}+\Theta_{tt}}{2}+|\Theta_{ta}|\Big)\nu^2\,dt\,da\,dx\\
&\leq Ms\Big(\frac{c_1}{(2-\gamma)k(1)}+c_1c_2\Big)\int_Q \Theta^{3/2}\nu^2\,dt\,da\,dx.
\end{aligned}
\tag{3.11}
$$

By Hölder, Young and Hardy-Poincaré inequalities (see [6]) and the fact that

$$
\exists M_1>0 \quad \text{such that } \Theta^2 \leq M_1\Theta^3, \tag{3.12}
$$

we conclude that

$$
\begin{aligned}
\int_0^1 \Theta^{3/2}\nu^2 dx &= \int_0^1 (\Theta^{1/2}\nu\frac{\sqrt{k}}{x})(\Theta\nu\frac{x}{\sqrt{k}})dx\\
&\leq C\Big(\int_0^1 \Theta k(x)\nu_x^2 dx\Big)^{1/2}\Big(\int_0^1 \Theta^2\nu^2\frac{x^2}{k(x)}dx\Big)^{1/2}\\
&\leq C\epsilon\int_0^1 \Theta k(x)\nu_x^2 dx + \frac{C_1}{4\epsilon}\int_0^1 \Theta^3\frac{x^2}{k(x)}\nu^2 dx.
\end{aligned}
$$

By this and (3.11), we infer that

$$
\begin{aligned}
&|\frac{s}{2}\int_Q (\Theta_{aa}+\Theta_{tt})\psi\nu^2\,dt\,da\,dx + s\int_Q \Theta_{ta}\psi\nu^2\,dt\,da\,dx|\\
&\leq sc_1C\epsilon\int_Q \Theta k(x)\nu_x^2\,dt\,da\,dx + \frac{sc_1}{4}\frac{C_2}{\epsilon}\int_Q \Theta^3\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx.
\end{aligned}
\tag{3.13}
$$

Taking ϵ small enough and s quite large, we conclude that

$$\begin{aligned}&\Big|\frac{s}{2}\int_Q(\Theta_{aa}+\Theta_{tt})\psi\nu^2 dx\,dt\,da+s\int_Q\Theta_{ta}\psi\nu^2\,dt\,da\,dx\Big|\\&\le\frac{sc_1}{4}\int_Q\Theta k(x)\nu_x^2\,dt\,da\,dx+\frac{c_1^3s^3}{4}\int_Q\Theta^3\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx.\end{aligned}\tag{3.14}$$

This involves, combining with the inequalities (3.8) and (3.9) that

$$S_1\ge K_1s^3\int_Q\Theta^3\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx+K_2s\int_Q\Theta k(x)\nu_x^2\,dt\,da\,dx.$$

Therefore,

$$\begin{aligned}2\langle L_s^+\nu,L_s^-\nu\rangle&\ge m\Big(s^3\int_Q\Theta^3\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx+s\int_Q\Theta k(x)\nu_x^2\,dt\,da\,dx\Big)\\&\quad-2sc_1k(1)\int_0^A\int_0^T\Theta\nu_x^2(a,t,1)\,dt\,da.\end{aligned}$$

Hence, we obtain the following Carleman estimate for (3.7)

$$\begin{aligned}&s^3\int_Q\Theta^3\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx+s\int_Q\Theta k(x)\nu_x^2\,dt\,da\,dx\\&\le C\Big(\int_Q h^2e^{2s\varphi}\,dt\,da\,dx+sk(1)\int_0^A\int_0^T\Theta\nu_x^2(a,t,1)\,dt\,da\Big).\end{aligned}$$

To return to system (3.4), we use the function change $\nu(t,a,x):=e^{s\varphi(t,a,x)}w(t,a,x)$. This implies that

$$\nu_x=s\varphi_xe^{s\varphi}w+e^{s\varphi}w_x,e^{2s\varphi}w_x^2\le2(\nu_x^2+s^2\varphi_x^2\nu^2).$$

Then, inequality (3.6) follows immediately for every solution of system (3.4). To show this inequality for the solutions of (3.5), we apply the last inequality for the function $\overline{h}=h+\mu w$. Hence, there are two positive constants C and s_0 such that, for all $s\ge s_0$, the following inequality holds

$$\begin{aligned}&s^3\int_Q\Theta^3\frac{x^2}{k(x)}w^2e^{2s\varphi}\,dt\,da\,dx+s\int_Q\Theta k(x)w_x^2e^{2s\varphi}\,dt\,da\,dx\\&\le C\Big(\int_Q|\,\overline{h}\,|^2\,e^{2s\varphi}\,dt\,da\,dx+sk(1)\int_0^A\int_0^T\Theta w_x^2(t,a,1)e^{2s\varphi(t,a,1)}\,dt\,da\Big).\end{aligned}\tag{3.15}$$

On the other hand, we have

$$\int_Q|\,\overline{h}\,|^2\,e^{2s\varphi}\,dt\,da\,dx\le2\Big(\int_Q|h|^2e^{2s\varphi}\,dt\,da\,dx+\|\mu\|_\infty^2\int_Q|w|^2e^{2s\varphi}\,dt\,da\,dx\Big).$$

Now, applying Hardy-Poincaré inequality to the function $\nu:=e^{s\varphi}w$, we obtain

$$\begin{aligned}\int_Q|w|^2e^{2s\varphi}\,dt\,da\,dx&\le\frac{1}{k(1)}\int_Q\frac{k(x)}{x^2}|w|^2e^{2s\varphi}\,dt\,da\,dx\\&\le\frac{C}{k(1)}\int_Q k(x)\nu_x^2\,dt\,da\,dx\\&\le\frac{C}{k(1)}\Big(\int_Q s^2c_1^2\Theta^2\frac{x^2}{k(x)}\nu^2+\int_Q k(x)e^{2s\varphi}w_x^2\,dt\,da\,dx\Big).\end{aligned}$$

Thus,

$$\int_Q |\,\overline{h}\,|^2\, e^{2s\varphi}\,dt\,da\,dx \le 2\Big[\int_Q |h|^2 e^{2s\varphi}\,dt\,da\,dx + \|\mu\|_\infty^2 \frac{C}{k(1)}\Big(\int_Q s^2c_1^2\Theta^2\frac{x^2}{k(x)}\nu^2 + \int_Q k(x)e^{2s\varphi}w_x^2\,dt\,da\,dx\Big)\Big].$$

This implies, using again (3.12) and taking s quite large, that

$$\begin{aligned}
&s^3\int_Q \Theta^3\frac{x^2}{k(x)}w^2e^{2s\varphi}\,dt\,da\,dx + s\int_Q \Theta k(x)w_x^2e^{2s\varphi}\,dt\,da\,dx\\
&\le D(\int_Q |h|^2e^{2s\varphi}\,dt\,da\,dx + sk(1)\int_0^A\int_0^T \Theta w_x^2(t,a,1)e^{2s\varphi(t,a,1)}\,dt\,da\\
&\quad + \int_Q s^2c_1^2\Theta^2\frac{x^2}{k(x)}\nu^2\,dt\,da\,dx + \int_Q k(x)e^{2s\varphi}w_x^2\,dt\,da\,dx)\\
&\le C\Big(\int_Q |h|^2e^{2s\varphi}\,dt\,da\,dx + sk(1)\int_0^A\int_0^T \Theta w_x^2(t,a,1)e^{2s\varphi(t,a,1)}\,dt\,da\Big).
\end{aligned}$$

This completes the proof. □

Now, we can provide the main result of this section, namely an ω-Carleman estimate of the model (3.2).

Theorem 3.3. *Assume that k satisfies hypotheses (2.1) and let $A > 0$ and $T > 0$ be given. Then there exist two positive constants C and s_0, such that every solution w of (3.2) satisfies, for all $s \ge s_0$, the inequality*

$$\int_Q (s\Theta k w_x^2 + s^3\Theta^3\frac{x^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx \le C\int_\omega\int_0^A\int_0^T w^2\,dt\,da\,dx. \tag{3.16}$$

Proof. Let us introduce the smooth cut-off function $\xi : \mathbb{R} \to \mathbb{R}$ defined as follows

$$\begin{gathered}
0 \le \xi(x) \le 1, \quad \forall x \in \mathbb{R},\\
\xi(x) = 0, \quad x \in [\frac{x_1+2x_2}{3}, 1],\\
\xi(x) = 1, \quad x \in [0, \frac{2x_1+x_2}{3}].
\end{gathered} \tag{3.17}$$

We define the function $v := \xi w$, where w is the solution of the system (3.2). Using the Carleman estimate obtained for the model (3.5) and Caccioppoli's inequality stated in Lemma 5.1, one can prove the existence of $C > 0$ such the following estimate holds

$$\int_0^1\int_0^A\int_0^T (s\Theta k v_x^2 + s^3\Theta^3\frac{x^2}{k}v^2)e^{2s\varphi}\,dt\,da\,dx \le C\int_\omega\int_0^A\int_0^T w^2\,dt\,da\,dx. \tag{3.18}$$

In $(x_1, 1)$, let us consider the function $z := \eta w$ with $\eta = 1 - \xi$. Since z is supported by $[0,T]\times[0,A]\times[x_1,1]$ and in this interval the equation (3.5) is uniformly parabolic, then we can replace the function k by a positive function belonging to $C^1([0,1])$ and which coincides with k on $(x_1, 1)$ denoted also by k and this implies that (3.5) is nondegenerate. Moreover, we can prove in a similar manner as in [2] the following result.

Proposition 3.4. *Let z be the solution of*

$$\begin{gathered}\frac{\partial z}{\partial t}+\frac{\partial z}{\partial a}+(k(x)z_x)_x-c(t,a,x)z=h \quad \text{in } Q_b,\\ z(t,a,1)=z(t,a,0)=0 \quad \text{on } (0,T)\times(0,A),\end{gathered} \tag{3.19}$$

with $h \in L^2(Q)$ and $k \in C^1([0,1])$ is a positive function. Then, there exist two positive constants c and s_0, such that for any $s \geq s_0$, z satisfies the estimate

$$\begin{aligned}&\int_Q (s^3\phi^3 z^2 + s\phi z_x^2)e^{2s\Phi}\,dt\,da\,dx\\ &\leq c\Big(\int_Q h^2 e^{2s\Phi}\,dt\,da\,dx+\int_\omega\int_0^A\int_0^T s^3\phi^3 z^2 e^{2s\Phi}\,dt\,da\,dx\Big),\end{aligned} \tag{3.20}$$

where $Q := (0,T)\times(0,A)\times(0,1)$, the functions ϕ and Φ are defined as follows

$$\begin{gathered}\phi(t,a,x)=\Theta(t,a)e^{\kappa\sigma(x)}, \quad \Theta(t,a)=\frac{1}{t^4(T-t)^4a^4},\\ \Phi(a,t,x)=\Theta(t,a)\Psi(x), \quad \Psi(x)=e^{\kappa\sigma(x)}-e^{2\kappa\|\sigma\|_\infty},\end{gathered} \tag{3.21}$$

$(t,a,x)\in Q$, $\kappa>0$, σ is a function satisfying

$$\begin{gathered}\sigma\in C^2([0,1]), \quad \sigma(x)>0 \quad \text{in } (0,1), \quad \sigma(0)=\sigma(1)=0,\\ \sigma_x(x)\neq 0 \quad \text{in } [0,1]\backslash\omega_0,\end{gathered} \tag{3.22}$$

where $\omega_0 \Subset \omega$ is an open subset.

The existence of the function σ is proved in [12]. Hence, applying Proposition 3.4 to the function z and $h=(k\eta_x w)_x + k\eta_x w_x$, using the definitions of η, σ, ϕ and Φ and thanks again to the Caccioppoli's inequality we obtain the estimate

$$\int_Q (s^3\phi^3 z^2 + s\phi z_x^2)e^{2s\Phi}\,dt\,da\,dx \leq C\int_\omega\int_0^A\int_0^T w^2 dt\,da\,dx. \tag{3.23}$$

Taking into account that $w=v+z$ and using the inequality (3.18), we obtain

$$\begin{aligned}&\int_Q (s\Theta k w_x^2 + s^3\Theta^3\frac{x^2}{k}w^2)e^{2s\varphi}\,dt\,da\,dx\\ &\leq 2\int_Q (s^3\Theta^3\frac{x^2}{k(x)}z^2 + s\Theta k(x)z_x^2)e^{2s\varphi}\,dt\,da\,dx\\ &\quad + 2\int_Q (s^3\Theta^3\frac{x^2}{k(x)}v^2 + s\Theta k(x)v_x^2)e^{2s\varphi}\,dt\,da\,dx\\ &\leq C\int_\omega\int_0^A\int_0^T w^2\,dt\,da\,dx + 2\int_Q (s^3\Theta^3\frac{x^2}{k(x)}z^2 + s\Theta k(x)z_x^2)e^{2s\varphi}\,dt\,da\,dx.\end{aligned} \tag{3.24}$$

On the other hand, by the definition of φ, taking

$$c_1 \geq \frac{k(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{c_2 k(1)(2-\gamma)-1},$$

one can prove the existence of $\varsigma>0$, such that, for all $(t,a,x)\in[0,T]\times[0,A]\times[x_1,1]$, we have

$$\Theta k(x)e^{2s\varphi} \leq \varsigma\phi e^{2s\Phi}, \Theta^3\frac{x^2}{k(x)}e^{2s\varphi} \leq \varsigma\phi^3 e^{2s\Phi}.$$

Using this and the relation (3.23) it follows that

$$
\begin{aligned}
&\int_Q (s^3\Theta^3 \frac{x^2}{k(x)} z^2 + s\Theta k(x) z_x^2) e^{2s\varphi}\,dt\,da\,dx \\
&= \int_{x_1}^1 \int_0^A \int_0^T (s^3\Theta^3 \frac{x^2}{k(x)} z^2 + s\Theta k(x) z_x^2) e^{2s\varphi}\,dt\,da\,dx \\
&\le \varsigma \int_{x_1}^1 \int_0^A \int_0^T (s^3\phi^3 z^2 + s\phi z_x^2) e^{2s\Phi}\,dt\,da\,dx \\
&\le \varsigma C \int_\omega \int_0^A \int_0^T w^2\,dt\,da\,dx.
\end{aligned}
$$

Finally, using the last inequality and (3.24) we obtain the Carleman estimate (3.16). ☐

3.2. **An observability inequality result.** This paragraph is devoted to the observability inequality of the system (3.2). This inequality is obtained by using our Carleman estimate (3.16) and Hardy-Poincaré inequality, see [6].

Proposition 3.5. *Assume that k satisfies the hypotheses* (2.1)*. Let $A > 0$, $T > 0$ and $0 < \delta \le \min(T, A)$. Take w_T such that*

$$
w_T(a,x) = 0 \quad \text{a.e. in } (0,\delta) \times (0,1). \tag{3.25}
$$

Then, there is $C_\delta > 0$ such that every solution w of (3.2) *satisfies the observability inequality*

$$
\begin{aligned}
&\int_0^1 \int_0^T w^2(t,0,x)\,dt\,dx + \int_0^1 \int_0^A w^2(0,a,x)\,da\,dx \\
&\le C_\delta \int_\omega \int_0^A \int_0^T w^2(t,a,x)\,dt\,da\,dx.
\end{aligned} \tag{3.26}
$$

For the proof, we need to show a crucial technical result. For this, consider the following wholes, see [17],

$$
\begin{gathered}
N_1 = \{(t,a) \in (0,T) \times (0,A); t \ge a + T - \delta\}, \\
N_2 = \{(t,a) \in (0,T) \times (0,A); t \le a + \delta - A\}, \\
D_1 = \{(t,a) \in (0,T) \times (0,A); t \le -\frac{T-\frac{\delta}{2}}{A-\frac{\delta}{2}} a + T - \frac{\delta}{2}\}, \\
D_2 = \{(t,a) \in (0,T) \times (0,A); a \ge -\frac{A-\frac{\delta}{2}}{T-\frac{\delta}{2}} t + A - \frac{\delta(\delta-2A)}{2(2T-\delta)}\}, \\
D_3 = (0,T) \times (0,A) - (D_1 \cup D_2), D_4 = \{(t,a) \in D_3; (a,t) \notin N_1 \cup N_2\}.
\end{gathered} \tag{3.27}
$$

See Figure 1.

Lemma 3.6. *Suppose that* (3.25) *holds. Then all solutions of* (3.2) *satisfy*

$$
w(t,a,x) = 0, \quad \text{a.e. in } (N_1 \cup N_2) \times (0,1).
$$

Proof. Let $(t_0, a_0) \in N_1$. Then, we have $t_0 = a_0 + T - \delta + d$ with $0 \le d < \delta$. Therefore $a_0 < \delta - d$. Let $S_d = \{(t_0 + r, a_0 + r), r \in (0, \delta - d - a_0)\}$ be a characteristic line in

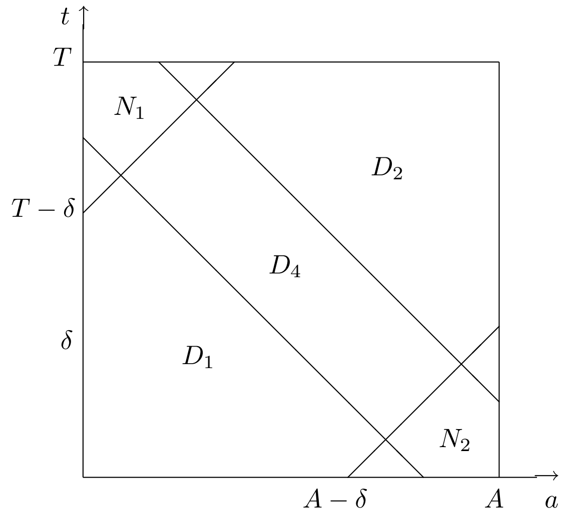

FIGURE 1. Decomposition of the region $(0,T)\times(0,A)$

N_1. Setting $\overline{w}(r,x)=w(t_0+r,a_0+r,x)$ and $\widetilde{\mu}(r,x)=\mu(t_0+r,a_0+r,x)$, where w is the solution of (3.2). Then, $\overline{w}$ solves

$$\begin{gathered}\frac{\partial\overline{w}}{\partial r}+(k(x)\overline{w}_x)_x-\widetilde{\mu}(r,x)\overline{w}=0,\quad \text{in } (0,\delta-d-a_0)\times(0,1),\\ \overline{w}(r,1)=\overline{w}(r,0)=0,\quad \text{on } (0,\delta-d-a_0),\\ \overline{w}(\delta-d-a_0,x)=w(T,\delta-d,x)=w_T(\delta-d,x),\quad \text{in } (0,1).\end{gathered}\tag{3.28}$$

Hence, $\overline{w}$ is given by

$$\overline{w}(r,\cdot)=L(\delta-d-a_0-r)\overline{w}(\delta-d-a_0,\cdot),\tag{3.29}$$

where $(L(l))_{l\geq 0}$ is the semigroup generated by the operator $C\overline{w}=(k\overline{w}_x)_x-\widetilde{\mu}\overline{w}$. Therefore, (3.25) and (3.29) lead to $\overline{w}=0$. Thus, for a. e. $d\in(0,\delta)$, $w=0$ on S_d. Subsequently, $w=0$ in $N_1\times(0,1)$. Arguing in the same way for N_2 and the fact that $w(t,A,x)=0$ in $(0,T)\times(0,1)$, we can show that $w=0$ in $N_2\times(0,1)$ and this achieves the proof. $\square$

Proof of Proposition 3.5. Consider a smooth cut-off function $\rho_1\in C_0^\infty(\mathbb{R}^2,[0,1])$ stated as follows $\rho_1(t,a)=1,(t,a)\in D_1,\rho_1(t,a)=0,(t,a)\in D_2,\rho_1>0,(t,a)\in D_3$. The function $\widetilde{w}=\rho_1 w$ satisfies the system

$$\begin{gathered}\frac{\partial\widetilde{w}}{\partial t}+\frac{\partial\widetilde{w}}{\partial a}+(k(x)\widetilde{w}_x)_x-\mu(t,a,x)\widetilde{w}=(\frac{\partial\rho_1}{\partial t}+\frac{\partial\rho_1}{\partial a})w\quad\text{in } Q,\\ \widetilde{w}(t,a,1)=\widetilde{w}(t,a,0)=0\quad \text{on } (0,T)\times(0,A),\\ \widetilde{w}(T,a,x)=0\quad \text{in } Q_A,\\ \widetilde{w}(t,A,x)=0\quad \text{in } Q_T.\end{gathered}\tag{3.30}$$

Multiplying (3.30) by $\widetilde{w}$, integrating over Q, using the definition of ρ_1 and Lemma 3.6, we obtain

$$\int_0^1\int_0^{A-\delta}w^2(0,a,x)\,da\,dx+\int_0^1\int_0^{T-\delta}w^2(t,0,x)\,dt\,dx$$

$$
\begin{aligned}
&\le -2\int_Q (\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1}{\partial a})\rho_1 w^2 \, dt\, da\, dx \\
&\le M_\delta \int_0^1 \int_{D_4} w^2 \, dt\, da\, dx.
\end{aligned}
$$

Thanks to Hardy-Poincaré inequality we conclude that

$$
\begin{aligned}
&\int_0^1 \int_0^{A-\delta} w^2(0,a,x)\, da\, dx + \int_0^1 \int_0^{T-\delta} w^2(t,0,x)\, dt\, dx \\
&\le d_\delta \int_0^1 \int_{D_4} k(x) w_x^2 \, dt\, da\, dx,
\end{aligned} \tag{3.31}
$$

with $d_\delta = \frac{CM_\delta}{k(1)}$. Observe that Θ is bounded in D_4 to infer that

$$
\begin{aligned}
&\int_0^1 \int_0^{A} w^2(0,a,x)\, da\, dx + \int_0^1 \int_0^{T} w^2(t,0,x)\, dt\, dx \\
&\le C_\delta \int_0^1 \int_{D_4} \Theta k(x) w_x^2 e^{2s\varphi} \, dt\, da\, dx.
\end{aligned} \tag{3.32}
$$

Taking s large and thanks to the Carleman inequality stated in Theorem 3.3, we obtain the observability inequality of system (3.2). $\square$

3.3. **Null controllability of the intermediate system.** This paragraph is devoted to study the null controllability of system (3.1). For this, let $\epsilon > 0$ and consider the following cost function

$$
J_\epsilon(\vartheta) = \frac{1}{2\epsilon}\int_0^1 \int_\delta^A y^2(T,a,x)\, da\, dx + \frac{1}{2}\int_q \vartheta^2(t,a,x)\, dt\, da\, dx.
$$

We can prove that J_ϵ is continuous, convex and coercive. Then, it admits at least one minimizer ϑ_ϵ and, arguing as in [5] or [7, Chapter 5], we have

$$
\vartheta_\epsilon = -w_\epsilon(t,a,x)\chi_\omega(x) \quad \text{in } Q, \tag{3.33}
$$

with w_ϵ is a solution of the following system

$$
\begin{gathered}
\frac{\partial w_\epsilon}{\partial t} + \frac{\partial w_\epsilon}{\partial a} + (k(x)(w_\epsilon)_x)_x - \mu(t,a,x) w_\epsilon = 0 \quad \text{in } Q, \\
w_\epsilon(t,a,1) = w_\epsilon(t,a,0) = 0 \quad \text{on } (0,T)\times(0,A), \\
w_\epsilon(T,a,x) = \frac{1}{\epsilon} y_\epsilon(T,a,x)\chi_{(\delta,A)}(a) \quad \text{in } Q_A, \\
w_\epsilon(t,A,x) = 0 \quad \text{in } Q_T,
\end{gathered} \tag{3.34}
$$

and y_ϵ is the solution of system (3.1) associated to the control ϑ_ϵ.

Multiplying (3.34) by y_ϵ, integrating over Q, using (3.33) and Young inequality we obtain that

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_0^1 \int_\delta^A y_\epsilon^2(T,a,x)\, da\, dx + \int_q \vartheta_\epsilon^2(t,a,x) dx\, dt\, da \\
&= \int_{Q_T} b(t,x) w_\epsilon(t,0,x)\, dt\, dx + \int_{Q_A} y_0(a,x) w_\epsilon(0,a,x)\, da\, dx \\
&\le \frac{1}{4C_\delta}\Big(\int_{Q_T} w_\epsilon^2(t,0,x)\, dt\, dx + \int_{Q_A} w_\epsilon^2(0,a,x)\, da\, dx\Big)
\end{aligned}
$$

$$+ C_\delta\Big(\int_{Q_T} b^2(t,x)\,dt\,dx + \int_{Q_A} y_0^2(a,x)\,da\,dx\Big),$$

with C_δ is the constant given in Proposition 3.5. Hence, by the observability inequality (3.26), we conclude that

$$\begin{aligned}&\frac{1}{\epsilon}\int_0^1\int_\delta^A y_\epsilon^2(T,a,x)\,da\,dx + \int_q \vartheta_\epsilon^2(t,a,x)\,dt\,da\,dx\\ &\le \frac{1}{4}\int_q w_\epsilon^2(t,a,x)\,dt\,da\,dx + C_\delta\Big(\int_{Q_T} b^2(t,x)\,dt\,dx + \int_{Q_A} y_0^2(a,x)\,da\,dx\Big).\end{aligned}$$

Hence, (3.33) yields

$$\begin{aligned}&\frac{1}{\epsilon}\int_0^1\int_\delta^A y_\epsilon^2(T,a,x)\,da\,dx + \frac{3}{4}\int_q \vartheta_\epsilon^2(t,a,x)\,dt\,da\,dx\\ &\le C_\delta\Big(\int_{Q_T} b^2(t,x)\,dt\,dx + \int_{Q_A} y_0^2(a,x)\,da\,dx\Big),\end{aligned} \tag{3.35}$$

and this yields

$$\begin{gathered}\int_0^1\int_\delta^A y_\epsilon^2(T,a,x)\,dx\,da \le C_\delta\epsilon\Big(\int_{Q_T} b^2(t,x)\,dx\,dt + \int_{Q_A} y_0^2(a,x)\,dx\,da\Big),\\ \int_q \vartheta_\epsilon^2(t,a,x)\,dt\,da\,dx \le \frac{4C_\delta}{3}\Big(\int_{Q_T} b^2(t,x)\,dt\,dx + \int_{Q_A} y_0^2(a,x)\,da\,dx\Big).\end{gathered} \tag{3.36}$$

Then, we can extract two subsequences of y_ϵ and ϑ_ϵ denoted also by ϑ_ϵ and y_ϵ that converge weakly towards ϑ and y in $L^2(q)$ and $L^2((0,T)\times(0,A);H_k^1(0,1))$ respectively. Furthermore, y is the unique solution of (3.1) that satisfies (1.2). In summary, we showed the following proposition.

Proposition 3.7. *For any $\delta > 0$ assumed to be small enough, for all $y_0 \in L^2(Q_A)$, there exists a control $\vartheta \in L^2(q)$ such that the associated solution of system* (3.1) *verifies* (1.2).

4. Main null controllability result

Now, after establishing the null controllability of system (3.1) we are ready to provide the one of the model (1.1). More precisely, we have the following theorem.

Theorem 4.1. *For any $\delta > 0$ assumed to be small enough, for all $y_0 \in L^2(Q_A)$, there exists a control $\vartheta \in L^2(q)$ such that the associated solution of system* (1.1) *verifies* (1.2).

To prove this result, let λ be a positive constant. A more precise restriction will be given later. Put $\widetilde{y} = e^{-\lambda t}y$. Then $\widetilde{y}$ solves

$$\begin{gathered}\frac{\partial\widetilde{y}}{\partial t} + \frac{\partial\widetilde{y}}{\partial a} - (k(x)\widetilde{y}_x)_x + \mu_1(t,a,x)\widetilde{y} = \widetilde{\vartheta}\chi_\omega \quad \text{in } Q,\\ \widetilde{y}(t,a,1) = \widetilde{y}(t,a,0) = 0 \quad \text{on } (0,T)\times(0,A),\\ \widetilde{y}(0,a,x) = y_0(a,x) \quad \text{in } Q_A,\\ \widetilde{y}(t,0,x) = \int_0^A \beta(t,a,x)\widetilde{y}(t,a,x)da \quad \text{in } Q_T,\end{gathered} \tag{4.1}$$

with $\widetilde{\vartheta} = e^{-\lambda t}\vartheta$ and $\mu_1 = \mu + \lambda$. Now, consider the system

$$\begin{gathered}
\frac{\partial \widetilde{y}}{\partial t} + \frac{\partial \widetilde{y}}{\partial a} - (k(x)\widetilde{y}_x)_x + \mu_1(t,a,x)\widetilde{y} = \widetilde{\vartheta}\chi_\omega \quad \text{in } Q, \\
\widetilde{y}(t,a,1) = \widetilde{y}(t,a,0) = 0 \quad \text{on } (0,T)\times(0,A), \\
\widetilde{y}(0,a,x) = y_0(a,x) \quad \text{in } Q_A, \\
\widetilde{y}(t,0,x) = b(t,x) \quad \text{in } Q_T,
\end{gathered} \tag{4.2}$$

with $b \in L^2(Q_T)$. Thus, showing Theorem 4.1 is equivalent to show the null controllability of system (4.1). For this, we consider the following multi-valued mapping

$$\Lambda_\delta : L^2(Q_T) \to \mathcal{P}(L^2(Q_T))$$

defined, for every small $\delta > 0$ and $R \in L^2(Q_T)$, by

$$\Lambda_\delta(R) = \Big\{ \int_0^A \beta\widetilde{y}da : \widetilde{y} \text{ satisfies (1.2) and (4.2) for } b = R, \text{ and } \widetilde{\vartheta} \text{ satisfies (3.36)}\Big\}.$$

To prove that model (4.1) is null controllable, it is sufficient to prove that the multivalued mapping admits a fixed point and this by using a generalization of the Leray-Schauder fixed point theorem stated in [8]. To use this generalization, we introduce the set

$$N_\delta = \{R \in L^2(Q_T) : \exists \rho \in (0,1), R \in \rho\Lambda_\delta(R)\}. \tag{4.3}$$

The existence of a fixed point of the multi-valued mapping Λ_δ is an immediate consequence of the following proposition.

Proposition 4.2. (i) *for all $R \in L^2(Q_T)$, $\Lambda_\delta(R)$ is a closed and convex set.*
(ii) *Λ_δ is upper semi-continuous on $L^2(Q_T)$.*
((iii) *$\Lambda_\delta : L^2(Q_T) \to P(L^2(Q_T))$ is a compact multivalued mapping.*
(iv) *N_δ is bounded in $L^2(Q_T)$.*

Proof. The proofs of (i) and (ii) are similar to the ones of (ii) and (iv) in [17], with R (respectively R_n) instead of $e^{-\lambda_0 t}F(e^{\lambda_0 t}R)$ (respectively $e^{-\lambda_0 t}F(e^{\lambda_0 t}R_n)$) and the convergence space of the subsequence of $\widetilde{y}_n$ is $L^2((0,A)\times(0,T), H_k^1(0,1))$ instead of the space $L^2((0,A)\times(0,T), H_0^1(0,1))$.

Now, we address the proof of (iii). Let $R \in L^2(Q_T)$ such that $\|R\|_{L^2(Q_T)} \le K$, $K > 0$. We have to prove that any sequence of elements of $\Lambda_\delta(R)$ admits a convergent subsequence. Let $(\rho_n)_n \subseteq \Lambda_\delta(R)$. From the definition of Λ_δ, for all n there exists $(\widetilde{\vartheta}_n, \widetilde{y}_n) \in L^2(q) \times L^2(Q)$ such that $\rho_n = \int_0^A \beta\widetilde{y}_n da$, $\widetilde{\vartheta}_n$ verifies (3.36) and $\widetilde{y}_n$, the associated solution of (4.2) verifies (1.2). Then, by (3.36) we have

$$\begin{aligned}
\int_q \widetilde{\vartheta}_n^2(t,a,x)\,dt\,da\,dx &\le \frac{4C_\delta}{3}\Big(\int_{Q_T} R^2(t,x)\,dt\,dx + \int_{Q_A} y_0^2(a,x)\,da\,dx\Big) \\
&\le \frac{4C_\delta}{3}\Big(K^2 + \int_{Q_A} y_0^2(a,x)\,da\,dx\Big).
\end{aligned} \tag{4.4}$$

Hence, $\widetilde{\vartheta}_n$ is bounded in $L^2(q)$. Thus, there exists a subsequence of $\widetilde{\vartheta}_n$ denoted by $\widetilde{\vartheta}_{n_k}$ that converges weakly towards $\widetilde{\vartheta}$ in $L^2(q)$. On the other hand, multiplying

(4.2) by $\widetilde{y}_n$, integrating over Q, using Young inequality, we infer that

$$\begin{aligned}
&\int_Q k(x)(\widetilde{y}_n)_x^2\,dt\,da\,dx+\lambda\int_Q \widetilde{y}_n^2\,dt\,da\,dx\\
&\leq \frac{1}{2\lambda}\int_q \widetilde{\vartheta}_n^2(t,a,x)\,dt\,da\,dx+\frac{\lambda}{2}\int_Q \widetilde{y}_n^2\,dt\,da\,dx\\
&\quad+\frac{1}{2}\Big(\int_{Q_A} y_0^2(a,x)\,da\,dx+\int_{Q_T} R^2(t,x)\,da\,dx\Big).
\end{aligned}\tag{4.5}$$

This implies

$$\begin{aligned}
&\int_Q k(x)(\widetilde{y}_n)_x^2\,dt\,da\,dx+\frac{\lambda}{2}\int_Q \widetilde{y}^2\,dt\,da\,dx\\
&\leq \frac{1}{2\lambda}\int_q \widetilde{\vartheta}_n^2(t,a,x)\,dt\,da\,dx+\frac{1}{2}\Big(\int_{Q_A} y_0^2(a,x)\,da\,dx+\int_{Q_T} R^2(t,x)\,da\,dx\Big).
\end{aligned}\tag{4.6}$$

Taking $\lambda\geq 2$ and using (4.4), (4.6) becomes

$$\int_Q k(x)(\widetilde{y}_n)_x^2\,dt\,da\,dx+\int_Q \widetilde{y}^2\,dt\,da\,dx\leq\Big(\frac{1}{2}+\frac{C_\delta}{3}\Big)\Big(K^2+\int_{Q_A} y_0^2(a,x)\,da\,dx\Big).\tag{4.7}$$

Therefore, $\widetilde{y}_n$ is bounded in $L^2((0,T)\times(0,A),H_k^1(0,1))$. Hence, we can extract a subsequence of $\widetilde{y}_n$ denoted by $\widetilde{y}_{n_{k_1}}$ that converges weakly toward $\widetilde{y}$ in $L^2((0,T)\times(0,A),H_k^1(0,1))$. Now, we consider $\rho_{n_{k_1}}=\int_0^A\beta\widetilde{y}_{n_{k_1}}\,da$ the subsequence of ρ_n associated to $\widetilde{y}_{n_{k_1}}$. Using (2.2), we conclude that $\rho_{n_{k_1}}$ satisfies the system

$$\begin{gathered}
\frac{\partial\rho_{n_{k_1}}}{\partial t}-(k(x)(\rho_{n_{k_1}})_x)_x+\int_0^A\beta\mu_1\widetilde{y}_{n_{k_1}}da=z_{n_{k_1}}\quad\text{in } Q_T,\\
\rho_{n_{k_1}}(t,1)=\rho_{n_{k_1}}(t,0)=0\quad\text{on } (0,T),\\
\rho_{n_{k_1}}(0,x)=\int_0^A\beta(0,a,x)y_0(a,x)da\quad\text{in } (0,1),
\end{gathered}\tag{4.8}$$

with,

$$\begin{aligned}
z_{n_{k_1}}&=\int_0^A\beta\widetilde{\vartheta}_{n_{k_1}}\chi_\omega da+\int_0^A(\beta_t+\beta_a)\widetilde{y}_{n_{k_1}}da\\
&\quad-\Big(\int_0^A k(x)\beta_x(\widetilde{y}_{n_{k_1}})_x da+\int_0^A(k(x)\beta_x\widetilde{y}_{n_{k_1}})_x da\Big).
\end{aligned}$$

Taking into account the assumptions on k, using Hardy-Poincaré and Minkowski's inequalities and exploiting the inequalities (4.4) and (4.7) for $\widetilde{\vartheta}_{n_{k_1}}$ and $\widetilde{y}_{n_{k_1}}$ respectively, we deduce that

$$\|z_{n_{k_1}}\|_{L^2(Q_T)}^2\leq D_\delta(K^2+\int_{Q_A} y_0^2(a,x)\,da\,dx).\tag{4.9}$$

Now, multiplying the first equation of system (4.8) by $\rho_{n_{k_1}}$, integrating over Q_T and using Young inequality, we obtain

$$\begin{aligned}
&\int_{Q_T} k(x)(\rho_{n_{k_1}})_x^2\,dt\,dx+\frac{\lambda}{2}\int_{Q_T}\rho_{n_{k_1}}^2\,dt\,dx\\
&\leq\frac{1}{2\lambda}\int_{Q_T} z_{n_{k_1}}^2\,dt\,dx+\frac{C_\beta}{2}\int_{Q_A} y_0^2(a,x)\,da\,dx.
\end{aligned}\tag{4.10}$$

Taking again $\lambda \geq 2$ in (4.10), we conclude by (4.9) that $\rho_{n_{k_1}}$ is bounded in $L^2((0,T);H^1_k(0,1))$. Now, thanks to Proposition 2.2 we infer that $\frac{\partial \rho_{n_{k_1}}}{\partial t}$ is bounded in $L^2((0,T);H^{-1}_k(0,1))$. Since $H^1_k(0,1)$ is compactly embedded in $L^2(0,1)$(see [6]), we conclude by Aubin-Lions lemma the existence of a subsequence of $\rho_{n_{k_1}}$ denoted by ρ_{n_j} that converges strongly towards ρ in $L^2(Q_T)$. This implies that ρ_{n_j} converges weakly towards ρ in $L^2(Q_T)$. Thus,

$$\int_{Q_T} g\rho_{n_j}\,dt\,dx \to \int_{Q_T} g\rho\,dt\,dx, \quad \forall g \in L^2(Q_T). \tag{4.11}$$

On the other hand, $\widetilde{y}_{n_{k_1}}$ converges weakly to $\widetilde{y}$ in $L^2((0,T)\times(0,A),H^1_k(0,1))$. Then, $\widetilde{y}_{n_{k_1}}$ converges weakly toward $\widetilde{y}$ in $L^2(Q)$ (because $L^2((0,T)\times(0,A),H^1_k(0,1)) \subseteq L^2(Q)$). Subsequently, $\widetilde{y}_{n_j}$ the subsequence of $\widetilde{y}_{n_{k_1}}$ associated to ρ_{n_j} converges weakly towards $\widetilde{y}$ as well. The fact that $g\beta \in L^2(Q)$, for all $g \in L^2(Q_T)$, implies that

$$\int_{Q_T} g\rho_{n_j}\,dx\,dt \to \int_{Q_T} g\Big(\int_0^A \beta\widetilde{y}da\Big)\,dx\,dt. \tag{4.12}$$

Accordingly, by (4.11) and (4.12), we infer that

$$\int_{Q_T} g\Big(\int_0^A \beta\widetilde{y}da - \rho\Big)\,dx\,dt = 0, \forall g \in L^2(Q_T). \tag{4.13}$$

Therefore,

$$\rho(t,x) = \int_0^A \beta\widetilde{y}\,da \quad \text{a.e. } (t,x) \in Q_T. \tag{4.14}$$

Furthermore, we can check by a standard argument that $\widetilde{y}$ satisfies (1.2) and solves (4.2) with R instead of b and $\widetilde{\vartheta} \in L^2(q)$ verifies (3.36) and this completes the proof of (iii). The rest follows as in [17]. □

5. Appendix

As is mentioned in the introduction, this section is devoted to a Caccioppoli's type inequality which played a crucial role to establish the Carleman estimate (3.16). This inequality is stated in the following lemma.

Lemma 5.1. *There exists a positive constant C such that*

$$\int_{\omega'}\int_0^A\int_0^T w_x^2 e^{2s\varphi}\,dt\,da\,dx \leq C\Big(\int_q s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx + \int_q h^2 e^{2s\varphi}\,dt\,da\,dx\Big), \tag{5.1}$$

where $\omega' = (\frac{2x_1+x_2}{3}, \frac{x_1+2x_2}{3})$ and w is the solution of (3.4).

Proof. Define the smooth cut-off function $\zeta : \mathbb{R} \to \mathbb{R}$ by

$$\begin{gathered} 0 \leq \zeta(x) \leq 1, \quad \text{if } x \in \mathbb{R}, \\ \zeta(x) = 0, \quad \text{if } x < x_1 \text{ and } x > x_2, \\ \zeta(x) = 1, \quad \text{if } x \in \omega', \\ \zeta(x) > 0, \quad \text{if } x > x_1 \text{ and } x < x_2. \end{gathered} \tag{5.2}$$

For the solution w of (3.4), we have

$$0 = \int_0^T \frac{d}{dt}\Big[\int_0^1\int_0^A \zeta^2 e^{2s\varphi} w^2\,da\,dx\Big]dt$$

$$
\begin{aligned}
&= 2s\int_0^1\int_0^A\int_0^T \zeta^2\varphi_t w^2 e^{2s\varphi}\,dt\,da\,dx + 2\int_0^1\int_0^A\int_0^T \zeta^2 w w_t e^{2s\varphi}\,dt\,da\,dx\\
&= 2s\int_0^1\int_0^A\int_0^T \zeta^2\varphi_t w^2 e^{2s\varphi}\,dt\,da\,dx\\
&\quad + 2\int_0^1\int_0^A\int_0^T \zeta^2 w(-(kw_x)_x - w_a + h + \mu w)e^{2s\varphi}\,dt\,da\,dx.
\end{aligned}
$$

Integrating by parts, we obtain

$$
\begin{aligned}
&2\int_Q k\zeta^2 e^{2s\varphi} w_x^2\,dt\,da\,dx\\
&= -2s\int_Q \zeta^2 w^2\psi(\Theta_a+\Theta_t)e^{2s\varphi}\,dt\,da\,dx - 2\int_Q \zeta^2 w h e^{2s\varphi}\,dt\,da\,dx\\
&\quad - 2\int_Q \zeta^2\mu w^2 e^{2s\varphi}\,dt\,da\,dx + \int_Q (k(\zeta^2 e^{2s\varphi})_x)_x w^2\,dt\,da\,dx.
\end{aligned}
$$

On the other hand, by the definitions of ζ, ψ and Θ, using Young inequality and taking s quite large there is a constant c such that

$$
\begin{gathered}
2\int_Q k\zeta^2 e^{2s\varphi} w_x^2\,dt\,da\,dx \ge 2\min_{x\in\omega'} k(x)\int_{\omega'}\int_0^A\int_0^T w_x^2 e^{2s\varphi}\,dt\,da\,dx,\\
\int_Q (k(\zeta^2 e^{2s\varphi})_x)_x w^2\,dt\,da\,dx \le c\int_{\omega'}\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx,\\
-2s\int_Q \zeta^2 w^2\psi(\Theta_a+\Theta_t)e^{2s\varphi}\,dt\,da\,dx \le c\int_{\omega'}\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx,\\
\begin{aligned}
&-2\int_Q \zeta^2 w h e^{2s\varphi}\,dt\,da\,dx\\
&\le c\Big(\int_{\omega'}\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx + \int_{\omega'}\int_0^A\int_0^T h^2 e^{2s\varphi}\,dt\,da\,dx\Big),
\end{aligned}\\
-2\int_Q \zeta^2\mu w^2 e^{2s\varphi}\,dt\,da\,dx \le c\int_{\omega'}\int_0^A\int_0^T s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx.
\end{gathered}
$$

This all together imply that there is $C>0$ such that

$$
\int_{\omega'}\int_0^A\int_0^T w_x^2 e^{2s\varphi}\,dt\,da\,dx \le C\Big(\int_q s^2\Theta^2 w^2 e^{2s\varphi}\,dt\,da\,dx + \int_q h^2 e^{2s\varphi}\,dt\,da\,dx\Big).
$$

Thus, the proof is complete. □

References

[1] B. Ainseba; *Corrigendum to "Exact and approximate controllability of the age and space population dynamics structured model [J. Math. Anal. Appl. 275 (2002), 562-574]"*, J. Math. Anal. Appl. 393 (2012), 328.

[2] B. Ainseba; *Exact and approximate controllability of the age and space population dynamics structured model*, J. Math. Anal. Appl. 275 (2002), 562-574.

[3] B. Ainseba, S. Anita; *Internal exact controllability of the linear population dynamics with diffusion*, Electronic Journal of Differential Equations, 2004 (2004) No. 112, 1-11.

[4] B. Ainseba, S. Anita; *Local exact controllability of the age-dependent population dynamics with diffusion*, Abstr. Appl. Anal. 6 (2001), 357-368.

[5] B. Ainseba, M. Langlais; *On a population dynamics control problem with age dependence and spatial structure*, Journal of Mathematical Analysis and Applications, 248 (2000), 455-474.

[6] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. evol. equ. 6 (2006), 161-204.
[7] S. Anita; *Analysis and control of age-dependent population dynamics*, Kluwer Academic Publishers, Dordrecht, (2000).
[8] C. Avramescu; *A fixed point theorem for multivalued mappings*, Electronic Journal of Qualitative Theory of Differential Equations 2004 (2004), 1-10.
[9] V. Barbu, M. Iannelli, M. Martcheva; *On the controllability of the Lotka-McKendrick model of population dynamics*, J. Math. Anal. Appl. 253 (2001), 142-165.
[10] M. Campiti, G. Metafune, D. Pallara; *Degenerate self-adjoint evolution equations on the unit interval*, Semigroup Forum. 57 (1998), pp. 1-36.
[11] P. Cannarsa, P. Martinez, J. Vancostenoble; *Null controllability of degenerate heat equations,* Adv. Differential Equations. 10 (2005), pp. 153-190.
[12] A. V. Fursikov, O. Yu. Imanuvilov; *Controllability of Evolution Equations*, Lecture Notes Series, vol. 34, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.
[13] M. Langlais; *A nonlinear problem in age-dependent population diffusion*, Siam J. Math. Anal. 16 (1985), 510-529.
[14] A. Rhandi, R. Schnaubelt; *Asymptotic behaviour of a non-autonomous population equation with diffusion in* L^1, Discrete Contin. Dynam. Systems 5 (1999), 663-683.
[15] R. Nagel, K. J. Engel; *One-Parameter semigroups for linear evolution equations*, Graduate texts in Mathematics, 194 (2000), XXI, 589 p.
[16] A. Pazy; *Semigroup of linear operator and applications to partial differential equations*, Applied Math. Sci. 44, Springer-Verlag, 1983.
[17] O. Traore; *Null controllability of a nonlinear population dynamics problem*, Int. J. Math. Sci. (2006), 1-20.
[18] M. Tucsnak, G. Weiss; *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts Basler Lehrbücher Series, 2009.
[19] G. F. Webb; *Population models structured by age, size, and spatial position.* Structured population models in biology and epidemiology, 149, Lecture Notes in Math. 1936, Springer, Berlin, 2008.

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