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A QUASISTATIC ELECTRO-ELASTIC CONTACT PROBLEM WITH NORMAL COMPLIANCE, FRICTION AND ADHESION

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Abstract. In this article we consider a mathematical model which describes the contact between a piezoelectric body and a deformable foundation. The constitutive law is assumed linear electro-elastic and the process is quasistatic. The contact is adhesive and frictional and is modelled with a version of normal compliance condition and the associated Coulomb's law of dry friction. The evolution of the bonding field is described by a first order differential equation. We derive a variational formulation for the model, in the form of a coupled system for the displacements, the electric potential and the bonding field. Under a smallness assumption on the coefficient of friction, we prove an existence result of the weak solution of the model. The proofs are based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1. Introduction

In this work, we study a frictional contact problem with adhesion between an elastic piezoelectric body and a deformable obstacle.

A piezoelectric material is one that produces an electric charge when a mechanical stress is applied (the material is squeezed or stretched). Conversely, a mechanical deformation (the material shrinks or expands) is produced when an electric field is applied. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, and those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. Different models have been developed to describe the interaction between the electric and mechanical fields (see [1, 13], [18]-[20], [28, 29]). General models for elastic materials with piezoelectric effect, called electro-elastic materials, can be found in [1, 13]. A static frictional contact problem for electric-elastic materials was considered in [2, 17] and a slip-dependent frictional contact problem for electro-elastic materials was studied in [26].

Adhesion may take place between parts of the contacting surfaces. It may be intentional, when surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces. The adhesive contact is modelled by the introduction

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of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimeas referred to as the intensity of adhesion. Following [9, 10], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. Basic modelling can be found in [9]–[11]. Analysis of models for adhesive contact can be found in [3, 4] and in the monographs [24, 25]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [22, 23]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Since frictional contact is so important in industry, there is a need to model and predict it accurately. However, the main industrial need is to effectively control the process of frictional contact. Currently, there is a considerable interest in frictional contact problems involving piezo-electric materials, see for instance [2, 15, 26].

The aim of this article is to continue the study of problems begun in [12, 21, 6]. The novelty of the present paper is to extend the result when the contact and friction are modelled by a normal compliance condition and a version of Coulomb's law of dry friction, respectively. Moreover, the adhesion is taken into account at the interface and the material behavior is assumed to be electro-elastic.

The paper is structured as follows. In Section 2 we present the electro-elastic contact model with normal compliance, friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In section 4, we present our main existence results.

2. Problem statement

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density f_0 and volume electric charges of density q_0 . It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2 and Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand., such that $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma a) > 0$. We assume that the body is clamped on Γ_1 and surface tractions of density f_2 act on Γ_2. On Γ_3 the body is in adhesive contact with an insulator obstacle, the so-called foundation. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and we use $\cdot$ and $\|\cdot\|$ for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Also, below ν represents the unit outward normal on Γ. With these assumptions, the classical model for the process is the following.

Problem $(\mathcal{P})$**.** Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\sigma = \mathcal{F}\varepsilon(u) - \mathcal{E}^* E(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$D = \mathcal{B}E(\varphi) + \mathcal{E}\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\operatorname{div} D = q_0 \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$\begin{gathered}
\|\sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau)\| \le \mu p_\nu(u_\nu), \\
\|\sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau)\| < \mu p_\nu(u_\nu) \Rightarrow \dot{u}_\tau = 0, \\
\|\sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau)\| = \mu p_\nu(u_\nu) \\
\Rightarrow \exists \lambda \ge 0 \text{ such that } \sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau) = -\lambda \dot{u}_\tau,
\end{gathered} \tag{2.8}$$

on $\Gamma_3 \times (0, T)$,

$$\dot{\beta}(t) = -[\beta(t)(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2) - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{2.9}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0, T), \tag{2.10}$$

$$D \cdot \nu = 0 \quad \text{on } \Gamma_b \times (0, T), \tag{2.11}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.12}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.13}$$

We now provide some comments on equations and conditions (2.1)–(2.13). Equations (2.1) and (2.2) represent the electro-elastic constitutive law in which $\varepsilon(u)$ denotes the linearized strain tensor, $E(\varphi) = -\nabla\varphi$ is the electric field, where φ is the electric potential, $\mathcal{F} = (\mathcal{F}_{ijkl})$ is a $4th$ rank tensor, called the elastic tensor and its components $\mathcal{F}_{ijkl}$ are called coefficients of elasticity, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed, $\mathcal{B}$ denotes the electric permittivity operator, and $D = (D_1, \dots, D_d)$ is the electric displacement vector. Details on the constitutive equations of the form (2.1) and (2.2) can be find, for instance, in [1] and in [2]. Next, equations ((2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which Div and div denote the divergence operator for tensor and vector valued functions, respectively. Equations (2.5) and (2.6) represent the displacement and traction boundary conditions. Conditions (2.10) and (2.11) represent the electric boundary conditions. Condition (2.7) describes contact with normal compliance and adhesion where u_ν is the normal displacement, σ_ν represents the normal stress, γ_ν denotes a given adhesion coefficient and R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0, \end{cases} \tag{2.14}$$

where $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with the operator R_τ defined below, is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Thus, by choosing L very large, we can assume that $R_\nu(u_\nu) = u_\nu$.

Here p_ν is a nonnegative prescribed function, called normal compliance function. Indeed, when $u_\nu < 0$ there is no contact and the normal pressure vanishes. When there is contact, u_ν is positive and is a measure of the interpenetration of the asperities. A commonly used example of the normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+,$$

where $c_\nu > 0$ is the surface stiffness coefficient and $r_+ = max\{0, r\}$ denotes the positive part of r. We can also consider the following truncated normal compliance function:

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if } r \le \alpha, \\ c_\nu \alpha & \text{if } r > \alpha, \end{cases}$$

where α is a positive coefficient related to the wear and hardness of the surface. In this case, the above equality means that when the penetration exceeds α the obstacle offers no additional resistance to penetration. It follows from (2.7) that the contribution of the adhesion to the normal traction is represented by the term $\gamma_\nu \beta^2 R_\nu(u_\nu)$, but as long as u_ν does not exceed the bond length L.

Condition (2.8) is the associated Coulomb's law of dry friction, where u_τ and σ_τ denote tangential components of vector u and tensor σ, respectively. Her μ is the coefficient of friction and R_τ is the truncation operator given by

$$R_\tau(\upsilon) = \begin{cases} \upsilon & \text{if } \|\upsilon\| \le L, \\ L\frac{\upsilon}{\|\upsilon\|} & \text{if } \|\upsilon\| > L. \end{cases} \tag{2.15}$$

This condition shows that the contribution of the adhesion to the tangential shear on the contact surface is represented by the term $\gamma_\tau \beta^2 R_\tau(u_\tau)$, but again, only up to the bond length L.

The evolution of the bonding field is governed by the differential equation (2.9) with given positive parameters γ_ν, γ_τ and ϵ_a. For more details about conditions (2.7)–(2.9), we refer the reader to [24] and [25]. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot\beta \le 0$. Finally, (2.12) and (2.13) represent the initial conditions where β_0 and u_0 are given.

3. Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$u \cdot \upsilon = u_i \upsilon_i, \quad \|\upsilon\| = (\upsilon \cdot \upsilon)^{\frac{1}{2}} \quad \forall u, \upsilon \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Everywhere below, we use the classical notation for L^p and *Sobolev* spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces

$$L^2(\Omega)^d = \{\upsilon = (\upsilon_i) : \upsilon_i \in L^2(\Omega)\}, \quad H^1(\Omega)^d = \{\upsilon = (\upsilon_i) : \upsilon_i \in H^1(\Omega)\},$$
$$\mathcal{H} = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\tau \in \mathcal{H} : \tau_{ij,j} \in L^2(\Omega)\}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products

$$(u,\upsilon)_{L^2(\Omega)^d} = \int_\Omega u\cdot \upsilon\,dx, \quad (u,\upsilon)_{H^1(\Omega)^d} = \int_\Omega u\cdot\upsilon\,dx + \int_\Omega \nabla u\cdot\nabla\upsilon\,dx,$$
$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma\cdot\tau\,dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = \int_\Omega \sigma\cdot\tau\,dx + \int_\Omega \mathrm{Div}\,\sigma\cdot\mathrm{Div}\,\tau\,dx,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla\upsilon = (\upsilon_{i,j}), \quad \varepsilon(\upsilon) = (\varepsilon_{ij}(\upsilon)), \quad \varepsilon_{ij}(\upsilon) = \frac{1}{2}(\upsilon_{i,j} + \upsilon_{j,i}) \quad \forall\upsilon \in H^1(\Omega)^d,$$
$$\mathrm{Div}\,\tau = (\tau_{ij,j}) \quad \forall\tau\in\mathcal{H}_1.$$

For every element $\upsilon \in H^1(\Omega)^d$. We also write υ for the trace of υ on Γ and we denote by υ_ν and υ_τ the normal and tangential components of υ on Γ given by $\upsilon_\nu = \upsilon\cdot\nu$, $\upsilon_\tau = \upsilon - \upsilon_\nu\nu$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\upsilon \in H^1(\Omega)^d \mid \upsilon = 0 \text{ on } \Gamma_1\}.$$

Since $\mathrm{meas}(\Gamma_1) > 0$, the following *Korn's* inequality holds

$$\|\varepsilon(\upsilon)\|_{\mathcal{H}} \ge c_K\,\|\upsilon\|_{H^1(\Omega)^d} \quad \forall\upsilon\in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(u,\upsilon)_V = (\varepsilon(u),\varepsilon(\upsilon))_{\mathcal{H}}, \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\upsilon\|_{L^2(\Gamma_3)^d} \le c_0\|\upsilon\|_V \quad \forall\upsilon\in V. \tag{3.3}$$

We also introduce the spaces

$$W = \{\psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\},$$
$$\mathcal{W}_1 = \{D = (D_i) \mid D_i \in L^2(\Omega),\ D_{i,i} \in L^2(\Omega)\}.$$

Since $\mathrm{meas}(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds

$$\|\nabla\psi\|_{L^2(\Omega)^d} \ge c_F\,\|\psi\|_{H^1(\Omega)} \quad \forall\psi\in W, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\psi = (\psi_{,i})$. Over the space W, we consider the inner product given by

$$(\varphi, \psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx,$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (3.4) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W, \|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\tilde{c}_0$, depending only on Ω, Γ_a and Γ_3, such that

$$\|\psi\|_{L^2(\Gamma_3)} \le \tilde{c}_0 \|\psi\|_W \quad \forall \psi \in W. \tag{3.5}$$

The space $\mathcal{W}_1$ is a real Hilbert space with the inner product

$$(D, E)_{\mathcal{W}_1} = \int_\Omega D \cdot E \, dx + \int_\Omega \operatorname{div} D \cdot \operatorname{div} E \, dx,$$

and the associated norm $\|\cdot\|_{\mathcal{W}_1}$.

Finally, for every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \le p \le \infty$, $k \ge 1$ and we also introduce the set

$$\mathcal{Q} = \{\beta \in L^\infty(0,T;L^2(\Gamma_3)) : 0 \le \beta(t) \le 1 \ \forall t \in [0,T], \text{ a.e. on } \Gamma_3\}.$$

In the study of problem $\mathcal{P}$, we consider the following assumptions on the problem data.

The elasticity operator $\mathcal{F}$, the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ satisfy the following conditions:

$$\begin{aligned}
&\text{(a) } \mathcal{F} = (\mathcal{F}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d,\\
&\text{(b) } \mathcal{F}_{ijkl} = \mathcal{F}_{klij} = \mathcal{F}_{jikl} \in L^\infty(\Omega),\\
&\text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that } \mathcal{F}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \ge m_{\mathcal{F}}\|\varepsilon\|^2 \text{ for all}\\
&\varepsilon \in \mathbb{S}^d, \text{ a.e. in } \Omega.
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d,\\
&\text{(b) } \mathcal{E}(x,\tau) = (e_{ijk}(x)\tau_{jk}) \text{ for all } \tau = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega,\\
&\text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega).
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\text{(a) } \mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d,\\
&\text{(b) } \mathcal{B}(x,E) = (b_{ij}(x)E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega,\\
&\text{(c) } b_{ij} = b_{ji} \in L^\infty(\Omega),\\
&\text{(d) There exists } m_{\mathcal{B}} > 0 \text{ such that } b_{ij}(x)E_iE_j \ge m_{\mathcal{B}}\|E\|^2 \text{ for all}\\
&E = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega.
\end{aligned} \tag{3.8}$$

From assumptions (3.7) and (3.8), we deduce that the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ are linear, have measurable bounded components denoted e_{ijk} and b_{ij}, respectively, and moreover, $\mathcal{B}$ is symmetric and positive definite.

Recall also that the transposed operator $\mathcal{E}^*$ is given by $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$, and

$$\mathcal{E}\sigma \cdot \upsilon = \sigma \cdot \mathcal{E}^*\upsilon \quad \forall \sigma \in \mathbb{S}^d, \ \upsilon \in \mathbb{R}^d. \tag{3.9}$$

The normal compliance function satisfies

(a) $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$,
(b) there exists $L_\nu > 0$ such that $\|p_\nu(x, r_1) - p_\nu(x, r_2)\| \le L_\nu |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$, a.e. $x \in \Gamma_3$. (3.10)
(c) $x \mapsto p_\nu(x, r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$.
(d) $x \mapsto p_\nu(x, r) = 0$ for all $r \le 0$ a.e. $x \in \Gamma_3$.

We also suppose that the body forces and surface tractions have the regularity

$$f_0 \in W^{1,\infty}(0, T; L^2(\Omega)^d), \quad f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d), \tag{3.11}$$

and the densities of electric charges satisfy

$$q_0 \in W^{1,\infty}(0, T; L^2(\Omega)), \quad q_2 \in W^{1,\infty}(0, T; L^2(\Gamma_b)). \tag{3.12}$$

Finally, we assume that

$$q_2(t) = 0 \quad \text{on } \Gamma_3 \ \forall t \in [0, T]. \tag{3.13}$$

Note that we need to impose assumption (3.13) for physical reasons; indeed, the foundation is supposed to be insulator and therefore the electric boundary conditions on Γ_3 do not have to change in function of the status of the contact, are the same on the contact and on the separation zone, and are included in the boundary condition (2.11).

The Riesz representation theorem implies the existence of two functions $f : [0, T] \to V$ and $q : [0, T] \to W$ such that

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da, \tag{3.14}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_b} q_2(t) \psi \, da, \tag{3.15}$$

for all $v \in V$, $\psi \in W$ and $t \in [0, T]$. We note that conditions (3.11) and (3.12) imply

$$f \in W^{1,\infty}(0, T; V), \quad q \in W^{1,\infty}(0, T; W). \tag{3.16}$$

The adhesion coefficients γ_ν, γ_τ and the limit bound ϵ_a satisfy the conditions

$$\gamma_\nu, \ \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_a \ge 0 \quad \text{a.e. on } \Gamma_3, \tag{3.17}$$

and the friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{3.18}$$

The initial condition β_0 satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1 \quad \text{a.e. on } \Gamma_3. \tag{3.19}$$

Next, we define the adhesion functional $j_{ad} : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} (-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau) \, da, \tag{3.20}$$

the normal compliance functional $V \times V \to \mathbb{R}$ by

$$j_{nc}(u, v) = \int_{\Gamma 3} p_\nu(u_\nu(t)) v_\nu \, da, \tag{3.21}$$

and the friction functional $V \times V \to \mathbb{R}$ by

$$j_{fr}(u, v) = \int_{\Gamma 3} \mu p_\nu(u_\nu) \|v_\tau\| \, da. \tag{3.22}$$

We consider the following assumptions on the conditions initials

$$u_0 \in V, \tag{3.23}$$

$$\begin{aligned}&(\mathcal{F}\varepsilon(u_0),\varepsilon(\upsilon))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_0,\varepsilon(\upsilon))_{\mathcal{H}} + j_{ad}(\beta_0,u_0,\upsilon) + j_{nc}(u_0,\upsilon) + j_{fr}(u_0,\upsilon)\\ &\geq (f(0),\upsilon)_V \quad \forall \upsilon \in V,\end{aligned} \tag{3.24}$$

$$(\mathcal{B}\nabla\varphi_0,\nabla\psi)_{L^2(\Omega)^d} = (\mathcal{E}\varepsilon(u_0),\nabla\psi)_{L^2(\Omega)^d} + (q(0),\psi)_W \quad \forall \psi \in W. \tag{3.25}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)–(2.13).

Problem ($\mathcal{P}^V$). Find a displacement field $u : [0,T] \to V$, an electric potential field $\varphi : [0,T] \to W$ and a bonding field $\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\begin{aligned}&(\mathcal{F}\varepsilon(u(t)),\varepsilon(\upsilon)-\varepsilon(\dot{u}(t)))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t),\varepsilon(\upsilon)-\varepsilon(\dot{u}(t)))_{\mathcal{H}}\\ &+ j_{ad}(\beta,u(t),\upsilon-\dot{u}(t)) + j_{nc}(u(t),\upsilon-\dot{u}(t))\\ &+ j_{fr}(u(t),\upsilon) - j_{fr}(u(t),\dot{u}(t))\\ &\geq (f(t),\upsilon-\dot{u}(t))_V \quad \forall \upsilon \in V \text{ a.e. } t \in [0,T],\end{aligned} \tag{3.26}$$

$$\begin{aligned}&(\mathcal{B}\nabla\varphi(t),\nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u(t)),\nabla\psi)_{L^2(\Omega)^d}\\ &= (q(t),\psi)_W \quad \forall \psi \in W \text{ a.e. } t \in [0,T],\end{aligned} \tag{3.27}$$

$$\dot{\beta}(t) = -[\beta(t)\big(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau\|R_\tau(u_\tau(t))\|^2\big) - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{3.28}$$

$$u(0) = u_0, \quad \beta(0) = \beta_0. \tag{3.29}$$

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, j_{nc} and j_{fr} which will be used in the following sections. Below in this section β_1 and β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta_1, \beta_2 \leq 1$ a.e. on Γ_3, u_1, $u_2, \upsilon_1, \upsilon_2$, u and υ represent elements of V and c is a generic positive constants which may depend on Ω, Γ_1 , Γ_3, p_ν, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $x \in \Omega \cup \Gamma_3$. Using (3.3), (3.10), (3.20) , (3.21) and the inequalities $|R_\nu(u_\nu)| \leq L$, $\|R_\tau(u_\tau)\| \leq L$, $|\beta_1| \leq 1$, $|\beta_2| \leq 1$, we obtain

$$\begin{aligned}&|j_{ad}(\beta_1,u_1,\omega) - j_{ad}(\beta_2,u_2,\omega) + j_{nc}(u_1,\omega) - j_{nc}(u_2,\omega)|\\ &\leq c(\|\beta_1-\beta_2\|_{L^2(\Gamma_3)} + \|u_1-u_2\|_V)\|\omega\|_V.\end{aligned} \tag{3.30}$$

Next, we use (3.22), (3.10) and (3.3) to obtain

$$j_{fr}(u,\upsilon-u) - j_{fr}(\upsilon,\upsilon-u) \leq c_0^2\|\mu\|_{L^\infty(\Gamma_3)}L_\nu\|u-\upsilon\|_V^2 \quad \forall u,\upsilon \in V. \tag{3.31}$$

$$\begin{aligned}&j_{fr}(u_1,\upsilon_1) - j_{fr}(u_1,\upsilon_2) + j_{fr}(u_2,\upsilon_2) - j_{fr}(u_2,\upsilon_1)\\ &\leq c_0^2L_\nu\|\mu\|_{L^\infty(\Gamma_3)}\|u_1-u_2\|_V\|\upsilon_1-\upsilon_2\|_V.\end{aligned} \tag{3.32}$$

Inequalities (3.30)–(3.32) will be used in various places in the rest of the paper.

4. Existence result

Our main result which states the solvability of Problem ($\mathcal{P}^V$), is the following.

Theorem 4.1. *Assume that* (3.6)–(3.8)*,* (3.10)–(3.13)*,* (3.17)–(3.19) *and* (3.23)–(3.25) *hold. Then there exists* $\mu_0 > 0$ *depending only on* Ω, Γ_1, Γ_3, Γ_a, $\mathcal{F}$, $\mathcal{B}$ *and* $\mathcal{E}$

such that, if $(L_\nu + L_\nu\|\mu\|_{L^\infty(\Gamma_3)} + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\nu\|_{L^\infty(\Gamma_3)}) < \mu_0$*, Problem* $(\mathcal{P}^V)$ *has at least one solution* (u, φ, β)*. Moreover, the solution satisfies*

$$u \in W^{1,\infty}(0,T;V), \tag{4.1}$$

$$\varphi \in W^{1,\infty}(0,T;W), \tag{4.2}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{4.3}$$

A "quintuplete" of functions (u, σ, φ, D, β) which satisfies (2.1), (2.2), (3.26)–(3.29) is called a *weak solution* of the contact problem $(\mathcal{P})$. To precise the regularity of the weak solution we note that the constitutive relations (2.1)-(2.2), the assumptions (3.6)–(3.8) and the regularities (4.1), (4.2) show that $\sigma \in W^{1,\infty}([0,T];\mathcal{H})$, $D \in W^{1,\infty}([0,T];L^2(\Omega)^d)$. By putting $v = \dot{u}(t) \pm \xi$, where $\xi \in C_0^\infty(\Omega)^d$ in (3.26) and $\psi \in C_0^\infty(\Omega)$ in (3.27) we obtain

$$\operatorname{Div}\sigma(t) + f_0(t) = 0, \quad \operatorname{div} D(t) = q_0(t), \quad \forall t \in [0,T].$$

It follows now from the regularities (3.11), (3.12) that $\operatorname{Div}\sigma \in W^{1,\infty}(0,T;L^2(\Omega)^d)$ and $\operatorname{div} D \in W^{1,\infty}(0,T;L^2(\Omega))$, which shows that

$$\sigma \in W^{1,\infty}(0,T;\mathcal{H}_1), \tag{4.4}$$

$$D \in W^{1,\infty}(0,T;\mathcal{W}_1). \tag{4.5}$$

We conclude that the weak solution $(u, \sigma, \varphi, D, \beta)$ of the piezoelectric contact problem $(\mathcal{P})$ has the regularity implied in (4.1), (4.2), (4.3), (4.4) and (4.5).

The proof of Theorem 4.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$.

Let $a : X \times X \to \mathbb{R}$ be a bilinear form on X, $j : X \times X \to \mathbb{R}$, $f : [0,T] \to X$ and $u_0 \in X$. With these data, we consider the following quasivariational problem: find $u : [0,T] \to X$ such that

$$\begin{aligned} &a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\geq (f(t), v - \dot{u}(t))_X \quad \forall v \in X, \text{ a.e. } t \in (0,T), \end{aligned} \tag{4.6}$$

$$u(0) = u_0. \tag{4.7}$$

To solve problem (4.6)–(4.7), we consider the following assumptions:

$a : X \times X \to \mathbb{R}$ is a bilinear symmetric form, and
(a) there exists $M > 0$ such that $|a(u,v)| \leq M\|u\|_X\|v\|_X$ for all $u, v \in X$,
(b) there exists $m > 0$ such that $a(v,v) \geq m\|v\|_X^2$ for all $v \in X$. (4.8)

For every $\zeta \in X$, $j(\zeta,.) : X \to \mathbb{R}$ is a positively homogeneous subadditive functional, i.e.
(a) $j(\zeta, \lambda u) = \lambda j(\zeta, u)$ for all $u \in X$, $\lambda \in \mathbb{R}_+$,
(b) $j(\zeta, u + v) \leq j(\zeta, u) + j(\zeta, v)$ for all $u, v \in X$, (4.9)

$$f \in W^{1,\infty}(0,T;X), \tag{4.10}$$

$$u_0 \in X. \tag{4.11}$$

$$a(u_0, v) + j(u_0, v) \geq (f(0), v)_X \quad \forall v \in X. \tag{4.12}$$

Keeping in mind (4.9), it results that for all $\zeta \in X$, $j(\zeta,.) : X \to \mathbb{R}$ is a convex functional. Therefore, there exists the directional derivative j_2' given by

$$j_2'(\zeta, u; v) = \lim_{\lambda \searrow 0} \frac{1}{\lambda}[j(\zeta, u + \lambda v) - j(\zeta, u)\ ] \quad \forall \zeta, u, v \in X. \tag{4.13}$$

We consider now the following additional assumptions on the functional j.

For every sequence $(u_n) \subset X$ with $\|u_n\|_X \to$, every sequence $(t_n) \subset [0,1]$ and each $\bar{u} \in X$ one has

$$\liminf_{n\to+\infty}[\frac{1}{\|u_n\|_X^2} j_2'(t_n u_n, u_n - \bar{u}; -u_n)] < m. \tag{4.14}$$

For every sequence $(u_n) \subset X$ with $\|u_n\|_X \to \infty$, every bounded sequence $(\zeta_n) \subset X$ and for each $\bar{u} \in X$, one has

$$\liminf_{n\to+\infty}[\frac{1}{\|u_n\|_X^2} j_2'(\zeta_n, u_n - \bar{u}; -u_n)] < m. \tag{4.15}$$

For all sequences $(u_n) \subset X$ and $(\zeta_n) \subset X$ such that $u_n \rightharpoonup u \in X$, $\zeta_n \rightharpoonup \zeta \in X$ and for every $v \in X$, we have

$$\limsup_{n\to+\infty}[j(\zeta_n, v) - j(\zeta_n, u_n)] \le j(\zeta, v) - j(\zeta, u). \tag{4.16}$$

There exists $k_0 \in (0, m)$ such that

$$j(u, v - u) - j(v, v - u) \le k_0 \|u - v\|_X^2 \quad \forall u, v \in X. \tag{4.17}$$

There exist two functions $a_1 : X \to \mathbb{R}$ and $a_2 : X \to \mathbb{R}$, which map bounded sets in X into bounded sets in $\mathbb{R}$ such that

$$|j(\zeta, u)| \le a_1(\zeta)\|u\|_X^2 + a_2(\zeta) \quad \forall \zeta, u \in X, \text{ and } a_1(0_X) < m - k_0. \tag{4.18}$$

For every sequence $(\zeta_n) \subset X$ with $\zeta_n \rightharpoonup \zeta \in X$ and every bounded sequence $(u_n) \subset X$ one has

$$\lim_{n\to+\infty}[j(\zeta_n, u_n) - j(\zeta, u_n)] = 0. \tag{4.19}$$

For every $s \in (0, T]$ and every pair of functions $u, v \in W^{1,\infty}(0, T; X)$, with $u(0) = v(0)$, $u(s) \ne v(s)$,

$$\begin{aligned} &\int_0^s [j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t))]dt \\ &< \frac{m}{2}\|u(s) - v(s)\|_X^2. \end{aligned} \tag{4.20}$$

There exists $\alpha \in (0, \frac{m}{2})$ such that for every $s \in (0, T]$ and for every functions $u, v \in W^{1,\infty}(0, T; X)$ with $u(s) \ne v(s)$, it holds that

$$\begin{aligned} &\int_0^s [j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t))]dt \\ &< \alpha\|u(s) - v(s)\|_X^2. \end{aligned} \tag{4.21}$$

For the study of the evolutionary problem (4.6)–(4.7), we recall the following result.

Theorem 4.2. *Assume* (4.8)–(4.12) *hold.*

(i) If assumptions (4.14)–(4.19) *are satisfied, then there exists at least a solution* $u \in W^{1,\infty}(0, T; X)$ *to problem* (4.6)–(4.7).

(ii) If assumptions (4.14)–(4.20) *are satisfied. then there exists a unique solution* $u \in W^{1,\infty}(0, T; X)$ *to problem* (4.6)–(4.7).

(iii) If assumptions (4.14)–(4.19) and (4.21) are satisfied, then there exists a unique solution $u \in W^{1,\infty}(0,T;X)$ to (4.6)–(4.7), and the mapping $(f,u_0) \to \mathbb{R}$ is Lipschitz continuous from $W^{1,\infty}(0,T;X) \times X$ to $L^\infty(0,T;X)$.

The proof can be find in [16], it is obtained in several steps and it is based on arguments of elliptic quasivariational inequalities and a time discretization method.

We return now to proof of theorem 4.1. To this end, we assume in the following that (3.6)–(3.8), (3.10)–(3.13), (3.17)–(3.19) and (3.23)–(3.25) hold. Below, c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, $\mathcal{F}$, p_ν, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $x \in \Omega \cup \Gamma_3$.

Using the Riesz's representation theorem, we define the operators $\mathcal{G} : W \to W$ and $\mathcal{R} : V \to W$ respectively by

$$(\mathcal{G}\varphi(t), \psi)_W = (\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} \quad \forall \varphi, \psi \in W, \tag{4.22}$$

$$(\mathcal{R}v, \varphi)_W = (\mathcal{E}\varepsilon(v), \nabla\varphi)_{L^2(\Omega)^d} \quad \forall \varphi \in W, v \in V. \tag{4.23}$$

We can show that $\mathcal{G}$ is a linear continuous symmetric positive definite operator. Therefore, $\mathcal{G}$ is an invertible operator on W. We can also prove that $\mathcal{R}$ is a linear continuous operator on V. Let $\mathcal{R}^*$ the adjoint of $\mathcal{R}$. Thus, from (3.9) we can write

$$(\mathcal{R}^*\varphi, v)_V = (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_{\mathcal{H}} \quad \forall \varphi \in W, \ v \in V. \tag{4.24}$$

By introducing (4.22)–(4.23) in (3.27) we obtain

$$(\mathcal{G}\varphi(t), \psi)_W = (\mathcal{R}u(t), \psi)_W + (q(t), \psi)_W \quad \forall \psi \in W,$$

and consequently

$$\mathcal{G}\varphi(t) = \mathcal{R}u(t) + q(t).$$

On the other hand, $\mathcal{G}$ is invertible where the previous equality gives us

$$\varphi(t) = \mathcal{G}^{-1}\mathcal{R}u(t) + \mathcal{G}^{-1}q(t). \tag{4.25}$$

Using (4.24)–(4.25) and (3.26) we obtain

$$\begin{aligned} &(\mathcal{F}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathcal{H}} + (\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}u(t), v - \dot{u}(t))_V \\ &+ j_{ad}(\beta, u(t), v - \dot{u}(t)) + j_{nc}(u(t), v - \dot{u}(t)) + j_{fr}(u(t), v) - j_{fr}(u(t), \dot{u}(t)) \\ &\geq (f(t) - \mathcal{R}^*\mathcal{G}^{-1}q(t), v - \dot{u}(t))_V \quad \forall v \in V, \text{ a.e. } t \in (0,T). \end{aligned} \tag{4.26}$$

Let now the operator $L : V \to V$ defined by

$$L(v) = \mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}(v), \ \forall v \in V. \tag{4.27}$$

Using the properties of the operators $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{R}^*$, we deduce that L is a linear symmetric positive operator on V. Indeed, we have

$$\begin{aligned} (Lu, v)_V &= (\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}u, v)_V \\ &= (\mathcal{G}^{-1}\mathcal{R}u, \mathcal{R}v)_W \\ &= (\mathcal{R}u, \mathcal{G}^{-1}\mathcal{R}v)_W \\ &= (u, \mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}v)_V \\ &= (u, Lv)_V \ \forall u, v \in V \end{aligned}$$

$$\begin{gathered} (Lv, v)_V = (\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}v, v)_V, \\ (Lv, v)_V = (\mathcal{G}^{-1}\mathcal{R}v, \mathcal{R}v)_W \geq 0 \ \ \forall v \in V. \end{gathered} \tag{4.28}$$

Now, let the bilinear form $a : V \times V \to \mathbb{R}$ be such that

$$a(u,v) = (\mathcal{F}\varepsilon(u(t)), \varepsilon(v))_{\mathcal{H}} + (Lu, v)_V \quad \forall u, v \in V. \tag{4.29}$$

The bilinear form a is continuous and coercive on V. Indeed, we have

$$|a(u,v)| \le (M + \|L\|)\|u\|_V \|v\|_V \quad \forall u, v \in V, \tag{4.30}$$

$$a(v,v) \ge m\|v\|_V^2 \quad \forall v \in V, \tag{4.31}$$

and the symmetry of $\mathcal{F}$ and L leads to the symmetry of a.

Let now the function $\mathbf{f} : [0\ T] \to V$ be defined by

$$\mathbf{f}(t) = f(t) - \mathcal{R}^* \mathcal{G}^{-1} q(t) \quad \forall t \in [0, T]. \tag{4.32}$$

From (3.16) we obtain

$$\mathbf{f} \in W^{1,\infty}(0, T, V). \tag{4.33}$$

The relations (4.26), (4.29), (4.32), (3.28) and (3.29) lead us to consider the following variational problem, in the terms of displacement and bonding fields.

Problem $\mathcal{P}_1^V$. Find a displacement field $u : [0, T] \to V$, and a bonding field $\beta : [0, T] \to L^2(\Gamma_3)$ such that

$$\begin{aligned} &a(u(t), v - \dot{u}(t)) + j_{ad}(\beta, u(t), v - \dot{u}(t)) + j_{nc}(u(t), v - \dot{u}(t)) \\ &+ j_{fr}(u(t), v) - j_{fr}(u(t), \dot{u}(t)) \\ &\ge (\mathbf{f}(t), v - \dot{u}(t))_V \quad \forall v \in V, \text{ a.e. t} \in (0, T), \end{aligned} \tag{4.34}$$

$$\dot{\beta}(t) = -[\beta(t)(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2) - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{4.35}$$

$$u(0) = u_0, \quad \beta(0) = \beta_0. \tag{4.36}$$

Theorem 4.3. *Assume that* (3.6)–(3.8), (3.10)–(3.13), (3.17)–(3.19) *and* (3.23)–(3.25) *hold. Then, there exists $\mu_0 > 0$ depending only on $\Omega, \Gamma_1, \Gamma_3, \Gamma_a$, $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{E}$ such that, if*

$$L_\nu + L_\nu \|\mu\|_{L^\infty(\Gamma_3)} + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}) < \mu_0,$$

then Problem $\mathcal{P}_1^V$ has at least one solution (u, β). Moreover, the solution satisfies

$$u \in W^{1,\infty}(0, T; V), \tag{4.37}$$

$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{4.38}$$

We assume in the following that the conditions of Theorem 4.3 hold. Let $\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Q}$ be given and $j_\beta : V \times V \to \mathbb{R}$ defined by

$$\begin{aligned} j_\beta(u, v) = &\int_{\Gamma 3} p_\nu(u_\nu(t)) v_\nu da + \int_{\Gamma 3} \mu p_\nu(u_\nu) \|v_\tau\| \, da \\ &+ \int_{\Gamma_3} \Big(- \gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau \Big) \, da, \end{aligned} \tag{4.39}$$

Now, we consider the following intermediate problem, in the term of displacement field.

Problem $\mathcal{P}_2^V$. Find the displacement field $u_\beta : [0,T] \to V$ such that

$$\begin{aligned} &a(u_\beta(t), \upsilon - \dot{u_\beta}(t)) + j_\beta(u_\beta(t), \upsilon) - j_\beta(u_\beta(t), \dot{u_\beta}(t)) \\ &\geq (\mathbf{f}(t), \upsilon - \dot{u_\beta}(t))_V \quad \forall \upsilon \in V, \text{ a.e. } t \in (0,T), \end{aligned} \tag{4.40}$$

$$u_\beta(0) = u_0, \tag{4.41}$$

Remark 4.4. From (3.24) and (3.25), we can deduce (4.12).

Theorem 4.5. *Assume that* (3.6)–(3.8)*,* (3.10)–(3.13)*,* (3.17)–(3.19) *and* (3.23)–(3.25) *hold. Then there exists* $\mu_0 > 0$ *depending only on* $\Omega, \Gamma_1, \Gamma_3, \Gamma_a$*,* $\mathcal{F}$*,* $\mathcal{B}$ *and* $\mathcal{E}$ *such that, if*

$$L_\nu + L_\nu \|\mu\|_{L^\infty(\Gamma_3)} + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} < \mu_0,$$

then Problem $\mathcal{P}_2^V$ *has at least one solution* $u_\beta \in W^{1,\infty}(0,T,V)$*.*

We will use the results given by the Theorem 4.2 to give a result of existence of solutions of problem $\mathcal{P}_2^V$. We remark that the functional j_β, given by (4.39), satisfies condition (4.9). In addition, we have the following results.

Lemma 4.6. *The functional* j_β *satisfies the assumptions* (4.14) *and* (4.15)*.*

Proof. Let $\zeta, u, \bar{u} \in V$ and let $\lambda \in]0,1]$. Using (3.22), it follows that j_β satisfies

$$\begin{aligned} &j_\beta(\zeta, u - \overline{u} - \lambda u) - j_\beta(\zeta, u - \overline{u}) \\ &\leq -\lambda \int_{\Gamma 3} p_\nu(\zeta_\nu) u_\nu \, da - \lambda \int_{\Gamma 3} \mu p_\nu(\zeta_\nu) \|u_\tau - \overline{u}_\tau\| \, da + \lambda \int_{\Gamma 3} \mu p_\nu(\zeta_\nu) \|\overline{u}_\tau\| \, da \\ &\quad + \lambda \int_{\Gamma 3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) u_\nu \, da - \lambda \int_{\Gamma 3} \gamma_\tau \beta^2 R_\tau(\zeta_\tau) \cdot u_\tau \, da, \end{aligned}$$

and as $\mu \geq 0$, $p_\nu \geq 0$ a.e. on Γ_3, we obtain

$$\begin{aligned} &j_\beta(\zeta, u - \overline{u} - \lambda u) - j_\beta(\zeta, u - \overline{u}) \\ &\leq -\lambda \int_{\Gamma 3} p_\nu(\zeta_\nu) u_\nu \, da + \lambda \int_{\Gamma 3} \mu p_\nu(\zeta_\nu) \|\overline{u}_\tau\| \, da + \lambda \int_{\Gamma 3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) u_\nu \, da \\ &\quad - \lambda \int_{\Gamma 3} \gamma_\tau \beta^2 R_\tau(\zeta_\tau) \cdot u_\tau \, da, \quad \forall \zeta, u, \bar{u} \in V. \end{aligned}$$

Moreover, we deduce from (4.13) that

$$\begin{aligned} &j_2'(\zeta, u - \overline{u}; -u) \\ &\leq - \int_{\Gamma 3} p_\nu(\zeta_\nu) u_\nu \, da + \int_{\Gamma 3} \mu p_\nu(\zeta_\nu) \|\overline{u}_\tau\| \, da \\ &\quad + \int_{\Gamma 3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) u_\nu \, da - \int_{\Gamma 3} \gamma_\tau \beta^2 R_\tau(\zeta_\tau) \cdot u_\tau \, da \quad \forall \zeta, u, \bar{u} \in V. \end{aligned} \tag{4.42}$$

Now consider the sequences $(u_n)_{n \in \mathbb{N}} \subset V$, $(t_n)_{n \in \mathbb{N}} \subset [0\ 1]$ and the element $\overline{u} \in V$. Using (3.3), (3.10), (3.18) and (4.42), we find

$$\begin{aligned} &j_2'(t_n u_n, u_n - \overline{u}; -u_n) \\ &\leq - \int_{\Gamma 3} p_\nu(t_n u_{n\nu}) u_{n\nu} + \int_{\Gamma 3} \mu p_\nu(t_n u_{n\nu}) \|\overline{u}_\tau\| \, da \\ &\quad + \int_{\Gamma 3} \gamma_\nu \beta^2 R_\nu(t_n u_{n\nu}) u_{n\nu} \, da - \int_{\Gamma 3} \gamma_\tau \beta^2 R_\tau(t_n u_{n\tau}) \cdot u_{n\tau} \, da \quad \forall \zeta, u, \bar{u} \in V \end{aligned} \tag{4.43}$$

Keeping in mind that $0 \leq \beta \leq 1$ a.e. on Γ_3 and using (3.10), (2.15) and (3.17) we obtain $p_\nu(t_n u_{n\nu})u_{n\nu} \geq 0$ and $\gamma_\tau \beta^2 R_\tau(t_n u_{n\tau}) \cdot u_\tau \geq 0$ p.p. on Γ_3. So (4.43) implies

$$j_2'(t_n u_n, u_n - \overline{u}; -u_n) \leq \int_{\Gamma 3} \mu p_\nu(t_n u_{n\nu}) \|\overline{u}_\tau\| \, da + \int_{\Gamma 3} \gamma_\nu \beta^2 R_\nu(t_n u_{n\nu}) u_{n\nu} \, da.$$

Now, using (3.10)(b), (3.3) and the fact that $|R_\nu(t_n u_{n\nu})| \leq L$ we obtain

$$\begin{aligned} &j_2'(t_n u_n, u_n - \overline{u}; -u_n) \\ &\leq \|\mu\|_{L^\infty(\Gamma_3)} L_\nu \int_{\Gamma 3} |u_{n\nu}| \|\overline{u}_\tau\| \, da + L \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \int_{\Gamma 3} |u_{n\nu}| \, da \\ &\leq c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} L_\nu \|u_n\|_V \|\overline{u}\|_V + c_0 L \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \operatorname{meas}(\Gamma_3) \|u_n\|_V. \end{aligned}$$

It follows from the previous inequality that if $\|u_n\|_V \to +\infty$, then

$$\liminf_{n \to +\infty} \Big[\frac{1}{\|u_n\|_V^2} j_2'(t_n u_n, u_n - \overline{u}; -u_n) \Big] \leq 0,$$

and we conclude that j_β satisfies assumption (4.14).

Now consider the sequences $(u_n)_{n\in\mathbb{N}} \subset V$, $(\zeta_n)_{n\in\mathbb{N}} \subset V$ such that

$$\|u_n\|_V \to +\infty, \tag{4.44}$$

$$\|\zeta_n\|_V \leq C \quad \forall n \in \mathbb{N}, \tag{4.45}$$

where $C > 0$. Let $\overline{u} \in V$. Using (3.3), (3.10), (3.18), (4.42) and (4.45) we obtain

$$\begin{aligned} j_2'(\zeta_n, u_n - \overline{u}; -u_n) \leq\ & c_0^2 L_\nu \|\zeta_n\|_V \|u_n\|_V + c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} L_\nu \|\zeta_n\|_V \|\overline{u}\|_V \\ & + c_0 L \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \operatorname{meas}(\Gamma_3) \|u_n\|_V \\ & + c_0 L \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \operatorname{meas}(\Gamma_3) \|u_n\|_V \quad \forall n \in \mathbb{N}. \end{aligned} \tag{4.46}$$

From (4.44) and (4.46), we conclude that

$$\liminf_{n \to +\infty} \big[\frac{1}{\|u_n\|_V^2} j_2'(\zeta_n, u_n - \overline{u}; -u_n) \big] \leq 0\,.$$

Thus, we deduce that j_β satisfies (4.15). □

Lemma 4.7. *The functional j_β satisfies the conditions* (4.16) *and* (4.19).

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset V$, $(\zeta_n)_{n\in\mathbb{N}} \subset V$ be two sequences such that $u_n \rightharpoonup u \in V$ and $\zeta_n \rightharpoonup \zeta \in V$. Using the compactness property of the trace map and (3.10), it follows that

$$p_\nu(\zeta_{n\nu}) \to p_\nu(\zeta_\nu) \quad \text{in } L^2(\Gamma_3), \tag{4.47}$$

$$u_n \to u \quad \text{in } L^2(\Gamma_3)^d. \tag{4.48}$$

$$\begin{aligned} R_\nu(\zeta_{n\nu}) \to R_\nu(\zeta_\nu) \quad \text{in } L^2(\Gamma_3). \\ R_\tau(\zeta_{n\tau}) \to R_\tau(\zeta_\tau) \quad \text{in } L^2(\Gamma_3)^d \end{aligned} \tag{4.49}$$

Therefore, we deduce from (4.47), (4.48) and (4.49) that

$$\begin{gathered} j_\beta(\zeta_n, \upsilon) \to j_\beta(\zeta, \upsilon) \quad \forall \upsilon \in V, \\ j_\beta(\zeta_n, u_n) \to j_\beta(\zeta, u), \end{gathered}$$

which show that the functional j_β satisfies

$$\limsup_{n \to +\infty} [j_\beta(\zeta_n, \upsilon) - j_\beta(\zeta_n, u_n)] \leq j_\beta(\zeta, \upsilon) - j_\beta(\zeta, u).$$

Thus, we deduce that j_β satisfies (4.16).

Now we consider $(u_n)_{n\in\mathbb{N}}$ a bounded sequence of V, i.e.

$$\|u_n\|_V \leq C \quad \forall n \in \mathbb{N}, \tag{4.50}$$

where $C > 0$. We have

$$\begin{aligned} |j_\beta(\zeta_n, u_n) - j_\beta(\zeta, u_n)| &\leq \int_{\Gamma 3} |p_\nu(\zeta_{n\nu}) - p_\nu(\zeta_\nu)||u_{n\nu}|\, da \\ &\quad + \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma 3} |p_\nu(\zeta_{n\nu}) - \mu p_\nu(\zeta_\nu)|\|u_{n\tau}\|\, da \\ &\quad + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |R_\nu(\zeta_{n\nu}) - R_\nu(\zeta_\nu))||u_{n\nu}|\, da \\ &\quad + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} \|R_\tau(u_{n\tau}) - R_\tau(u_{n\tau})\|\|u_{n\tau}\|\, da, \end{aligned}$$

using (3.3), we obtain

$$\begin{aligned} &|j_\beta(\zeta_n, u_n) - j_\beta(\zeta, u_n)| \\ &\leq c_0(\|p_\nu(\zeta_{n\nu}) - p_\nu(\zeta_\nu)\|_{L^2(\Gamma_3)} + \|\mu\|_{L^\infty(\Gamma_3)}\|p_\nu(\zeta_{n\nu}) - p_\nu(\zeta_\nu)\|_{L^2(\Gamma_3)} \\ &\quad + \|\gamma_\nu\|_{L^\infty(\Gamma_3)}\|R_\nu(\zeta_{n\nu}) - R_\nu(\zeta_\nu))\|_{L^2(\Gamma_3)} \\ &\quad + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}\|R_\tau(u_{n\tau}) - R_\tau(u_{n\tau})\|_{L^2(\Gamma_3)})\|u_n\|_V, \end{aligned} \tag{4.51}$$

Thus, from (4.47), (4.49), (4.50) and (4.51), we conclude that j_β satisfies

$$\lim_{n\to+\infty}[j_\beta(\zeta_n, u_n) - j_{fr}(\zeta, u_n)] = 0.$$

So, we deduce that j_β satisfies (4.19). □

Lemma 4.8. *The functional j_β satisfies the assumption* (4.18) *for all $k_0 \in (0, m)$. Moreover,*

$$\begin{aligned} &j_{fr}(u, \upsilon - u) - j_{fr}(\upsilon, \upsilon - u) \\ &\leq c_0^2(L_\nu + \|\mu\|_{L^\infty(\Gamma_3)}L_\nu + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)})\|u - \upsilon\|_V^2 \end{aligned} \tag{4.52}$$

Proof. Let $\zeta, u \in V$. Using (3.10), (3.18) and (4.39), we obtain

$$\begin{aligned} |j_\beta(\zeta, u)| &\leq L_\nu\|\zeta_\nu\|_{L^2(\Gamma_3)}\|u_\nu\|_{L^2(\Gamma_3)} \\ &\quad + \|\mu\|_{L^\infty(\Gamma_3)}L_\nu\|\zeta_\nu\|_{L^2(\Gamma_3)}\|u_\tau\|_{L^2(\Gamma_3)^d} \\ &\quad + \|\gamma_\nu\|_{L^\infty(\Gamma_3)}\|R_\nu(\zeta_\nu)\|_{L^2(\Gamma_3)}\|u_\nu\|_{L^2(\Gamma_3)} \\ &\quad + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}\|R_\tau(\zeta_\tau)\|_{L^2(\Gamma_3)^d}\|u_\tau\|_{L^2(\Gamma_3)^d}. \end{aligned}$$

Keeping in mind (3.3) and that R_τ, R_ν are Lipschitz continuous operators, we obtain

$$\begin{aligned} |j_\beta(\zeta, u)| &\leq c_0^2 L_\nu\|\zeta_\nu\|_V\|u\|_V + c_0^2\|\mu\|_{L^\infty(\Gamma_3)}L_\nu\|\zeta\|_V\|u\|_V \\ &\quad + c_0^2\|\gamma_\nu\|_{L^\infty(\Gamma_3)}\|\zeta\|_V\|u\|_V + c_0^2\|\gamma_\tau\|_{L^\infty(\Gamma_3)}\|\zeta\|_V\|u\|_V, \end{aligned}$$

Finally, we obtain

$$|j_\beta(\zeta, u)| \leq c_0^2(L_\nu + \|\mu\|_{L^\infty(\Gamma_3)}L_\nu + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)})\|\zeta\|_V\|u\|_V$$

which implies condition (4.18), for all $k_0 \in (0, m)$. Now let $u, \upsilon \in V$. Using again the assumptions (3.10), (3.18) and (4.39) we find

$$j_\beta(u, \upsilon - u) - j_\beta(\upsilon, \upsilon - u)$$

$$
\begin{aligned}
&= \int_{\Gamma 3} (p_\nu(u_\nu) - p_\nu(\upsilon_\nu))(\upsilon_\nu - u_\nu)\, da + \int_{\Gamma 3} \mu(p_\nu(u_\nu) - p_\nu(\upsilon_\nu))\|\upsilon_\tau - u_\tau\|\, da \\
&\quad + \int_{\Gamma 3} \gamma_\nu \beta^2 (R_\nu(u_\nu) - R_\nu(\upsilon_\nu))(\upsilon_\nu - u_\nu)\, da \\
&\quad + \int_{\Gamma 3} \gamma_\tau \beta^2 (R_\tau(u_\tau) - R_\tau(\upsilon_\tau)).(\upsilon_\tau - u_\tau)\, da
\end{aligned}
$$

Keeping in mind (3.10), (3.3) and that R_τ, R_ν are Lipschitz continuous operators, we obtain

$$
\begin{aligned}
&j_\beta(u, \upsilon - u) - j_\beta(\upsilon, \upsilon - u) \\
&\le L_\nu \int_{\Gamma 3} |\upsilon_\nu - u_\nu|^2\, da + \|\mu\|_{L^\infty(\Gamma_3)} L_\nu \int_{\Gamma 3} |\upsilon_\nu - u_\nu| \|\upsilon_\tau - u_\tau\|\, da \\
&\quad + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} \int_{\Gamma 3} |\upsilon_\nu - u_\nu|^2\, da + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} \int_{\Gamma 3} \|\upsilon_\tau - u_\tau\|^2\, da
\end{aligned}
$$

It follows from the previous inequality that

$$
\begin{aligned}
&j_{fr}(u, \upsilon - u) - j_{fr}(\upsilon, \upsilon - u) \\
&\le c_0^2 (L_\nu + \|\mu\|_{L^\infty(\Gamma_3)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}) \|u - \upsilon\|_V^2
\end{aligned}
$$

which implies (4.52). □

Proof of Theorem 4.5. Using the symmetry of $\mathcal{F}$ and L and (4.31), we see that the bilinear form a defined by (4.29) is symmetric and coercive.

Let $\mu_0 = \frac{m}{c_0^2}$. Clearly, μ_0 depends only on Ω, Γ_1, Γ_3, Γ_a, $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{B}$. Now assume that

$$
L_\nu + \|\mu\|_{L^\infty(\Gamma_3)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)} < \mu_0.
$$

We deduce that

$$
c_0^2 (L_\nu + \|\mu\|_{L^\infty(\Gamma_3)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}) < m.
$$

Then, there exists a real k_0 such that

$$
c_0^2 (L_\nu + \|\mu\|_{L^\infty(\Gamma_3)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Gamma_3)} + \|\gamma_\tau\|_{L^\infty(\Gamma_3)}) \le k_0 < m.
$$

Using (4.52) we deduce that (4.17) is verified. Using Lemmas 4.6–4.8, (3.23), Remark 4.4 and Theorem 4.2(i), we deduce that problem $\mathcal{P}_2^V$ has at least one solution $u_\beta \in W^{1,\infty}(0, T; V)$. □

As in [5], we adopt the following time-discretization. For all $n \in \mathbf{N}^*$, we set $t_i = i\Delta t$, $0 \le i \le n$, and $\Delta t = T/n$. We denote respectively by $u^i = u(t_i)$ where u is the solution of Problem $\mathcal{P}_1^V$ and β^i the approximation of β at time t_i and $\Delta u(t_i) = u(t_{i+1}) - u(t_i)$, $\Delta\beta^i = \beta^{i+2} - \beta^i$. For a continuous function $w(t)$, we use the notation $w^i = w(t_i)$. Then we obtain a sequence of time-discretized problems P_n^i of Problem $\mathcal{P}_1^V$ defined for $u(0) = u_0$ and $\beta^0 = \beta_0$ by:

Problem P_n^i. For $u(t_i) \in V$, $\beta^i \in L^\infty(\Gamma_3)$, find $u(t_{i+1}) \in V$, $\beta^{i+1} \in L^\infty(\Gamma_3)$ such that

$$\begin{aligned}
&a(u(t_{i+1}), w - u(t_{i+1})) + j_{ad}(\beta^{i+1}, u(t_{i+1}), w - u(t_{i+1}))\\
&+ j_{nc}(u(t_{i+1}), w - u(t_{i+1})) + j_{fr}(u(t_{i+1}), w - u(t_i)) - j_{fr}(u(t), \Delta u(t_i))\\
&\geq (\mathbf{f}(t_{i+1}), w - u(t_{i+1}))_V,\\
&\frac{\beta^{i+1} - \beta^i}{\Delta t} = -[\beta^{i+1}(\gamma_\nu (R_\nu(u_\nu^{i+1}))^2 + \gamma_\tau(|R_\tau(u_\tau^{i+1})|)^2) - \varepsilon_a]_+ \quad \text{a.e. on } \Gamma_3.
\end{aligned} \tag{4.53}$$

We have the following result.

Proposition 4.9. *There exists $\mu_c > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_c$, Problem P_n^i has a unique solution.*

For the proof of the above proposition, it suffices to invoke [23, Proposition 4.4] In the next step, we use the displacement field u_β obtained in Theorem 4.5, let $u = u_\beta$ and denote by u_ν, u_τ its normal and tangential components, and we consider the following initial value problem.

Problem $\mathcal{P}_3^{\beta_u}$. Find a bonding field $\beta_u : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot\beta_u(t) = -[\beta_u(t)(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2) - \varepsilon_a]_+ \quad \text{a.e. } t \in (0,T), \tag{4.54}$$

$$\beta_u(0) = \beta_0. \tag{4.55}$$

We obtain the following result.

Lemma 4.10. *There exists a unique solution β_u to Problem $\mathcal{P}_3^{\beta_u}$ and it satisfies $\beta_u \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$.*

Proof. Consider the mapping $F : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F(t, \beta_u) = -[\beta_u(t)(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2) - \varepsilon_a]_+, \tag{4.56}$$

for all $t \in [0,T]$ and $\beta_u \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators R_ν and R_τ that F is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\beta_u \in L^2(\Gamma_3)$, the mapping $t \mapsto F(t, \beta_u)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of Cauchy-Lipschitz theorem, see [25, page 48], we obtain the existence of a unique function $\beta_u \in W^{1,\infty}(0,T,L^2(\Gamma_3))$ which solves (4.54), 4.55. We note that the restriction $0 \leq \beta_u \leq 1$ is implicitly included in the Cauchy problem $\mathcal{P}_3^{\beta_u}$. Indeed, (4.54) and (4.55) guarantee that $\beta_u(t) \leq \beta_0$ and, therefore, assumption (3.19) shows that $\beta_u(t) \leq 1$ for $t \geq 0$, a.e. on Γ_3. On the other hand, if $\beta_u(t_0) = 0$ at $t = t_0$, then it follows from (4.54) and (4.55) that $\dot\beta_u(t) = 0$ for all $t \geq t_0$ and therefore, $\beta_u(t) = 0$ for all $t \geq t_0$, a.e. on Γ_3. We conclude that $0 \leq \beta_u(t) \leq 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Q}$, we find that $\beta_u \in \mathcal{Q}$. Then, it follows that $\beta_u \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$, which concludes the proof of Lemma 4.10. ☐

Now we introduce the sequences of functions $\beta^n(t)$ and $u^n(t)$ defined on $[0;T]$ by $\beta^n(t) = \beta^{i+1}$, $u^n(t) = u^{i+1} = u(t_{i+1})$, $\tilde u^n(t) = u^i + \frac{(t-t_i)}{\Delta t}\Delta u^i$ and $f^n(t) = f^{i+1} = f(t_{i+1})$ for all $t \in]t_i, t_{i+1}[$; $i = 0, \dots, n-1$; and $\beta^n(0) = \beta_0$, $u^n(0) = u_0$, $f^n(0) = f_0$.

Lemma 4.11. *Let u and β be the solutions to Problem $\mathcal{P}_2^V$ and Problem $\mathcal{P}_3^{\beta_u}$, respectively. Then we have:*

(i) $u^n \to u$ *and* $\tilde{u}^n \to \dot{u}$ *strongly in* $L^\infty(0,T;V)$*, For* $t \in (t_i, t_{i+1})$*,*
(ii) $\beta^n \to \beta$ *strongly in* $L^\infty(0,T;L^2(\Gamma_3))$*, For* $t \in (t_i, t_{i+1})$

Proof. (i) Since $u \in W^{1,\infty}(0,T,V)$, we deduce that $u^n \to u$ and $\widetilde{\dot{u}}^n \to \dot{u}$ strongly in $L^\infty(0,T;V)$, For $t \in (t_i, t_{i+1})$.

(ii) For $t \in (t_i, t_{i+1})$ we have

$$\|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)} \le \|\beta^n(t) - \beta(t_{i+1})\|_{L^2(\Gamma_3)} + \|\beta(t_{i+1}) - \beta(t)\|_{L^2(\Gamma_3)}.$$

As $\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$, we have

$$\|\beta(t_{i+1}) - \beta(t)\|_{L^2(\Gamma_3)} \le \frac{T}{n}\|\dot{\beta}\|_{L^\infty(0,T;L^2(\Gamma_3))}.$$

Using the properties of R_ν and R_τ, in [5], we have

$$\lim_{n\to\infty} \max_{i=0,\dots,n} \|\beta^i - \beta(t_i)\|_{L^2(\Gamma_3)} = 0.$$

So we deduce that

$$\lim_{n\to\infty} \max_{t\in[0,T]} \|\beta^n(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0.$$

□

Now we have all the ingredients to prove the following proposition.

Proposition 4.12. (u,β) *is a solution to Problem* $\mathcal{P}_1^V$*.*

Proof. In the inequality (4.53), for $v \in V$ set $w = u(t_i) + v\Delta t$ and divide by Δt; we obtain

$$\begin{aligned}
&a(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{nc}(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{fr}(u(t_{i+1}), v) \\
&- j_{fr}(u(t), \frac{\Delta u(t_i)}{\Delta t}) + j_{ad}(\beta^{i+1}, u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) \\
&\ge (f^{i+1}, v - \frac{\Delta u(t_i)}{\Delta t})_V\,.
\end{aligned}$$

Whence for any $v \in L^2(0,T;V)$, we have

$$\begin{aligned}
&a(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{nc}(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{fr}(u(t_{i+1}), v) \\
&- j_{fr}(u(t_{i+1}), \frac{\Delta u(t_i)}{\Delta t}) + j_{ad}(\beta^{i+1}, u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) \\
&\ge (f^{i+1}, v - \frac{\Delta u(t_i)}{\Delta t})_V
\end{aligned}$$

Integrating both sides of the above inequality on $(0,T)$, we obtain

$$\begin{aligned}
&a(u^n(t), v(t) - \widetilde{\dot{u}}^n) + j_{fr}(u^n(t), v(t)) - j_{fr}(u^n(t), \widetilde{\dot{u}}^n(t)) \\
&+ j_{nc}(u^n(t), v(t) - \widetilde{\dot{u}}^n(t)) + j_{ad}(\beta^n(t), u^n(t), v(t) - \widetilde{\dot{u}}^n(t)) \\
&\ge (f^n(t), v(t) - \widetilde{\dot{u}}^n(t))
\end{aligned} \tag{4.57}$$

To pass to the limit in this inequality we need to establish the following properties. After whihc the proof will be complete. □

Lemma 4.13. *We have the following properties for $v \in L^2(0,T;V)$:*

$$\lim_{n\to\infty}\int_0^T a(u^n(t), v(t)-\widetilde{\dot{u}}^n)dt = \int_0^T a(u(t), v(t)-\dot{u}(t))dt, \tag{4.58}$$

$$\liminf_{n\to\infty}\int_0^T j_{fr}(u^n(t), \widetilde{\dot{u}}^n(t))dt \geq \int_0^T j_{fr}(u(t), \dot{u}(t))dt, \tag{4.59}$$

$$\lim_{n\to\infty}\int_0^T j_{fr}(u^n(t), v(t))dt = \int_0^T j_{fr}(u(t), v(t))dt, \tag{4.60}$$

$$\lim_{n\to\infty}\int_0^T j_{nc}(u^n(t), v(t)-\widetilde{\dot{u}}^n(t))dt \geq \int_0^T j_{nc}(u(t), v(t)-\dot{u}(t))dt, \tag{4.61}$$

$$\lim_{n\to\infty}\int_0^T (f^n(t), v(t)-\widetilde{\dot{u}}^n(t))_V dt = \int_0^T (f(t), v(t)-\dot{u}(t))_V dt, \tag{4.62}$$

$$\lim_{n\to\infty}\int_0^T j_{ad}(\beta^n(t), u^n(t), v(t)-\widetilde{\dot{u}}^n(t))dt = \int_0^T j_{ad}(\beta(t), u(t), v(t)-\dot{u}(t))dt. \tag{4.63}$$

Proof. For (4.58) and (4.62) we refer the reader to [30, Lemma 4.6]. To prove (4.59) and (4.61) it suffices to see [16, Lemma 3.5]. To prove (4.60), it suffices to use Lemma 4.11(i). Finally for the proof of (4.63) we refer the reader to [5, Lemma 3.8] and use the properties of operators R_τ, R_ν.

Now using lemma 4.11(ii) and Lemma 4.13 we pass to the limit as $n \to +\infty$ in the inequality (4.57) to obtain

$$\begin{aligned}
&\int_0^T a(u(t), v(t)-\dot{u}(t))dt + \int_0^T j_{fr}(u(t), v(t))dt - \int_0^T j_{fr}(u(t), \dot{u}(t))dt \\
&+ \int_0^T j_{nc}(u(t), v(t)-\dot{u}(t))dt + \int_0^T j_{ad}(\beta(t), u(t), v(t)-\dot{u}(t))dt \\
&\geq \int_0^T (f(t), v(t)-\dot{u}(t))_V dt,
\end{aligned}$$

from which we deduce (4.34) and also that β is the unique solution of the differential equation (4.35). □

Proof of Theorem 4.1. Let (u, β) be the solution of Problem $\mathcal{P}_1^V$. It follows from (4.32), (4.29), (4.27), (4.25), (4.24), (4.23) and (4.22) that (u, φ, β) is, at least, a solution of Problem $\mathcal{P}^V$. Property (4.1), (4.2) and (4.3) follow from Theorem 4.3 and (4.25). □

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