1. Introduction

The stationary Navier-Stokes problem may be written in the form

\[-\nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega\]
\[\text{div } u = 0 \quad \text{in } \Omega\]
\[u = 0 \quad \text{on } \Gamma = \partial \Omega\]  

(1.1)

This equation describes the motion of an incompressible fluid contained in $$\Omega$$ and subjected to an outside forces $$f$$, $$u$$ is the velocity of fluid flow, $$p$$ is the pressure and $$\nu$$ its viscosity.

The variational formulation of the Navier Stokes equations in the classic form is well studied in [8, 9, 16]. In most publications they uses a trilinear form in the variational formulation for studying the nonlinear term presented in the equation of momentum.

This paper is devoted to give another idea: we construct a sequence of a Newton-linearized problems and we show, using Lax-Milgramm theorem, that the variational formulation of each one has an unique solution. We show then that the sequence of weak solutions converges towards the solution of the nonlinear one in a quadratic way.

The outline of the paper is as follows: In Section 2 we start by a Newton-linearisation of the Navier Stokes equations. We obtain a sequence of linear problems and we show the existence of a weak solution. In Section 3 we show the quadratic convergence of the sequence of the solutions in Theorem 3.3. In section 4 the nonhomogeneous problem is treated.

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2. Linearized problems

Linearization. Let $\Omega$ a bounded domain of $\mathbb{R}^2$ with Lipschitz-continuous boundary $\Gamma$, and let

$$V = \{ v \in (H^1_0(\Omega))^2, \text{div} v = 0 \}$$

with norm $\|u\|_V = \max\{\|u_1\|_{H^1_0(\Omega)}, \|u_2\|_{H^1_0(\Omega)}\}$. We set $L^2_0 = (L^2(\Omega))^2$, and $H^1_0(\Omega) = (H^1(\Omega))^2$ with norm $\|u\|_{H^1_0} = \max\{\|u_1\|_{H^1_0}, \|u_2\|_{H^1_0}\}$ and $W = H^1_0(\Omega) \times L^2_0$.

The nonlinear term

$$(u \cdot \nabla)u = \left( \begin{array}{c} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \end{array} \right)$$

can be written as

$$(u \cdot \nabla)u = \frac{1}{2} \nabla|u|^2 + \text{rot} u \wedge u.$$
Therefore we have $K(v) = 0$ for all $v \in V$, then de Rham’s theorem implies that there exists a unique function $p_{n+1} \in L^2(\Omega)/\mathbb{R}$ such that

$$a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) - L_n(v) = \int_\Omega p_{n+1} \text{div} \ v \ dx \quad \forall v \in H^1_0(\Omega);$$

therefore,

$$a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a^n(u_{n+1}, v) - \int_\Omega p_{n+1} \text{div} v \ dx = L_n(v) \quad \forall v \in H^1_0(\Omega).$$

Which gives the desired result. \hfill \Box

Let us now show that problem (2.3) has an unique solution for each $n$. For this, we need the following lemma.

**Lemma 2.2.** For fixed $u_n \in V$ the form $(u, v) \rightarrow a_n(u, v)$ and $(u, v) \rightarrow a^n(u, v)$ are continuous on $H^1_0(\Omega)$.

**Proof.** We have

$$a_n(u, v) = \sum_{i,j=1}^2 \int_\Omega u_{n,j} \frac{\partial u_i}{\partial x_j} v_i \ dx,$$

$$a^n(u, v) = \sum_{i,j=1}^2 \int_\Omega u_j \frac{\partial u_n,i}{\partial x_j} v_i \ dx$$

by Hölder’s inequality we have

$$\left| \int_\Omega u_{n,j} \frac{\partial u_i}{\partial x_j} v_i \ dx \right| \leq \|u_{n,j}\|_{L^4} \|v_i\|_{L^4} \left\| \frac{\partial u_i}{\partial x_j} \right\|_{L^2} \quad (2.4)$$

According to the Sobolev Imbedding Theorem, the space $H^1(\Omega)$ is continuously embedded in $L^4(\Omega)$. Then there exists $C_1 > 0$ such that

$$|a_n(u, v)| \leq C_1 \|u\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)} \|u_n\|_{H^1_0(\Omega)} \quad (2.5)$$

The same result holds with the term $a^n$,

$$|a^n(u, v)| \leq C_2 \|u\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)} \|u_n\|_{H^1_0(\Omega)} \quad (2.6)$$

To show the coercivity of the form $a = a_0 + a_n + a^n$ we have the following lemma.

**Lemma 2.3.** We have $a_n(u, u) = 0$ for all $u \in V$.

**Proof.** Note that

$$a_n(u, u) = \int_{\Omega} (u_n, \nabla) u \ u \ dx = \frac{1}{2} \int_{\Omega} u_n \nabla(|u|^2) \ dx \quad (2.7)$$

where

$$\nabla(|u|^2) = \left( \frac{\partial (u_1^2 + u_2^2)}{\partial x_1}, \frac{\partial (u_1^2 + u_2^2)}{\partial y} \right)$$

Using Green’s formula and $\text{div} u_n = 0$ and boundary conditions we have

$$2a_n(u, u) = \int_{\Omega} \nabla |u|^2 u_n \ dx = - \int_{\Omega} \text{div} u_n |u|^2 \ dx = 0. \quad (2.8)$$

\hfill \Box
Therefore, the discriminant of the polynomial

$$P(\alpha^*) = (C + C_2)\alpha^* - \nu C_3 \alpha^* + \|f\|_2 \leq 0,$$ \quad $\alpha^* < \frac{\nu C_3}{C_2}$

Therefore, the discriminant of the polynomial $P(\alpha^*)$ must verify

$$\Delta = \nu^2 C_3^2 - 4(C + C_2)\|f\|_2 > 0.$$ \quad (2.15)
Then
\[ \|f\|_2 < \frac{\nu^2 C_3^2}{4(C + C_2)} \]
and hence \( P(\alpha^*) \) has two roots
\[ \alpha_1 = \frac{\nu C_3 - \sqrt{\Delta}}{2(C + C_2)}, \quad \alpha_2 = \frac{\nu C_3 + \sqrt{\Delta}}{2(C + C_2)} \]
Since \( \alpha_2 > 0 \) we can choose \( 0 < \alpha^* < \min(\frac{\nu C_3}{C_2}, \alpha_2) \).

**Theorem 2.6.** (1) For \( f \in (L^2(\Omega))^2 \) satisfying (2.14), problem (2.3) has a unique solution \( u_{n+1} \in V \cap B_{\alpha^*} \).

(2) If \( u_0 \in B_{\alpha^*} \cap H^2(\Omega) \), then \( u_{n+1} \in H^2(\Omega) \).

**Proof.** (1) Since \( u_n \in B_{\alpha^*} \), we have
\[ \|L_n(v)\| \leq (\|f\|_2 + C\alpha^2)\|v\|_{H^1_0(\Omega)} \]
which gives the continuity of \( L_n \) and using Lemma 2.1, Lemma 2.3 and Lemma 2.4 with Lax-Milgram Theorem we obtain the result.

(2) We assume that \( u_n \in H^2(\Omega) \) then \( (u_n \nabla)u_n \in (L^2(\Omega))^2 \), which implies that \( f_n = f + (u_n \nabla)u_n \in (L^2(\Omega))^2 \) for \( f \in (L^2(\Omega))^2 \), and by classical regularity Theorem we have \( u_{n+1} \in H^2(\Omega) \).

3. Convergence

The sequence \( (u_n)_{n \in \mathbb{N}}, \) solutions of (2.3) with \( n \) instead of \( n + 1 \), satisfy
\[ \|u_n\|_{H^1_0(\Omega)} \leq \alpha^* \quad \forall n \geq 0, \]
which implies that the sequence \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H^1_0(\Omega) \). Then there exist a subsequence that converges weakly to \( \phi \) in \( H^1_0(\Omega) \). Since the injection of \( H^1_0(\Omega) \) in \( (L^2(\Omega))^2 \) is compact, there exists a subsequence still noted \( u_n \) which converges strongly to \( \phi \) in \( (L^2(\Omega))^2 \).

We need the following result.

**Lemma 3.1.** For \( v \in V \), we have:

1. \( \lim_{n \to \infty} a_0(u_{n+1}, v) = a_0(\phi, v) \).
2. \( \lim_{n \to \infty} a_n(u_{n+1}, v) = a_\infty(\phi, v) = \int_\Omega (\phi \nabla)\phi v dx \).
3. \( \lim_{n \to \infty} a^n(u_{n+1}, v) = a^\infty(\phi, v) = \int_\Omega (\phi \nabla)\phi v dx \).
4. We have \( \lim_{n \to \infty} L_n(v) = L_\infty(v) = \int_\Omega [f + (\phi \nabla)\phi] v dx \).

**Proof.** (1) Since \( u_n \to \phi \), and by linearity of \( u \to a_0(u, v) \) we have \( a_0(u_{n+1}, v) \to a_0(\phi, v) \) for all \( v \in V \).

(2) Let
\[ E = \|a^n(u_{n+1}, v) - a^\infty(\phi, v)\| = \int_\Omega \|(u_{n+1} \nabla)u_n - (\phi \nabla)\phi\| v dx \] (3.2)
We can write
\[ (u_{n+1} \cdot \nabla)u_n - (\phi \nabla)\phi = ((u_{n+1} - \phi) \cdot \nabla)u_n + (\phi \cdot \nabla)(u_n - \phi) \] (3.3)
which gives with \( u_n \in H^2(\Omega) \) and using Green’s theorem,
\[ E \leq C\|u_{n+1} - \phi\|_2 \|u_n\|_{H^1} \|v\|_{H^1} + \|u_n - \phi\|_2 \|\nabla v\|_{H^1} \|\phi\|_{H^2} + \|\nabla v\|_{H^1} \|\phi\|_{H^2} \|v\|_{H^1} \] (3.4)
Since \( u_n \) converges strongly to \( \phi \) in \( (L^2(\Omega))^2 \), it follows that \( E \to 0 \).
Let

\[ F = \left| a_n(u_{n+1}, v) - a_\infty(\phi, v) \right| = \int_{\Omega} \{(u_n \cdot \nabla)u_{n+1} - (\phi \cdot \nabla)\phi \} v \, dx. \]

Then

\[ F \leq C[\|u_n - \phi\|_2 \|u_{n+1}\|_{H^1} + \|u_{n+1} - \phi\|_2(\|v\|_{H^1} \|\nabla \phi\|_{H^1} + \|\phi\|_{H^1} \|\nabla v\|_{H^1})]; \]

thus \( F \to 0. \)

(4) Let

\[ G = |L_n(v) - L_\infty(v)| \leq \int_{\Omega} |(u_n \nabla u_n) - (\phi \nabla \phi)| \|\nabla v\| \, dx. \]

Then

\[ G \leq C[\|u_n - \phi\|_2(\|u_n\|_{H^1} \|v\|_{H^1} + \|\phi\|_{H^1} \|\nabla v\|_{H^1} + \|\nabla \phi\|_{H^1} \|v\|_{H^1})]. \]

Then Lemma 3.1 gives the desired result.

For using de Rham’s Theorem, let \( L \) a continuous linear form on \((H^1_0(\Omega))^2\) which vanishes on \( V \) if and only if there exists a unique function \( \varphi \in L^2(\Omega)/\mathbb{R} \) such that for all \( v \in H^1_0(\Omega), \)

\[ L(v) = \int_{\Omega} \varphi \, \text{div} \, v \, dx. \]

**Theorem 3.2.** We have \( \lim_{n \to \infty} u_n = \phi \) in \( V \) then \( \phi \) is a solution of (1.1).

**Proof.** It follows from Lemma 3.1 that

\[ \lim_{n \to \infty} a_0(u_{n+1}, v) + a_n(u_{n+1}, v) + a_n(u_{n+1}, v) = a_0(\phi, v) + 2a_\infty(\phi, v) = L_\infty(v). \]

Let the linear form \( L(v) = a_0(\phi, v) + a_\infty(\phi, v) - L_\infty(v) \). Therefore \( L(v) = 0 \) for all \( v \in V \), then de Rham’s theorem implies that there exists a unique function \( p \in L^2(\Omega)/\mathbb{R} \) such that

\[ a_0(\phi, v) + 2a_\infty(\phi, v) - L_\infty(v) = \int_{\Omega} p \, \text{div} \, v \, dx \quad \forall v \in H^1_0(\Omega) \]

which gives

\[ \nu \int_{\Omega} \nabla \phi \, \nabla v \, dx + \int_{\Omega} (\phi \nabla)\phi \, dx - \int_{\Omega} p \, \text{div} \, v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^1_0(\Omega), \]

\[ \int_{\Omega} (-\nu \Delta \phi + (\phi \nabla)\phi + \nabla p - f) v \, dx = 0 \quad \forall v \in H^1_0(\Omega). \]

Then in \( \mathcal{D}'(\Omega), \)

\[ -\nu \Delta \phi + (\phi \nabla)\phi + \nabla p - f = 0. \]

Since \( \phi \in V \) we conclude that \( \phi \) is the solution of (1.1). \( \Box \)

**Theorem 3.3.** Let \( u_{n+1} \) be the solution of (2.2), and \( \phi \) be the solution of (1.1). Then convergence of the sequence \((u_{n+1})_{n \in \mathbb{N}}\) towards \( \phi \) is quadratic; i.e.,

\[ \|u_{n+1} - \phi\|_{H^1_0(\Omega)} \leq C_2\|u_n - \phi\|_{H^1_0(\Omega)}^2 \]

(3.11)
Proof. Let \( \omega_n = u_n - \phi \) and \( \chi_n = p_n - p \). Subtracting problem (2.1) from (1.1) we obtain

\[
-\nu \Delta \omega_{n+1} + (\omega_{n+1} \nabla) u_n + (u_n \nabla) \omega_{n+1} + \nabla \chi_{n+1} = (\omega_n \nabla) \omega_n \quad \text{in} \ \Omega
\]

\[
\text{div} \omega_{n+1} = 0 \quad \text{in} \ \Omega
\]

\[
\omega_{n+1} = 0 \quad \text{on} \ \Gamma
\]

The variational formulation of (3.12) is

\[
\text{Find} \ (\omega_{n+1}, \chi_{n+1}) \in W \text{ such that}
\
a(\omega_{n+1}, v) + b(\chi_{n+1}, v) = F_n(v) \quad \forall v \in H^1_0(\Omega)
\]

where \( a = a_0 + a^n + a_n \) and

\[
b(q, \omega_{n+1}) = -\int_{\Omega} q \text{div} \omega_{n+1} \, dx, \quad F_n(v) = \int_{\Omega} (\omega_n \nabla) \omega_n v \, dx.
\]

Since \( \text{div} \omega_{n+1} = 0 \), using Lemma 2.1 and Lemma 2.5 for \( u_n \in B_{\alpha^*} \) and \( v = \omega_{n+1} \), we obtain

\[
(\nu C_1 - C\alpha^*) \|\omega_{n+1}\|_{H^1_0(\Omega)}^2 \leq a(\omega_{n+1}, \omega_{n+1}) = F(\omega_{n+1})
\]

\[
\leq C \|\omega_n\|_{H^1_0(\Omega)}^2 \|\omega_{n+1}\|_{H^1_0(\Omega)}.
\]

This gives (3.10) with \( C_2 = \frac{C}{(\nu C_1 - C\alpha^*)} \) and the convergence is quadratic. \( \square \)

4. Nonhomogeneous Problem

We are concerned now with the nonhomogeneous problem

\[
-\nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in} \ \Omega
\]

\[
\text{div} u = 0 \quad \text{in} \ \Omega
\]

\[
u
\]

\[
\omega_{n+1} = 0 \quad \text{on} \ \Gamma
\]

Where the state \( u \) is sought in the space \((H^1(\Omega))^2 \cap V\).

Throughout this section \( \Omega \) denotes a bounded domain in \( \mathbb{R}^2 \), with Lipschitz-continuous boundary \( \Gamma = \cap \Gamma_i \), \( i = 1, \ldots, 4 \). We assume in this section that

\[
\int_{\Gamma_i} g \cdot n_i \, d\sigma = 0 \quad \text{with} \ g \in H = (H^{1/2}(\Gamma))^2 \text{ and } f \in K = (H^{-1}(\Omega))^2.
\]

We assume also that for a given \( g \in H \) satisfying (3.11) for any \( c > 0 \) there exists a function \( w_0 \in (H^1(\Omega))^2 \) such that

\[
\text{div} w_0 = 0, \quad w_0|\Gamma = g,
\]

\[
|a_n(w_0, u_n)| \leq c\|u_n\|_{H^1_0(\Omega)}^2 \quad \forall u_n \in V.
\]

The existence of \( w_0 \) satisfying (3.11) (3.14) is a technical result due to Hopf [11].

**Theorem 4.1.** Given \((g, f) \in K \times H \) satisfying (3.14), there exists a pair \((u, p) \in (H^1(\Omega))^2 \times L^2_0(\Omega)\) which is a solution of (4.1).

**Proof.** Let \( \xi_0 = u_0 - w_0 \) where \( w_0 \) verify (3.11) (3.14) and an arbitrary \( u_0 \in V \). We consider the sequence of linear problems

\[
-\nu \Delta \xi_{n+1} + (\xi_{n+1} \cdot \nabla) \xi_n + (\xi_n \cdot \nabla) \xi_{n+1} + \nabla p_{n+1} = f_n \quad \text{in} \ \Omega
\]

\[
\text{div} \xi_{n+1} = 0 \quad \text{in} \ \Omega
\]

\[
\xi_{n+1} = 0 \quad \text{on} \ \Gamma
\]
with $\xi_{n+1} = u_{n+1} - w_0$, $u_{n+1} \in H^1_0(\Omega)$ and $f_n = f + (\xi_n \nabla)\xi_n + \nu \Delta w_0 - (w_0 \cdot \nabla)w_0$. Then $\xi_{n+1}$ is a solution of the variational problem

Find $\xi_{n+1} \in V$ such that

$$a(\xi_{n+1}, v) = L_n(v) \quad \forall v \in V,$$

where $a(\xi, v) = a_0(\xi, v) + a_n(\xi, v) + a^0(\xi, v) + a^*(\xi, v)$ with

$$a^*(\xi, v) = \int_{\Omega} (\xi \nabla)w_0v \, dx + \int_{\Omega} (w_0 \nabla)\xi v \, dx$$

and $L_n(v) = \langle f_n, v \rangle$.

Taking $c > \nu$ and using 2.9 we obtain

$$|a(\xi_{n+1}, \xi_{n+1})| \geq (\nu - c)\|\xi_{n+1}\|^2_{H^1_0(\Omega)}$$

(4.7)

Thus we have the coercivity and $L$ is obviously continuous on $V$. We observe that problem (4.6) fits into the framework of section 1 and therefore the sequence $\xi_n$ converges towards a solution of (4.1). □

References


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