A QUASISTATIC ELECTRO-ELASTIC CONTACT PROBLEM WITH NORMAL COMPLIANCE, FRICTION AND ADHESION

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Abstract. In this article we consider a mathematical model which describes the contact between a piezoelectric body and a deformable foundation. The constitutive law is assumed linear electro-elastic and the process is quasistatic. The contact is adhesive and frictional and is modelled with a version of normal compliance condition and the associated Coulomb’s law of dry friction. The evolution of the bonding field is described by a first order differential equation. We derive a variational formulation for the model, in the form of a coupled system for the displacements, the electric potential and the bonding field. Under a smallness assumption on the coefficient of friction, we prove an existence result of the weak solution of the model. The proofs are based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1. Introduction

In this work, we study a frictional contact problem with adhesion between an elastic piezoelectric body and a deformable obstacle.

A piezoelectric material is one that produces an electric charge when a mechanical stress is applied (the material is squeezed or stretched). Conversely, a mechanical deformation (the material shrinks or expands) is produced when an electric field is applied. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, and those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. Different models have been developed to describe the interaction between the electric and mechanical fields (see [1, 13, 18]–[20, 28, 29]). General models for elastic materials with piezoelectric effect, called electro-elastic materials, can be found in [1, 13]. A static frictional contact problem for electric-elastic materials was considered in [2, 17] and a slip-dependent frictional contact problem for electro-elastic materials was studied in [29].

Adhesion may take place between parts of the contacting surfaces. It may be intentional, when surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces. The adhesive contact is modelled by the introduction
of a surface internal variable, the bonding field, denoted in this paper by $\beta$; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [9, 10], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction $\beta$ of the bonds is active. Basic modelling can be found in [9]–[11]. Analysis of models for adhesive contact can be found in [3, 4] and in the monographs [24, 25]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [22, 23]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Since frictional contact is so important in industry, there is a need to model and predict it accurately. However, the main industrial need is to effectively control the process of frictional contact. Currently, there is a considerable interest in frictional contact problems involving piezo-electric materials, see for instance [2, 15, 26].

The aim of this article is to continue the study of problems begun in [12, 21, 6]. The novelty of the present paper is to extend the result when the contact and friction are modelled by a normal compliance condition and a version of Coulomb’s law of dry friction, respectively. Moreover, the adhesion is taken into account at the interface and the material behavior is assumed to be electro-elastic.

The paper is structured as follows. In Section 2 we present the electro-elastic contact model with normal compliance, friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In section 4, we present our main existence results.

## 2. Problem statement

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a smooth boundary $\partial \Omega = \Gamma$. The body is submitted to the action of body forces of density $f_0$ and volume electric charges of density $q_0$. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of $\Gamma$ into three measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$ on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts $\Gamma_a$ and $\Gamma_b$, on the other hand, such that $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma_a) > 0$. We assume that the body is clamped on $\Gamma_1$ and surface tractions of density $f_2$ act on $\Gamma_2$. On $\Gamma_3$ the body is in adhesive contact with an insulator obstacle, the so-called foundation. We also assume that the electrical potential vanishes on $\Gamma_a$ and a surface electric charge of density $q_2$ is prescribed on $\Gamma_b$. We denote by $S^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and we use $\cdot$ and $\|\cdot\|$ for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $S^d$, respectively. Also, below $\nu$ represents the unit outward normal on $\Gamma$. With these assumptions, the classical model for the process is the following.

**Problem** ($P$). Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \rightarrow S^d$, an electric potential $\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}$, an electric displacement
field $D : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Omega \times [0, T] \to \mathbb{R}$ such that
\begin{align*}
\sigma &= \mathcal{F}\varepsilon(u) - \mathcal{E}^* E(\varphi) \quad \text{in} \quad \Omega \times (0, T), \\
D &= B\varepsilon(\varphi) + \mathcal{E}\varepsilon(u) \quad \text{in} \quad \Omega \times (0, T), \\
\text{Div} \sigma + f_0 &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\text{div} D &= q_0 \quad \text{in} \quad \Omega \times (0, T), \\
u &= 0 \quad \text{on} \quad \Gamma_1 \times (0, T), \\
\sigma\nu &= f_2 \quad \text{on} \quad \Gamma_2 \times (0, T), \\
-\sigma_{\nu} &= p_{\nu}(u_{\nu}) - \gamma_{\nu}\beta^2 R_{\nu}(u_{\nu}) \quad \text{on} \quad \Gamma_3 \times (0, T), \\
\|\sigma_{\tau} + \gamma_{\tau}\beta^2 R_{\tau}(u_{\tau})\| &\leq \mu p_{\nu}(u_{\nu}), \\
\|\sigma_{\tau} + \gamma_{\tau}\beta^2 R_{\tau}(u_{\tau})\| &< \mu p_{\nu}(u_{\nu}) \Rightarrow \dot{u}_{\tau} = 0, \\
\|\sigma_{\tau} + \gamma_{\tau}\beta^2 R_{\tau}(u_{\tau})\| &= \mu p_{\nu}(u_{\nu}).
\end{align*}

We now provide some comments on equations and conditions (2.1)–(2.13). Equations (2.7) and (2.8) represent the electro-elastic constitutive law in which $\varepsilon(u)$ denotes the linearized strain tensor, $E(\varphi) = -\nabla \varphi$ is the electric field, where $\varphi$ is the electric potential, $\mathcal{F} = (\mathcal{F}_{ijkl})$ is a 4th rank tensor, called the elastic tensor and its components $\mathcal{F}_{ijkl}$ are called coefficients of elasticity, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed, $B$ denotes the electric permittivity operator, and $D = (D_1, \ldots, D_d)$ is the electric displacement vector. Details on the constitutive equations of the form (2.1) and (2.2) can be found, for instance, in [1] and in [2]. Next, equations (2.9)–(2.11) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which Div and div denote the divergence operator for tensor and vector valued functions, respectively. Equations (2.12) and (2.13) represent the displacement and traction boundary conditions. Conditions (2.10) and (2.11) represent the electric boundary conditions. Condition (2.7) describes contact with normal compliance and adhesion where $u_{\nu}$ is the normal displacement, $\sigma_{\nu}$ represents the normal stress, $\gamma_{\nu}$ denotes a given adhesion coefficient and $R_{\nu}$ is the truncation operator defined by
\begin{equation}
R_{\nu}(s) = \begin{cases}
L & \text{if } s < -L, \\
\frac{s}{L} & \text{if } -L \leq s \leq 0, \\
0 & \text{if } s > 0,
\end{cases}
\end{equation}

where $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator $R_{\nu}$, together with the operator $R_{\tau}$ defined below, is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the
parameter $L$ is made in what follows. Thus, by choosing $L$ very large, we can assume that $R_\nu(u_\nu) = u_\nu$.

Here $p_\nu$ is a nonnegative prescribed function, called normal compliance function. Indeed, when $u_\nu < 0$ there is no contact and the normal pressure vanishes. When there is contact, $u_\nu$ is positive and is a measure of the interpenetration of the asperities. A commonly used example of the normal compliance function $p_\nu$ is

$$p_\nu(r) = c_\nu r_+, \text{ where } c_\nu > 0 \text{ is the surface stiffness coefficient and } r_+ = \max\{0, r\} \text{ denotes the positive part of } r.$$ 

We can also consider the following truncated normal compliance function:

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if } r \leq \alpha, \\ c_\nu \alpha & \text{if } r > \alpha, \end{cases}$$

where $\alpha$ is a positive coefficient related to the wear and hardness of the surface. In this case, the above equality means that when the penetration exceeds $\alpha$ the obstacle offers no additional resistance to penetration. It follows from (2.7) that the contribution of the adhesion to the normal traction is represented by the term $\gamma_\nu \beta^2 R_\nu(u_\nu)$, but as long as $u_\nu$ does not exceed the bond length $L$.

Condition (2.8) is the associated Coulomb’s law of dry friction, where $u_\tau$ and $\sigma_\tau$ denote tangential components of vector $u$ and tensor $\sigma$, respectively. Here $\mu$ is the coefficient of friction and $R_\tau$ is the truncation operator given by

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \leq L, \\ \frac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases} \quad (2.15)$$

This condition shows that the contribution of the adhesion to the tangential shear on the contact surface is represented by the term $\gamma_\tau \beta^2 R_\tau(u_\tau)$, but again, only up to the bond length $L$.

The evolution of the bonding field is governed by the differential equation (2.9) with given positive parameters $\gamma_\nu, \gamma_\tau$ and $\epsilon_a$. For more details about conditions (2.7)–(2.9), we refer the reader to [24] and [25]. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$. Finally, (2.12) and (2.13) represent the initial conditions where $\beta_0$ and $u_0$ are given.

### 3. Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, $i, j, k, l$ run from 1 to $d$, summation over repeated indices is applied and the index that follows a comma represents the partial
derivative with respect to the corresponding component of the spatial variable, e.g. 
\[ u_{i,j} = \frac{\partial^2 u}{\partial x_i \partial x_j}. \]

Everywhere below, we use the classical notation for \( L^p \) and Sobolev spaces associated to \( \Omega \) and \( \Gamma \). Moreover, we use the notation \( L^2(\Omega)^d, H^1(\Omega)^d, \mathcal{H} \) and \( \mathcal{H}_1 \) for the following spaces
\[ L^2(\Omega)^d = \{ v = (v_i) : v_i \in L^2(\Omega) \}, \quad H^1(\Omega)^d = \{ v = (v_i) : v_i \in H^1(\Omega) \}, \]
\[ \mathcal{H} = \{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega) \}, \quad \mathcal{H}_1 = \{ \tau \in \mathcal{H} : \tau_{ij,j} \in L^2(\Omega) \}. \]
The spaces \( L^2(\Omega)^d, H^1(\Omega)^d, \mathcal{H} \) and \( \mathcal{H}_1 \) are real Hilbert spaces endowed with the canonical inner products
\[ (u, v)_{L^2(\Omega)^d} = \int_\Omega u \cdot v \, dx, \quad (u, v)_{H^1(\Omega)^d} = \int_\Omega u \cdot v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx, \]
\[ (\sigma, \tau)_\mathcal{H} = \int_\Omega \sigma \cdot \tau \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = \int_\Omega \sigma \cdot \tau \, dx + \int_\Omega \text{Div} \sigma \cdot \text{Div} \tau \, dx, \]
and the associated norms \( \| \cdot \|_{L^2(\Omega)^d}, \| \cdot \|_{H^1(\Omega)^d}, \| \cdot \|_\mathcal{H} \) and \( \| \cdot \|_{\mathcal{H}_1} \), respectively.

Here and below we use the notation
\[ \nabla v = (v_{i,j}), \quad \varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall v \in H^1(\Omega)^d, \]
\[ \text{Div} \tau = (\tau_{ij,j}) \quad \forall \tau \in \mathcal{H}_1. \]

For every element \( v \in H^1(\Omega)^d \). We also write \( v \) for the trace of \( v \) on \( \Gamma \) and we denote by \( v_\nu \) and \( v_\tau \) the normal and tangential components of \( v \) on \( \Gamma \) given by \( v_\nu = v \cdot \nu, v_\tau = v - v_\nu \nu \).

Let now consider the closed subspace of \( H^1(\Omega)^d \) defined by
\[ V = \{ v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_1 \}. \]
Since \( \text{meas}(\Gamma_1) > 0 \), the following Korn’s inequality holds
\[ \| \varepsilon(v) \|_\mathcal{H} \geq c_K \| v \|_{H^1(\Omega)^d} \quad \forall v \in V, \quad (3.1) \]
where \( c_K > 0 \) is a constant which depends only on \( \Omega \) and \( \Gamma_1 \). Over the space \( V \) we consider the inner product given by
\[ (u, v)_V = (\varepsilon(u), \varepsilon(v))_\mathcal{H}, \quad (3.2) \]
and let \( \| \cdot \|_V \) be the associated norm. It follows from Korn’s inequality \((3.1)\) that \( \| \cdot \|_{H^1(\Omega)^d} \) and \( \| \cdot \|_V \) are equivalent norms on \( V \) and, therefore, \( (V, \| \cdot \|_V) \) is a real Hilbert space. Moreover, by the Sobolev trace theorem, \((3.1)\) and \((3.2)\), there exists a constant \( c_0 \) depending only on the domain \( \Omega, \Gamma_1 \) and \( \Gamma_3 \) such that
\[ \| v \|_{L^2(\Gamma_3)^d} \leq c_0 \| v \|_V \quad \forall v \in V. \quad (3.3) \]

We also introduce the spaces
\[ W = \{ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a \}, \]
\[ W_1 = \{ D = (D_i) \mid D_i \in L^2(\Omega), \ D_{i,i} \in L^2(\Omega) \}. \]
Since \( \text{meas}(\Gamma_a) > 0 \), the following Friedrichs-Poincaré inequality holds
\[ \| \nabla \psi \|_{L^2(\Omega)^d} \geq c_F \| \psi \|_{H^1(\Omega)} \quad \forall \psi \in W, \quad (3.4) \]
where \( c_F > 0 \) is a constant which depends only on \( \Omega \) and \( \Gamma \), and \( \nabla \psi = (\psi, i) \). Over the space \( W \), we consider the inner product given by

\[
(\varphi, \psi)_W = \int_{\Omega} \nabla \varphi \cdot \nabla \psi \, dx,
\]

and let \( \| \cdot \|_W \) be the associated norm. It follows from \( (3.4) \) that \( \| \cdot \|_{H^1(\Omega)} \) and \( \| \cdot \|_W \) are equivalent norms on \( W \) and therefore \( (W, \| \cdot \|_W) \) is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant \( \tilde{c}_0 \), depending only on \( \Omega, \Gamma \), such that

\[
\| \psi \|_{L^2(\Gamma_3)} \leq \tilde{c}_0 \| \psi \|_W \quad \forall \psi \in W. \quad (3.5)
\]

The space \( W_1 \) is a real Hilbert space with the inner product

\[
(D, E)_{W_1} = \int_{\Omega} D \cdot E \, dx + \int_{\Omega} \text{div} D \cdot \text{div} E \, dx,
\]

and the associated norm \( \| \cdot \|_{W_1} \).

Finally, for every real Hilbert space \( X \) we use the classical notation for the spaces \( L^p(0,T;X) \) and \( W^{k,p}(0,T;X) \), \( 1 \leq p \leq \infty \), \( k \geq 1 \) and we also introduce the set

\[
Q = \{ \beta \in L^\infty(0,T;L^2(\Gamma_3)) : 0 \leq \beta(t) \leq 1 \ \forall t \in [0,T], \ \text{a.e. on } \Gamma_3 \}.
\]

In the study of problem \( P \), we consider the following assumptions on the problem data.

The elasticity operator \( F \), the piezoelectric operator \( E \) and the electric permittivity operator \( B \) satisfy the following conditions:

(a) \( F = (F_{ijkl}) : \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}^d \),
(b) \( F_{ijkl} = F_{klij} = F_{jikl} \in L^\infty(\Omega) \),
(c) There exists \( m_F > 0 \) such that \( F_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq m_F \| \varepsilon \|^2 \) for all \( \varepsilon \in \mathbb{S}^d \), a.e. in \( \Omega \). \quad (3.6)

(a) \( E : \Omega \times \mathbb{S}^d \rightarrow \mathbb{R}^d \),
(b) \( E(x, \tau) = (e_{ijk}(x) \tau_{jk}) \) for all \( \tau = (\tau_{ij}) \in \mathbb{S}^d \), a.e. \( x \in \Omega \),
(c) \( e_{ijk} = e_{ikj} \in L^\infty(\Omega) \). \quad (3.7)

(a) \( B : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d \),
(b) \( B(x, E) = (b_{ij}(x) E_j) \) for all \( E = (E_i) \in \mathbb{R}^d \), a.e. \( x \in \Omega \),
(c) \( b_{ij} = b_{ji} \in L^\infty(\Omega) \),
(d) There exists \( m_B > 0 \) such that \( b_{ij}(x) E_i E_j \geq m_B \| E \|^2 \) for all \( E = (E_i) \in \mathbb{R}^d \), a.e. \( x \in \Omega \). \quad (3.8)

From assumptions \( (3.7) \) and \( (3.8) \), we deduce that the piezoelectric operator \( E \) and the electric permittivity operator \( B \) are linear, have measurable bounded components denoted \( e_{ijk} \) and \( b_{ij} \), respectively, and moreover, \( B \) is symmetric and positive definite.

Recall also that the transposed operator \( E^* \) is given by \( E^* = (e^*_{ijk}) \) where \( e^*_{ijk} = e_{kij} \), and

\[
E \sigma \cdot v = \sigma \cdot E^* v \quad \forall \sigma \in \mathbb{S}^d, \ v \in \mathbb{R}^d. \quad (3.9)
\]
Finally, we assume that
\[ p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+, \]
(b) there exists \( L_\nu > 0 \) such that \( \|p_\nu(x, r_1) - p_\nu(x, r_2)\| \leq L_\nu |r_1 - r_2| \) for all \( r_1, r_2 \in \mathbb{R}, \) a.e. \( x \in \Gamma_3. \)
(c) \( x \mapsto p_\nu(x, r) \) is measurable on \( \Gamma_3 \) for all \( r \in \mathbb{R}. \)
(d) \( x \mapsto p_\nu(x, r) = 0 \) for all \( r \leq 0 \) a.e. \( x \in \Gamma_3. \)

We also suppose that the body forces and surface tractions have the regularity
\[ f_0 \in W^{1,\infty}(0, T; L^2(\Omega)^d), \quad f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_3)^d), \] (3.11)
and the densities of electric charges satisfy
\[ q_0 \in W^{1,\infty}(0, T; L^2(\Omega)), \quad q_2 \in W^{1,\infty}(0, T; L^2(\Gamma_3)). \] (3.12)

Finally, we assume that
\[ q_2(t) = 0 \quad \text{on} \Gamma_3 \forall t \in [0, T]. \] (3.13)

Note that we need to impose assumption (3.13) for physical reasons; indeed, the foundation is supposed to be insulator and therefore the electric boundary conditions on \( \Gamma_3 \) do not have to change in function of the status of the contact, are the same on the contact and on the separation zone, and are included in the boundary condition (2.11).

The Riesz representation theorem implies the existence of two functions \( f : [0, T] \to V \) and \( q : [0, T] \to W \) such that
\[
(f(t), v)_V = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_3} f_2(t) \cdot v \, da, \tag{3.14}
\]

\[
(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_3} q_2(t) \psi \, da, \tag{3.15}
\]
for all \( v \in V, \) \( \psi \in W \) and \( t \in [0, T]. \) We note that conditions (3.11) and (3.12) imply
\[ f \in W^{1,\infty}(0, T; V), \quad q \in W^{1,\infty}(0, T; W). \] (3.16)

The adhesion coefficients \( \gamma_\nu, \gamma_\tau \) and the limit bound \( \epsilon_\sigma \) satisfy the conditions
\[ \gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_\sigma \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_\sigma \geq 0 \quad \text{a.e. on} \Gamma_3, \] (3.17)
and the friction coefficient \( \mu \) is such that
\[ \mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on} \Gamma_3. \] (3.18)

The initial condition \( \beta_0 \) satisfies
\[ \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad \text{a.e. on} \Gamma_3. \] (3.19)

Next, we define the adhesion functional \( j_{ad} : L^2(\Gamma_3) \times V \times V \to \mathbb{R} \) by
\[ j_{ad}(\beta, u, v) = \int_{\Gamma_3} (-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau) \, da, \] (3.20)
the normal compliance functional \( V \times V \to \mathbb{R} \) by
\[ j_{nc}(u, v) = \int_{\Gamma_3} p_\nu(u_\nu(t)) v_\nu \, da, \] (3.21)
and the friction functional \( V \times V \to \mathbb{R} \) by
\[ j_{fr}(u, v) = \int_{\Gamma_3} \mu p_\nu(u_\nu) \|v_\tau\| \, da. \] (3.22)
We consider the following assumptions on the conditions initials

\begin{equation}
 u_0 \in V, \quad (3.23)
\end{equation}

\begin{equation}
 (\mathcal{F}(u_0), \varepsilon(v))_\mathcal{H} + (E^* \nabla \varphi_0, \varepsilon(v))_\mathcal{H} + j_{ad}(\beta_0, u_0, v) + j_{nc}(u_0, v) + j_{fr}(u_0, v) \geq (f(0), v)_V \quad \forall v \in V,
\end{equation}

\begin{equation}
 (B \nabla \varphi, \nabla \psi)_{L^2(\Omega)^d} = (E \varepsilon(u_0), \nabla \psi)_{L^2(\Omega)^d} + (q(0), \psi)_W \quad \forall \psi \in W. \quad (3.25)
\end{equation}

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)–(2.13).

**Problem (PV).** Find a displacement field \( u : [0, T] \to V \), an electric potential field \( \varphi : [0, T] \to W \) and a bonding field \( \beta : [0, T] \to L^2(\Gamma_3) \) such that

\begin{equation}
 (\mathcal{F}(u(t)), \varepsilon(v) - \varepsilon(\bar{u}(t)))_\mathcal{H} + (E^* \nabla \varphi(t), \varepsilon(v) - \varepsilon(\bar{u}(t)))_\mathcal{H} + j_{ad}(\beta, u(t), v - \bar{u}(t)) + j_{nc}(u(t), v - \bar{u}(t)) + j_{fr}(u(t), v - \bar{u}(t)) \geq (f(t), v - \bar{u}(t))_V \quad \forall v \in V \text{ a.e. } t \in [0, T],
\end{equation}

\begin{equation}
 (B \nabla \varphi(t), \nabla \psi)_{L^2(\Omega)^d} - (E \varepsilon(u(t)), \nabla \psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \quad \forall \psi \in W \text{ a.e. } t \in [0, T],
\end{equation}

\begin{equation}
 \dot{\beta}(t) = -[\beta(t) (\gamma_\nu R_\nu(u_\nu(t)))^2 + \gamma_\tau ||R_\tau(u_\tau(t))||^2 - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0, T), \quad (3.28)
\end{equation}

\begin{equation}
 u(0) = u_0, \quad \beta(0) = \beta_0. \quad (3.29)
\end{equation}

In the rest of this section, we derive some inequalities involving the functionals \( j_{ad}, j_{nc} \) and \( j_{fr} \) which will be used in the following sections. Below in this section \( \beta_1 \) and \( \beta_2 \) denote elements of \( L^2(\Gamma_3) \) such that \( 0 \leq \beta_1, \beta_2 \leq 1 \) a.e. on \( \Gamma_3 \), \( u_1, u_2, v_1, v_2, u \) and \( v \) represent elements of \( V \) and \( c \) is a generic positive constants which may depend on \( \Omega, \Gamma_1, \Gamma_3, p_\nu, \gamma_\nu, \gamma_\tau \) and \( L \), whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on \( x \in \Omega \cup \Gamma_3 \). Using (3.3), (3.10), (3.21) and the inequalities \( ||R_\nu(u_\nu)\|_L \leq L, ||R_\tau(u_\tau)|| \leq L, ||\beta_1|| \leq 1, ||\beta_2|| \leq 1 \), we obtain

\begin{equation}
 |j_{ad}(\beta_1, u_1, \omega) - j_{ad}(\beta_2, u_2, \omega) + j_{nc}(u_1, \omega) - j_{nc}(u_2, \omega)| \leq c(||\beta_1 - \beta_2||_{L^2(\Gamma_3)} + ||u_1 - u_2||_V) ||\omega||_V. \quad (3.30)
\end{equation}

Next, we use (3.22), (3.10) and (3.3) to obtain

\begin{equation}
 j_{fr}(u, v - u) - j_{fr}(v, v - u) \leq c_0^2 ||\mu||_{L^\infty(\Gamma_3)} L \mu ||u - v||_V^2 \quad \forall u, v \in V. \quad (3.31)
\end{equation}

\begin{equation}
 j_{fr}(u_1, v_1) - j_{fr}(u_2, v_2) - j_{fr}(u_2, v_1) \leq c_0^2 L \nu ||\mu||_{L^\infty(\Gamma_3)} ||u_1 - u_2||_V ||v_1 - v_2||_V. \quad (3.32)
\end{equation}

Inequalities (3.30)–(3.32) will be used in various places in the rest of the paper.

4. **Existence result**

Our main result which states the solvability of Problem (PV), is the following.

**Theorem 4.1.** Assume that (3.6)–(3.8), (3.10), (3.13), (3.17)–(3.19) and (3.23)–(3.25) hold. Then there exists \( \mu_0 > 0 \) depending only on \( \Omega, \Gamma_1, \Gamma_3, \mu, \mathcal{F}, \mathcal{B} \) and \( \mathcal{E} \)
such that, if \((L_\nu + L_\upsilon)\|u\|_{L^2(\Gamma_3)} + \|\gamma_\nu\|_{L^2(\Gamma_3)} + \|\gamma_\upsilon\|_{L^2(\Gamma_3)} < \mu_0\), Problem (P\(^V\)) has at least one solution \((u, \varphi, \beta)\). Moreover, the solution satisfies

\[
\begin{align*}
   u &\in W^{1,\infty}(0, T; V), \\
   \varphi &\in W^{1,\infty}(0, T; W), \\
   \beta &\in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap Q. 
\end{align*}
\] (4.1)

(3.29) is called a weak solution of the contact problem (P). To precise the regularity of the weak solution we note that the constitutive relations (2.1), (2.2), the assumptions (3.6), (3.8) and the regularities (4.1), (4.2) show that, if \((\psi, \sigma, \varphi, D, \beta)\) of the piezoelectric contact problem (4.11) which satisfies (2.1), (2.2), \((3.26)\) and \((3.29)\) is called a weak solution of the contact problem (P). To precise the regularity implied in (4.1), (4.2), (4.3), (4.4) and (4.5) we obtain

\[
\begin{align*}
   \text{Div } \sigma(t) + f_0(t) = 0, \\
   \text{div } D(t) = q_0(t), \quad \forall t \in [0, T].
\end{align*}
\]

It follows now from the regularities (3.11), (3.12) that \(\text{Div } \sigma \in W^{1,\infty}(0, T; L^2(\Omega)^d)\) and \(\text{div } D \in W^{1,\infty}(0, T; L^2(\Omega))\), which shows that

\[
\begin{align*}
   \sigma &\in W^{1,\infty}(0, T; \mathcal{H}_1), \\
   D &\in W^{1,\infty}(0, T; W_1).
\end{align*}
\] (4.4) (4.5)

We conclude that the weak solution \((u, \varphi, \beta)\) of the piezoelectric contact problem (P) has the regularity implied in (4.1), (4.2), (4.3), (4.4) and (4.5).

The proof of Theorem 4.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let \(X\) be a real Hilbert space with the inner product \((\cdot, \cdot)_X\) and the associated norm \(\|\cdot\|_X\).

Let \(a : X \times X \to \mathbb{R}\) be a bilinear form on \(X\), \(j : X \times X \to \mathbb{R}\), \(f : [0, T] \to X\) and \(u_0 \in X\). With these data, we consider the following quasivariational problem: find \(u : [0, T] \to X\) such that

\[
\begin{align*}
   a(u(t), v - \dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \\
   &\geq (f(t), v - \dot{u}(t))_X \quad \forall v \in X, \ a.e. \ t \in (0, T), \\
   u(0) &= u_0.
\end{align*}
\] (4.6) (4.7)

To solve problem (4.6)–(4.7), we consider the following assumptions:

\[
\begin{align*}
   a &\colon X \times X \to \mathbb{R}\ 	ext{is a bilinear symmetric form, and} \\
   (a) &\text{ there exists } M > 0 \text{ such that } |a(u, v)| \leq M\|u\|_X\|v\|_X \text{ for all } u, v \in X, \\
   (b) &\text{ there exists } m > 0 \text{ such that } a(v, v) \geq m\|v\|_X^2 \text{ for all } v \in X. 
\end{align*}
\] (4.8)

For every \(\zeta \in X\), \(j(\zeta, \cdot) : X \to \mathbb{R}\) is a positively homogeneous subadditive functional, i.e.

\[
\begin{align*}
   (a) &\ j(\lambda u, \zeta) = \lambda j(u, \zeta) \text{ for all } u \in X, \lambda \in \mathbb{R}_+, \\
   (b) &\ j(u + v, \zeta) \leq j(u, \zeta) + j(v, \zeta) \text{ for all } u, v \in X. 
\end{align*}
\] (4.9)

\[
\begin{align*}
   f &\in W^{1,\infty}(0, T; X), \\
   u_0 &\in X. 
\end{align*}
\] (4.10) (4.11)

\[
\begin{align*}
   a(u_0, v) + j(u_0, v) &\geq (f(0), v)_X \quad \forall v \in X. 
\end{align*}
\] (4.12)
Keeping in mind (4.9), it results that for all \( \zeta \in X \), \( j(\zeta, \cdot) : X \to \mathbb{R} \) is a convex functional. Therefore, there exists the directional derivative \( j' \) given by

\[
j'_2(\zeta, u; v) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[ j(\zeta, u + \lambda v) - j(\zeta, u) \right] \quad \forall \zeta, u, v \in X. \tag{4.13}\]

We consider now the following additional assumptions on the functional \( j \).

For every sequence \((u_n) \subset X\) with \( \|u_n\|_X \to \infty \), every bounded sequence \((\zeta_n) \subset X\) and each \( \tilde{u} \in X \) one has

\[
\liminf_{n \to +\infty} \frac{1}{\|u_n\|_X^2} j'_2(t_n u_n, u_n - \tilde{u}; -u_n) < m. \tag{4.14}\]

For every sequence \((u_n) \subset X\) with \( \|u_n\|_X \to \infty \), every bounded sequence \((\zeta_n) \subset X\) and for each \( \tilde{u} \in X \), one has

\[
\liminf_{n \to +\infty} \frac{1}{\|u_n\|_X^2} j'_2(\zeta_n, u_n - \tilde{u}; -u_n) < m. \tag{4.15}\]

For all sequences \((u_n) \subset X\) and \((\zeta_n) \subset X\) such that \( u_n \rightharpoonup u \in X \), \( \zeta_n \rightharpoonup \zeta \in X \) and for every \( v \in X \), we have

\[
\limsup_{n \to +\infty} [j(\zeta_n, v) - j(\zeta_n, u_n)] \leq j(\zeta, v) - j(\zeta, u). \tag{4.16}\]

There exists \( k_0 \in (0, m) \) such that

\[
j(u, v - u) - j(v, v - u) \leq k_0 \|u - v\|^2_X \quad \forall u, v \in X. \tag{4.17}\]

There exist two functions \( a_1 : X \to \mathbb{R} \) and \( a_2 : X \to \mathbb{R} \), which map bounded sets in \( X \) into bounded sets in \( \mathbb{R} \) such that

\[
|j(\zeta, u)| \leq a_1(\zeta) \|u\|^2_X + a_2(\zeta) \quad \forall \zeta, u \in X, \text{ and } a_1(0_X) < m - k_0. \tag{4.18}\]

For every sequence \((\zeta_n) \subset X\) with \( \zeta_n \rightharpoonup \zeta \in X \) and every bounded sequence \((u_n) \subset X\) one has

\[
\lim_{n \to +\infty} [j(\zeta_n, u_n) - j(\zeta, u_n)] = 0. \tag{4.19}\]

For every \( s \in (0, T] \) and every pair of functions \( u, v \in W^{1,\infty}(0, T; X) \), with \( u(0) = v(0), u(s) \neq v(s) \),

\[
\int_0^s [j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t))] dt
\]

\[
\leq \frac{m}{2} \|u(s) - v(s)\|^2_X. \tag{4.20}\]

There exists \( \alpha \in (0, m/2) \) such that for every \( s \in (0, T] \) and for every functions \( u, v \in W^{1,\infty}(0, T; X) \) with \( u(s) \neq v(s) \), it holds that

\[
\int_0^s [j(u(t), \dot{v}(t)) - j(u(t), \dot{u}(t)) + j(v(t), \dot{u}(t)) - j(v(t), \dot{v}(t))] dt
\]

\[
\leq \alpha \|u(s) - v(s)\|^2_X. \tag{4.21}\]

For the study of the evolutionary problem (4.6)–(4.7), we recall the following result.

\textbf{Theorem 4.2.} Assume (4.8)–(4.12) hold.

(i) If assumptions (4.14)–(4.19) are satisfied, then there exists at least a solution \( u \in W^{1,\infty}(0, T; X) \) to problem (4.6)–(4.7).

(ii) If assumptions (4.14)–(4.20) are satisfied, then there exists a unique solution \( u \in W^{1,\infty}(0, T; X) \) to problem (4.6)–(4.7).
(iii) If assumptions (4.14)–(4.19) and (4.21) are satisfied, then there exists a unique solution \( u \in W^{1,\infty}(0, T; X) \) to (4.6)–(4.7), and the mapping \( (f, u_0) \to \mathbb{R} \) is Lipschitz continuous from \( W^{1,\infty}(0, T; X) \times X \) to \( L^{\infty}(0, T; X) \).

The proof can be found in [10], it is obtained in several steps and it is based on arguments of elliptic quasivariational inequalities and a time discretization method.

We return now to proof of theorem 4.1. To this end, we assume in the following that (3.6)–(3.8), (3.10)–(3.13), (3.17)–(3.19) and (4.23)–(4.25) hold. Below, \( c \) is a generic positive constants which may depend on \( \Omega, \Gamma_1, \Gamma_3, \mathcal{F}, \nu, \gamma_p, \) and \( L \), whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on \( x \in \Omega \cup \Gamma_3 \).

Using the Riesz’s representation theorem, we define the operators \( \mathcal{G} : W \to W \) and \( \mathcal{R} : V \to V \) respectively by

\[
\mathcal{G}\varphi(t, \psi)_W = (\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)} \quad \forall \varphi, \psi \in W, \quad (4.22)
\]

\[
\mathcal{R}v, \varphi)_W = (\mathcal{E}\varphi(t), \nabla\varphi)_{L^2(\Omega)} \quad \forall \varphi \in W, \quad v \in V. \quad (4.23)
\]

We can show that \( \mathcal{G} \) is a linear continuous symmetric positive definite operator. Therefore, \( \mathcal{G} \) is an invertible operator on \( W \). We can also prove that \( \mathcal{R} \) is a linear continuous operator on \( V \). Let \( \mathcal{R}^{*} \) the adjoint of \( \mathcal{R} \). Thus, from (3.9) we can write

\[
(\mathcal{R}^{*}\varphi, v)_V = (\mathcal{E}^{*}\nabla\varphi, \varepsilon(v))_H \quad \forall \varphi \in W, \quad v \in V. \quad (4.24)
\]

By introducing (4.22)–(4.23) in (4.27) we obtain

\[
(\mathcal{G}\varphi(t, \psi)_W = (\mathcal{R}u(t), \psi)_W + (q(t), \psi)_W \quad \forall \psi \in W;
\]

and consequently

\[
\mathcal{G}\varphi(t) = \mathcal{R}u(t) + q(t).
\]

On the other hand, \( \mathcal{G} \) is invertible where the previous equality gives us

\[
\varphi(t) = \mathcal{G}^{*}\mathcal{R}u(t) + \mathcal{G}^{*}q(t). \quad (4.25)
\]

Using (4.24)–(4.25) and (3.26) we obtain

\[
(\mathcal{F}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\hat{u}(t)))_H + \left( \mathcal{R}^{*}\mathcal{G}^{*}\mathcal{R}u(t), v - \hat{u}(t) \right)_V
\]

\[
+ \left( \mathcal{R}^{*}\mathcal{G}^{*}q(t), v - \hat{u}(t) \right)_V \quad \forall v \in V, \text{ a.e. } t \in (0, T). \quad (4.26)
\]

Let now the operator \( L : V \to V \) defined by

\[
L(v) = \mathcal{R}^{*}\mathcal{G}^{*}\mathcal{R}(v), \quad \forall v \in V. \quad (4.27)
\]

Using the properties of the operators \( \mathcal{G}, \mathcal{R} \) and \( \mathcal{R}^{*} \), we deduce that \( L \) is a linear symmetric positive operator on \( V \). Indeed, we have

\[
(Lu, v)_V = (\mathcal{R}^{*}\mathcal{G}^{*}\mathcal{R}u, v)_V
\]

\[
= (\mathcal{G}^{-1}\mathcal{R}u, \mathcal{R}v)_W
\]

\[
= (\mathcal{R}u, \mathcal{G}^{-1}\mathcal{R}v)_W
\]

\[
= (u, \mathcal{R}^{*}\mathcal{G}^{*}\mathcal{R}v)_V
\]

\[
= (u, Lu)_V \quad \forall u, v \in V
\]

\[
(Lu, v)_V = (\mathcal{R}^{*}\mathcal{G}^{*}\mathcal{R}u, v)_V, \quad (4.28)
\]

\[
(Lv, v)_V = (\mathcal{G}^{-1}\mathcal{R}v, \mathcal{R}v)_W \geq 0 \quad \forall v \in V.
\]
Now, let the bilinear form $a : V \times V \to \mathbb{R}$ be such that
\[
a(u, v) = (\mathcal{F}(\varepsilon(t)), \varepsilon(v))_{\mathcal{H}} + (Lu, v)_V \quad \forall u, v \in V. \tag{4.29}
\]
The bilinear form $a$ is continuous and coercive on $V$. Indeed, we have
\[
|a(u, v)| \leq (M + \|L\|)\|u\|_V \|v\|_V \quad \forall u, v \in V, \tag{4.30}
\]
and the symmetry of $\mathcal{F}$ and $L$ leads to the symmetry of $a$.

Let now the function $f : [0, T] \to V$ be defined by
\[
f(t) = f(t) - \mathcal{R}^{s}G^{-1}q(t) \quad \forall t \in [0, T]. \tag{4.32}
\]
From (3.16) we obtain
\[
f \in W^{1, \infty}(0, T, V). \tag{4.33}
\]
The relations (4.26), (4.29), (4.32), (3.28) and (3.29) lead us to consider the following intermediate problem, in the terms of displacement and bonding fields.

**Problem $\mathcal{P}_1^V$.** Find a displacement field $u : [0, T] \to V$, and a bonding field $\beta : [0, T] \to L^2(\Gamma_3)$ such that
\[
a(u(t), v - \dot{u}(t)) + j_{ad}(\beta, u(t), v - \dot{u}(t)) + j_{nc}(u(t), v - \dot{u}(t))
+ j_{fr}(u(t), v) - j_{fr}(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V, \text{ a.e. } t \in (0, T), \tag{4.34}
\]
\[
\dot{\beta}(t) = -[\beta(t)(\gamma_v R_v(u_v(t)))^2 + \gamma_r \|R_r(u_r(t))\|^2] - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{4.35}
\]
\[
u(0) = u_0, \quad \beta(0) = \beta_0. \tag{4.36}
\]

**Theorem 4.3.** Assume that (3.6), (3.8), (3.10), (3.13), (3.17), (3.19) and (3.23)–(3.25) hold. Then, there exists $\mu_0 > 0$ depending only on $\Omega, \Gamma_1, \Gamma_3, \Gamma_a, \mathcal{F}, \mathcal{B}$ and $\mathcal{E}$ such that, if
\[
L_v + L_v \|\mu\|_{L^\infty(\Gamma_3)} + \|\gamma_v\|_{L^\infty(\Gamma_3)} + \|\gamma_r\|_{L^\infty(\Gamma_3)} < \mu_0,
\]
then Problem $\mathcal{P}_1^V$ has at least one solution $(u, \beta)$. Moreover, the solution satisfies
\[
u \in W^{1, \infty}(0, T; V), \tag{4.37}
\]
\[
\beta \in W^{1, \infty}(0, T; L^2(\Gamma_3)) \cap Q. \tag{4.38}
\]

We assume in the following that the conditions of Theorem 4.3 hold. Let $\beta \in W^{1, \infty}(0, T; L^2(\Gamma_3)) \cap Q$ be given and $j_\beta : V \times V \to \mathbb{R}$ defined by
\[
j_\beta(u, v) = \int_{\Gamma_3} \rho_v(u_v(t)) v_v da + \int_{\Gamma_3} \mu \rho_v(u_v) \|v_r\| da
+ \int_{\Gamma_3} \left(-\gamma_v \beta^2 R_v(u_v) v_v + \gamma_r \beta^2 R_r(u_r) \cdot v_r \right) da, \tag{4.39}
\]
Now, we consider the following intermediate problem, in the term of displacement field.
Problem $P^V_{2}$. Find the displacement field $u_\beta : [0, T] \to V$ such that
\begin{align*}
a(u_\beta(t), v - u_\beta(t)) + j_\beta(u_\beta(t), v) - j_\beta(u_\beta(t), u_\beta(t)) & \geq (f(t), v - u_\beta(t))_\nu \quad \forall v \in V, \text{ a.e. } t \in (0, T), \\
u_\beta(0) = u_0,
\end{align*}
(4.40)

Remark 4.4. From (3.24) and (3.25), we can deduce (4.12).

Theorem 4.5. Assume that (3.6)–(3.8), (3.10)–(3.13), (3.17)–(3.19) and (3.23)–(3.25) hold. Then there exists $\mu_0 > 0$ depending only on $\Omega, \Gamma_1, \Gamma_3, \Gamma_a, F, B$ and $E$ such that, if
\[ L_\nu + L_\nu \mu \| \mu \|_{L^\infty(\Gamma_3)} + \| \gamma_\nu \|_{L^\infty(\Gamma_3)} + \| \gamma_\nu \|_{L^\infty(\Gamma_3)} < \mu_0, \]
then Problem $P^V_{2}$ has at least one solution $u_\beta \in W^{1,\infty}(0, T, V)$.

Proof. Let $\zeta, u, \bar{u} \in V$ and let $\lambda \in [0, 1]$. Using (3.22), it follows that $j_\beta$ satisfies
\begin{align*}
j_\beta(\zeta, u - \bar{u} - \mu - \lambda u) - j_\beta(\zeta, u - \bar{u}) & \leq -\lambda \int_{\Gamma_3} p_\nu(\zeta_\nu) u_\nu \, da - \lambda \int_{\Gamma_3} \mu p_\nu(\zeta_\nu) \| u_\nu - \bar{u}_\nu \| \, da + \lambda \int_{\Gamma_3} \mu p_\nu(\zeta_\nu) \| \bar{u}_\nu \| \, da \\
& \quad + \lambda \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) u_\nu \, da - \lambda \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) \cdot u_\nu \, da,
\end{align*}
and as $\mu \geq 0$, $p_\nu \geq 0$ a.e. on $\Gamma_3$, we obtain
\begin{align*}
j_\beta(\zeta, u - \bar{u} - \lambda u) - j_\beta(\zeta, u - \bar{u}) & \leq -\lambda \int_{\Gamma_3} p_\nu(\zeta_\nu) u_\nu \, da + \lambda \int_{\Gamma_3} \mu p_\nu(\zeta_\nu) \| \bar{u}_\nu \| \, da + \lambda \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) u_\nu \, da \\
& \quad - \lambda \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) \cdot u_\nu \, da, \quad \forall \zeta, u, \bar{u} \in V.
\end{align*}
Moreover, we deduce from (4.13) that
\begin{align*}
j_\beta'(\zeta, u - \bar{u}; -u) & \leq -\int_{\Gamma_3} p_\nu(\zeta_\nu) u_\nu \, da + \int_{\Gamma_3} \mu p_\nu(\zeta_\nu) \| \bar{u}_\nu \| \, da \\
& \quad + \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) u_\nu \, da - \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(\zeta_\nu) \cdot u_\nu \, da \quad \forall \zeta, u, \bar{u} \in V.
\end{align*}
Now consider the sequences $(u_n)_{n \in \mathbb{N}} \subset V$, $(t_n)_{n \in \mathbb{N}} \subset [0, 1]$ and the element $\bar{u} \in V$. Using (3.3), (3.10), (3.18) and (4.42), we find
\begin{align*}
j_\beta'(t_n u_n, u_n - \bar{u}; -u_n) & \leq -\int_{\Gamma_3} p_\nu(t_n u_{n\nu}) u_{n\nu} \, da + \int_{\Gamma_3} \mu p_\nu(t_n u_{n\nu}) \| \bar{u}_{n\nu} \| \, da \\
& \quad + \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(t_n u_{n\nu}) u_{n\nu} \, da - \int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(t_n u_{n\nu}) \cdot u_{n\nu} \, da \quad \forall \zeta, u, \bar{u} \in V
\end{align*}
(4.43)
Now, using (3.10)(b), (3.3) and the fact that $0 \leq 1 N. \text{CHOUGUI, S. DRABLA EJDE-2014/257}$

Therefore, we deduce from (4.47), (4.48) and (4.49) that

Thus, we deduce that $j_\beta$ satisfies assumption (4.14).

Now consider the sequences $(u_n)_{n \in \mathbb{N}} \subset V$, $(\zeta_n)_{n \in \mathbb{N}} \subset V$ such that

and we conclude that $j_\beta$ satisfies assumption (4.15).

Thus, we deduce that $j_\beta$ satisfies assumption (4.16).

\textbf{Lemma 4.7.} The functional $j_\beta$ satisfies the conditions (4.16) and (4.19).

Proof. Let $(u_n)_{n \in \mathbb{N}} \subset V$, $(\zeta_n)_{n \in \mathbb{N}} \subset V$ be two sequences such that $u_n \to u \in V$ and $\zeta_n \to \zeta \in V$. Using the compactness property of the trace map and (3.10), it follows that

\begin{align*}
p_\nu(t_n u_n')u_n' & \to p_\nu(\zeta_n') \text{ in } L^2(\Gamma_3), \\
u_n & \to u \text{ in } L^2(\Gamma_3)^d, \\
R_\nu(t_n u_n') & \to R_\nu(\zeta_n') \text{ in } L^2(\Gamma_3). \\
R_\tau(t_n u_n') & \to R_\tau(\zeta_n') \text{ in } L^2(\Gamma_3)^d.
\end{align*}

Therefore, we deduce from (4.17), (4.18) and (4.19) that

\begin{align*}
_j(\zeta_n, v) & \to j_\beta(\zeta, v) \forall v \in V, \\
j_\beta(\zeta_n, u_n) & \to j_\beta(\zeta, u),
\end{align*}

which show that the functional $j_\beta$ satisfies

\begin{align*}
\limsup_{n \to \infty} [j_\beta(\zeta_n, v) - j_\beta(\zeta_n, u_n)] & \leq j_\beta(\zeta, v) - j_\beta(\zeta, u).
\end{align*}

Thus, we deduce that $j_\beta$ satisfies (4.16).
Now we consider \((u_n)_{n \in \mathbb{N}}\) a bounded sequence of \(V\), i.e.
\[
\|u_n\|_V \leq C \quad \forall n \in \mathbb{N},
\] (4.50)
where \(C > 0\). We have
\[
|j_\beta(\zeta_n, u_n) - j_\beta(\zeta, u_n)| \leq \int_{\Gamma_3} |p_{\nu}(\zeta_{nv}) - p_{\nu}(\zeta)| |u_{nv}| \, da
+ \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |p_{\nu}(\zeta_{nv}) - \mu p_{\nu}(\zeta)| |u_{nv}| \, da
+ \|\gamma_{\nu}\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |R_{\nu}(\zeta_{nv}) - R_{\nu}(\zeta)| |u_{nv}| \, da
+ \|\gamma_{\tau}\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |R_{\tau}(u_{nt}) - R_{\tau}(u_{nt})| |u_{nt}| \, da.
\] (4.51)

Thus, from (4.47), (4.49), (4.50) and (4.51), we conclude that \(j_\beta\) satisfies
\[
\lim_{n \to +\infty} [j_\beta(\zeta_n, u_n) - j_\beta(\zeta, u_n)] = 0.
\]
So, we deduce that \(j_\beta\) satisfies (4.19). \(\square\)

**Lemma 4.8.** The functional \(j_\beta\) satisfies the assumption (4.18) for all \(k_0 \in (0, m)\).
Moreover,
\[
j_{fr}(u, v - u) - j_{fr}(v, v - u) 
\leq c_0^2 L_{\nu} + \|\mu\|_{L^\infty(\Gamma_3)} L_{\nu} + \|\gamma_{\nu}\|_{L^\infty(\Gamma_3)} + \|\gamma_{\tau}\|_{L^\infty(\Gamma_3)} \|u - v\|_V^2
\] (4.52)

**Proof.** Let \(\zeta, u \in V\). Using (3.10), (3.18) and (4.39), we obtain
\[
|j_\beta(\zeta, u)| \leq L_{\nu} \|\zeta_{\nu}\|_{L^2(\Gamma_3)} \|u_{\nu}\|_{L^2(\Gamma_3)}
+ \|\mu\|_{L^\infty(\Gamma_3)} L_{\nu} \|\zeta_{\nu}\|_{L^2(\Gamma_3)} \|u_{\nu}\|_{L^2(\Gamma_3)}
+ \|\gamma_{\nu}\|_{L^\infty(\Gamma_3)} \|R_{\nu}(\zeta_{\nu})\|_{L^2(\Gamma_3)} \|u_{\nu}\|_{L^2(\Gamma_3)}
+ \|\gamma_{\tau}\|_{L^\infty(\Gamma_3)} \|R_{\tau}(\zeta_{\tau})\|_{L^2(\Gamma_3)} \|u_{\nu}\|_{L^2(\Gamma_3)}
\]

Keeping in mind (3.3) and that \(R_{\nu}, R_{\tau}\) are Lipschitz continuous operators, we obtain
\[
|j_\beta(\zeta, u)| \leq c_0^2 L_{\nu} \|\zeta_{\nu}\|_V \|u\|_V + c_0^2 \|\mu\|_{L^\infty(\Gamma_3)} L_{\nu} \|\zeta\|_V \|u\|_V
+ c_0^2 \|\gamma_{\nu}\|_{L^\infty(\Gamma_3)} \|\zeta\|_V \|u\|_V + c_0^2 \|\gamma_{\tau}\|_{L^\infty(\Gamma_3)} \|\zeta\|_V \|u\|_V
\]

Finally, we obtain
\[
|j_\beta(\zeta, u)| \leq c_0^2 (L_{\nu} + \|\mu\|_{L^\infty(\Gamma_3)} L_{\nu} + \|\gamma_{\nu}\|_{L^\infty(\Gamma_3)} + \|\gamma_{\tau}\|_{L^\infty(\Gamma_3)}) \|\zeta\|_V \|u\|_V
\]
which implies condition (4.18), for all \(k_0 \in (0, m)\). Now let \(u, v \in V\). Using again the assumptions (3.10), (3.18) and (4.39) we find
\[
j_\beta(u - v) - j_\beta(v - u)
\]
Proof of Theorem 4.5. □

which implies (4.52).

It follows from the previous inequality that

\[ \Delta u \text{ is the solution of Problem } P. \]

Using (4.52) we deduce that (4.17) is verified. Using Lemmas 4.6–4.8, (3.23), Remark 4.4 and Theorem 4.2(i), we deduce that problem \( P^V_2 \) has at least one solution \( u_\beta \) exists in \( W^{3,\infty}(0, T; V) \).

As in [3], we adopt the following time-discretization. For all \( n \in \mathbb{N}^* \), we set \( t_i = i \Delta t, 0 \leq i \leq n \), and \( \Delta t = T/n \). We denote respectively by \( u^i = u(t_i) \) where \( u \) is the solution of Problem \( P^V_1 \) and \( \beta^i \) the approximation of \( \beta \) at time \( t_i \) and \( \Delta u(t_i) = u(t_{i+1}) - u(t_i) \), \( \Delta \beta^i = \beta^{i+1} - \beta^i \). For a continuous function \( w(t) \), we use the notation \( w^i = w(t_i) \). Then we obtain a sequence of time-discretized problems \( P^\gamma_n \) of Problem \( P^V_1 \) defined for \( u(0) = u_0 \) and \( \beta^0 = \beta_0 \) by:

\[
\frac{1}{2} \int_{\Omega} (p_\nu(u_\nu) - p_\nu(u_\nu))(v_\nu - u_\nu) \, dx + \int_{\Omega} \mu(p_\nu(u_\nu) - p_\nu(v_\nu)) ||v_\nu - u_\nu|| \, dx \\
+ \int_{\Omega} \gamma_\nu \beta^2 (R_\nu(u_\nu) - R_\nu(v_\nu))(v_\nu - u_\nu) \, dx \\
+ \int_{\Omega} \gamma_\nu \beta^2 (R_\nu(u_\nu) - R_\nu(v_\nu))(v_\nu - u_\nu) \, dx
\]

Keeping in mind (3.10), (3.3) and that \( R_\nu, R_\nu \) are Lipschitz continuous operators, we obtain

\[
j_\beta(u, v - u) - j_\beta(v, v - u)
\leq L_\nu \int_{\Omega} |v_\nu - u_\nu|^2 \, dx + \|\mu\|_{L^\infty(\Omega)} L_\nu \int_{\Omega} |v_\nu - u_\nu||v_\nu - u_\nu| \, dx
\]

It follows from the previous inequality that

\[
j_{\beta}(u, v - u) - j_{\beta}(v, v - u)
\leq \frac{c^2_0(L_\nu + \|\mu\|_{L^\infty(\Omega)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Omega)} + \|\gamma_\tau\|_{L^\infty(\Omega)})(\|u - v\|^2
\]

which implies (4.52). □

Proof of Theorem 4.5. Using the symmetry of \( F \) and \( L \) and (4.31), we see that the bilinear form \( a \) defined by (4.20) is symmetric and coercive.

Let \( \mu_0 = \frac{m}{\gamma_0} \). Clearly, \( \mu_0 \) depends only on \( \Omega, \Gamma_1, \Gamma_3, \Gamma_a, F, E \) and \( B \). Now assume that

\[ L_\nu + \|\mu\|_{L^\infty(\Omega)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Omega)} + \|\gamma_\tau\|_{L^\infty(\Omega)} < \mu_0. \]

We deduce that

\[ \frac{c^2_0(L_\nu + \|\mu\|_{L^\infty(\Omega)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Omega)} + \|\gamma_\tau\|_{L^\infty(\Omega)}) < m. \]

Then, there exists a real \( k_0 \) such that

\[ \frac{c^2_0(L_\nu + \|\mu\|_{L^\infty(\Omega)} L_\nu + \|\gamma_\nu\|_{L^\infty(\Omega)} + \|\gamma_\tau\|_{L^\infty(\Omega)}) \leq k_0 < m. \]

Using (4.52) we deduce that (4.17) is verified. Using Lemmas 4.6–4.8, (3.23), Remark 4.4 and Theorem 4.2(i), we deduce that problem \( P^V_2 \) has at least one solution \( u_\beta \) exists in \( W^{3,\infty}(0, T; V) \). □
Problem $P^n_a$. For $u(t_i) \in V$, $\beta^i \in L^\infty(\Gamma_3)$, find $u(t_{i+1}) \in V$, $\beta^{i+1} \in L^\infty(\Gamma_3)$ such that
\[
a(u(t_{i+1}), w - u(t_{i+1})) + j_{ad}(\beta^{i+1}, u(t_{i+1}), w - u(t_{i+1})) \\
+ j_{nc}(u(t_{i+1}), w - u(t_{i+1})) + j_{fr}(u(t_{i+1}), w - u(t_i)) - j_{fr}(u(t), \Delta u(t_i)) \\
\geq (f(t_{i+1}), w - u(t_{i+1}))v, \\
\frac{\beta^{i+1} - \beta^i}{\Delta t} = -[\beta^{i+1}(\gamma_R(u'_{\tau}^{i+1}))^2 + \gamma_T(|R_T(u'_{\tau}^{i+1})|)^2] - \varepsilon_3 a.e. \text{ on } \Gamma_3.
\]
(4.53)

We have the following result.

Proposition 4.9. There exists $\mu_c > 0$ such that for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_c$, Problem $P^n_a$ has a unique solution.

For the proof of the above proposition, it suffices to invoke [23, Proposition 4.4]. In the next step, we use the displacement field $u_{3\beta}$ obtained in Theorem 4.5 and let $u = u_3$ and denote by $u_\gamma$, $u_\tau$ its normal and tangential components, and we consider the following initial value problem.

Problem $P_{3\beta}^\beta$. Find a bonding field $\beta_a : [0, T] \to L^2(\Gamma_3)$ such that
\[
\dot{\beta}_a(t) = -[\beta_a(t)(\gamma_R(u_\gamma(t)))^2 + \gamma_T(|R_T(u_\tau(t))|)^2] - \varepsilon_3 a.e. \text{ t} \in (0, T), \quad (4.54) \\
\beta_a(0) = \beta_0.
\]
(4.55)

We obtain the following result.

Lemma 4.10. There exists a unique solution $\beta_a$ to Problem $P_{3\beta}^\beta$ and it satisfies $\beta_a \in W^{1,\infty}(0, T, L^2(\Gamma_3)) \cap Q$.

Proof. Consider the mapping $F : [0, T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by
\[
F(t, \beta_a) = -[\beta_a(t)(\gamma_R(u_\gamma(t)))^2 + \gamma_T(|R_T(u_\tau(t))|)^2] - \varepsilon_3 a.e. \text{ for all } t \in [0, T] \text{ and } \beta_a \in L^2(\Gamma_3).
\]
(4.56)

for all $t \in [0, T]$ and $\beta_a \in L^2(\Gamma_3)$. It follows from the properties of the truncation operators $R_\gamma$ and $R_\tau$ that $F$ is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\beta_a \in L^2(\Gamma_3)$, the mapping $t \mapsto F(t, \beta_a)$ belongs to $L^\infty(0, T; L^2(\Gamma_3))$. Using now a version of Cauchy-Lipschitz theorem, see [23, page 48], we obtain the existence of a unique function $\beta_a \in W^{1,\infty}(0, T, L^2(\Gamma_3))$ which solves (4.54), (4.55). We note that the restriction $0 \leq \beta_a \leq 1$ is implicitly included in the Cauchy problem $P_{3\beta}^\beta$. Indeed, (4.54) and (4.55) guarantee that $\beta_a(t) \leq \beta_0$ and, therefore, assumption (4.19) shows that $\beta_a(t) \leq 1$ for $t \geq 0$, a.e. on $\Gamma_3$. On the other hand, if $\beta_a(t_0) = 0$ at $t = t_0$, then it follows from (4.54) and (4.55) that $\beta_a(t) = 0$ for all $t \geq t_0$ and therefore, $\beta_a(t) = 0$ for all $t \geq t_0$, a.e. on $\Gamma_3$. We conclude that $0 \leq \beta_a(t) \leq 1$ for all $t \in [0, T]$, a.e. on $\Gamma_3$. Therefore, from the definition of the set $Q$, we find that $\beta_a \in Q$. Then, it follows that $\beta_a \in W^{1,\infty}(0, T, L^2(\Gamma_3)) \cap Q$, which concludes the proof of Lemma 4.10.

Now we introduce the sequences of functions $\beta^n(t)$ and $u^n(t)$ defined on $[0; T]$ by $\beta^n(t) = \beta^{n+1}$, $u^n(t) = u^{n+1} = u(t_{i+1})$, $\tilde{u}^n(t) = u^i + \frac{(t-i)}{\Delta t} u' \text{ and } f^n(t) = f(t_{i+1}) = f(t_{i+1})$ for all $t \in [t_i, t_{i+1}]; \ i = 0, \ldots, n-1$; and $\beta^n(0) = \beta_0$, $u^n(0) = u_0$, $f^n(0) = f_0$.

Lemma 4.11. Let $u$ and $\beta$ be the solutions to Problem $P_2^\beta$ and Problem $P_{3\beta}^\beta$, respectively. Then we have:
(i) \( u^n \to u \) and \( \tilde{u}^n \to \tilde{u} \) strongly in \( L^\infty(0, T; V) \), for \( t \in (t_i, t_{i+1}) \),
(ii) \( \beta^n \to \beta \) strongly in \( L^\infty(0, T; L^2(\Gamma_3)) \), for \( t \in (t_i, t_{i+1}) \)

Proof. (i) Since \( u \in W^{1, \infty}(0, T, V) \), we deduce that \( u^n \to u \) and \( \tilde{u}^n \to \tilde{u} \) strongly in \( L^\infty(0, T; V) \), for \( t \in (t_i, t_{i+1}) \).
(ii) For \( t \in (t_i, t_{i+1}) \) we have
\[
\| \beta^n(t) - \beta(t) \|_{L^2(\Gamma_3)} \leq \| \beta^n(t) - \beta(t_{i+1}) \|_{L^2(\Gamma_3)} + \| \beta(t_{i+1}) - \beta(t) \|_{L^2(\Gamma_3)}.
\]
As \( \beta \in W^{1, \infty}(0, T; L^2(\Gamma_3)) \), we have
\[
\| \beta(t_{i+1}) - \beta(t) \|_{L^2(\Gamma_3)} \leq \frac{T}{n} \| \beta \|_{L^\infty(0, T; L^2(\Gamma_3))}.
\]
Using the properties of \( R_u \) and \( R_\tau \), in [5], we have
\[
\lim_{n \to \infty} \max_{t \in [0, T]} \| \beta^i - \beta(t_i) \|_{L^2(\Gamma_3)} = 0.
\]
So we deduce that
\[
\lim_{n \to \infty} \max_{t \in (0, T]} \| \beta^n(t) - \beta(t) \|_{L^2(\Gamma_3)} = 0.
\]
\( \square \)

Now we have all the ingredients to prove the following proposition.

**Proposition 4.12.** \((u, \beta)\) is a solution to Problem \( P_V^\gamma \).

Proof. In the inequality (4.53), for \( v \in V \) set \( w = u(t_i) + v \Delta t \) and divide by \( \Delta t \); we obtain
\[
a(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{nc}(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{fr}(u(t_{i+1}), v)
- j_{fr}(u(t_i), \frac{\Delta u(t_i)}{\Delta t}) + j_{ad}(\beta^{i+1}, u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t})
\geq (f^{i+1}, v - \frac{\Delta u(t_i)}{\Delta t})_V.
\]
Whence for any \( v \in L^2(0, T; V) \), we have
\[
a(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{nc}(u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t}) + j_{fr}(u(t_{i+1}), v)
- j_{fr}(u(t_{i+1}), \frac{\Delta u(t_i)}{\Delta t}) + j_{ad}(\beta^{i+1}, u(t_{i+1}), v - \frac{\Delta u(t_i)}{\Delta t})
\geq (f^{i+1}, v - \frac{\Delta u(t_i)}{\Delta t})_V.
\]
Integrating both sides of the above inequality on \((0, T)\), we obtain
\[
a(u^n(t), v(t) - \tilde{u}^n(t)) + j_{fr}(u^n(t), v(t)) - j_{fr}(u^{n}(t), \tilde{u}^n(t))
+ j_{nc}(u^n(t), v(t) - \tilde{u}^n(t)) + j_{ad}(\beta^n(t), u^n(t), v(t) - \tilde{u}^n(t))
\geq (f^n(t), v(t) - \tilde{u}^n(t))
\]
To pass to the limit in this inequality we need to establish the following properties. After while the proof will be complete. \( \square \)
Lemma 4.13. We have the following properties for $v \in L^2(0,T;V)$:
\[
\lim_{n \to \infty} \int_0^T a(u^n(t), v(t) - \tilde{u}^n) \, dt = \int_0^T a(u(t), v(t) - \dot{u}(t)) \, dt, \tag{4.58}
\]
\[
\liminf_{n \to \infty} \int_0^T j_{fr}(u^n(t), \tilde{u}^n(t)) \, dt \geq \int_0^T j_{fr}(u(t), \dot{u}(t)) \, dt, \tag{4.59}
\]
\[
\lim_{n \to \infty} \int_0^T j_{nc}(u^n(t), v(t) - \tilde{u}^n(t)) \, dt \geq \int_0^T j_{nc}(u(t), v(t) - \dot{u}(t)) \, dt, \tag{4.60}
\]
\[
\lim_{n \to \infty} \int_0^T (f^n(t), v(t) - \tilde{u}^n(t))_V \, dt \geq \int_0^T (f(t), v(t) - \dot{u}(t))_V \, dt, \tag{4.61}
\]
\[
\lim_{n \to \infty} \int_0^T j_{ad}(\beta^n(t), u^n(t), v(t) - \tilde{u}^n(t)) \, dt = \int_0^T j_{ad}(\beta(t), u(t), v(t) - \dot{u}(t)) \, dt. \tag{4.62}
\]
\[
\lim_{n \to \infty} \int_0^T j_{nc}(u^n(t), v(t) - \tilde{u}^n(t)) \, dt \geq \int_0^T j_{nc}(u(t), v(t) - \dot{u}(t)) \, dt, \tag{4.63}
\]

Proof. For (4.58) and (4.62) we refer the reader to [30, Lemma 4.6]. To prove (4.59) and (4.61) it suffices to see [16, Lemma 3.5]. To prove (4.60), it suffices to use Lemma 4.11(i). Finally for the proof of (4.63) we refer the reader to [5, Lemma 3.8] and use the properties of operators $R_{\tau}$, $R_{\nu}$.

Now using lemma 4.11(ii) and Lemma 4.13 we pass to the limit as $n \to +\infty$ in the inequality (4.57) to obtain
\[
\int_0^T a(u(t), v(t) - \dot{u}(t)) \, dt + \int_0^T j_{fr}(u(t), v(t)) \, dt - \int_0^T j_{fr}(u(t), \dot{u}(t)) \, dt
\]
\[
+ \int_0^T j_{nc}(u(t), v(t) - \dot{u}(t)) \, dt + \int_0^T j_{ad}(\beta(t), u(t), v(t) - \dot{u}(t)) \, dt
\]
\[
\geq \int_0^T (f(t), v(t) - \dot{u}(t))_V \, dt,
\]
from which we deduce (4.34) and also that $\beta$ is the unique solution of the differential equation (4.35).

Proof of Theorem 4.1. Let $(u, \beta)$ be the solution of Problem $\mathcal{P}_Y^V$. It follows from (4.32), (4.29), (4.27), (4.25), (4.24), (4.23) and (4.22) that $(u, \varphi, \beta)$ is, at least, a solution of Problem $\mathcal{P}_Y^V$. Property (4.1), (4.2) and (4.3) follow from Theorem 4.3.

References


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