Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 258, pp. 1–29.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

PRODUCT MEASURABILITY WITH APPLICATIONS TO A STOCHASTIC CONTACT PROBLEM WITH FRICTION

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Abstract. A new product measurability result for evolution equations with random inputs, when there is no uniqueness of the ω-wise problem, is established using results on measurable selection theorems for measurable multifunctions. The abstract result is applied to a general stochastic system of ODEs with delays and to a frictional contact problem in which the gap between a viscoelastic body and the foundation and the motion of the foundation are random processes. The existence and uniqueness of a measurable solution for the problem with Lipschitz friction coefficient, and just existence for a discontinuous one, is obtained by using a sequence of approximate problems and then passing to the limit. The new result shows that the limit exists and is measurable. This new result opens the way to establish the existence of measurable solutions for various problems with random inputs in which the uniqueness of the solution is not known, which is the case in many problems involving frictional contact.

1. Introduction

This article establishes the product measurability of solutions to evolution equations having random coefficients, that is, the various operators occurring in the equations are assumed to be stochastic processes depending on the random variable ω that belongs to a probability space $(\Omega, \mathcal{F}, P)$. In many problems described by nonlinear partial differential equations and inclusions, this is an important generalization. We apply our theory to a system of ordinary delay-differential equations involving inputs that are stochastic processes and to a problem of frictional contact between a viscoelastic body and a reactive foundation. In the latter problem the gap between the body and the foundation in the reference configuration is assumed to be a random process, and so is the speed of the foundation.

This abstract result, Theorem 1, opens the way to study a host of models set as differential inclusions or equations that arise in many applications in which some of the input parameters are naturally random or known with some uncertainty, which is the case in most applied continuous systems. This general result on the measurability of the solution is based on the use of theorems on measurable multifunctions. This approach allows one to essentially consider the problem for one fixed

2000 *Mathematics Subject Classification.* 60H15, 34F05, 35R60, 60H10, 74M10.
Key words and phrases. Stochastic differential equations; product measurability; dynamic contact.

Submitted August 27, 2014. Published December 11, 2014.

value of the random variable ω and then to conclude the existence of a measurable solution. A related contact problem for the vibrations of a Gao beam when the gap is random has been studied in [13], however since the setting there was simpler, the result was obtained directly. This abstract result applies directly to the stochastic Navier-Stokes equations studied in [4].

From the applied point of view, it is natural to allow various parameters in the problem to be random variables or stochastic processes. However, such generalizations lead to the difficulty of showing that the solutions are measurable, although showing measurability is relatively straightforward when the ω-wise problems obtained by fixing ω has a unique solution. Our approach is more general and it does not require the uniqueness of the solution of the problem with fixed ω. In addition, we do not make any special assumptions on the underlying probability space $(\Omega, \mathcal{F}, P)$, in contrast to what was done in the important paper [4] in the case of the Navier-Stokes equations. However, we use the same measurable selection theorem for measurable multi-functions but in a very different context.

The main application of Theorem 1 in this work is to a stochastic version of a model for the dynamic frictional contact between a viscoelastic body and a reactive foundation when the coefficient of friction is slip-rate dependent, that was studied in [15]. The problem without stochastic input but with reactive foundation and slip-rate independent friction coefficient was first studied in [20] and then in [10, 11] where the static case was considered, and since then in many papers with models of various degrees of complexity, see, e.g., [2, 7, 12, 14, 16, 17, 24] and the references therein. General references about various versions of related contact problems with friction are, e.g. [5, 6, 8, 21, 22, 25] and the references therein. In addition to adding stochastic inputs, we also present an improved result for the case when the friction coefficient is a discontinuous function of the slip-rate or even a graph, than in our earlier papers [14, 17]. These methods open the way to study a variety of contact problems in which the various parameters and inputs are random. We foresee that it will be used in a number of publications.

Following the Introduction, the main theorem, Theorem 1, is formulated and proved in Section 2. It provides a general approach that allows one to use standard techniques for evolution equations and inclusions for fixed value of the random variable ω. The main constraints are that one must start with measurable functions and that subsequences converge weakly to weakly continuous functions, which is usually the case in evolution problems. An application of the theorem to the measurability of the solutions of systems of ordinary differential equations or inclusions with delays, when some of the inputs are random, is provided in Section 3. The contact problem with random gap and sliding rate is studied in Section 4, where the problem data is given, too. Subsection 4.1 provides an abstract form of the problem, and contains some results from the literature on compact sets in function spaces. To deal with the friction term, which is a set-inclusion, the problem is regularized in Subsection 4.2 and the Galerkin method is used to obtain approximate measurable solutions. Then, by obtaining the necessary a priori estimates, we pass to the limit and obtain the unique measurable solution of the problem in the case when the coefficient of friction is a Lipschitz function of the slip-rate. Finally, in Subsection 4.3 the case when the friction coefficient is discontinuous, has a jump from a static value to a dynamic value when relative motion commences, is studied. In this case the uniqueness of the solution is not known, and seems to be unlikely.

Using the new tools, we establish the existence of a measurable solution to the problem. As noted above, this is an improvement of the result in [15].

2. The measurable selection theorem

In this section we study the problem of obtaining product measurable solutions to evolution equations in the context when either there is no uniqueness to the non-stochastic problem obtained by fixing a given ω in the probability space, or uniqueness is not known. This is of considerable interest because there are many important problems in which the existence of solutions is known but not their uniqueness. This often occurs when weak limits are used to obtain existence but there is insufficient monotonicity to show uniqueness. For example, the equations describing a vibrating purely elastic Gao beam appear to fail to have uniqueness, see [13]. Another well known example is the three-dimensional Navier-Stokes equations for an incompressible viscous fluid with Dirichlet boundary conditions in a bounded domain.

We make essential use of the ideas of measurable multi-functions having values in a complete separable metric space, i.e., a Polish space, [9, Vol. 1, p.141].

Definition 2.1. Let X be a Polish space and let $(\Omega, \mathcal{F})$ be a measurable space and let $F : \Omega \to 2^X$ be a multi-function assumed to have values that are non-empty and closed sets. Then, F is said to be *measurable* if for every open set U in 2^X,

$$F^-(U) = \{\omega : F(\omega) \cap U \neq \emptyset\} \in \mathcal{F}.$$

The multi-function is said to be strongly measurable if for every closed set C in 2^X,

$$F^-(C) = \{\omega : F(\omega) \cap C \neq \emptyset\} \in \mathcal{F}.$$

One can show that strong measurability implies measurability and that measurability is sufficient to obtain the existence of a measurable selection, which is a function $\gamma(\omega)$ that is $\mathcal{F}$ measurable and $\gamma(\omega) \in F(\omega)$ for each ω. In the case when the values of F are compact sets, it can be shown that the two versions of measurability are equivalent (the proof can be found in [9, Vol. 1, p.143]).

We now introduce some notation. We describe randomness by a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the *sample space* with elements ω, $\mathcal{F}$ is a given σ-algebra of subsets of Ω, and P is the *probability measure* on $\mathcal{F}$. The usual Borel σ-algebra of open sets in $[0, T]$ is denoted by $\mathcal{B}([0, T])$ and $P = \mu_L$ is the usual Lebesgue measure. Next, $C = C(\alpha, \dots, \beta)$ denotes a positive constant that depends only on the problem data and on $\alpha, \dots, \beta$, whose value may change from place to place. Also, $C^{0,1}([0, T])$ denotes the Hölder space with $\gamma = 1$, so that the norm is

$$\|f\|_{0,1} = \sup_{t\in[0,T]} |f(t)| + \sup\Big\{\frac{|f(t) - f(s)|}{|t - s|} : s \neq t\Big\}.$$

The following abstract theorem is the main result in this work.

Theorem 2.2. *Let V be a reflexive separable Banach space with dual V', and let p, p' be such that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and $\omega \in \Omega$. Let the functions $t \to u_n(t, \omega)$, for $n \in \mathbb{N}$, be in $L^{p'}([0, T]; V')$ and $(t, \omega) \to u_n(t, \omega)$ be $\mathcal{B}([0, T]) \times \mathcal{F} \equiv \mathcal{P}$ measurable into V'. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then*

$$\sup_{t\in[0,T]} \|u_n(t, \omega)\|_{V'} \leq C(\omega), \tag{2.1}$$

for all n. Also, suppose for each $\omega \notin N$, each subsequence of $\{u_n\}$ has a further subsequence that converges weakly in $L^{p'}([0,T];V')$ to $v(\cdot,\omega) \in L^{p'}([0,T];V')$ such that the function $t \to v(t,\omega)$ is weakly continuous into V'.

Then, there exists a product measurable function u such that $t \to u(t,\omega)$is weakly continuous into V' for each $\omega \notin N$. Moreover, there exists, for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot,\omega) \to u(\cdot,\omega)$ weakly in $L^{p'}([0,T];V')$.

We prove the theorem in steps given below. We let $X = \prod_{k=1}^{\infty} C([0,T])$ and note that when it is equipped with the product topology, then one can consider X as a metric space using the metric

$$d(\mathbf{f},\mathbf{g}) \equiv \sum_{k=1}^{\infty} 2^{-k} \frac{\|f_k - g_k\|}{1 + \|f_k - g_k\|},$$

where $\mathbf{f} = (f_1, f_2, \dots)$, $\mathbf{g} = (g_1, g_2, \dots) \in X$, and the norm is the maximum norm in $C([0,T])$. With this metric, X is complete and separable.

The next lemma claims that if $\{\mathbf{f}_n\}$ has each component bounded in $C^{0,1}([0,T])$ then it is pre-compact in X.

Lemma 2.3. *Let $\{\mathbf{f}_n\}$ be a sequence in X and suppose that each one of the components f_{nk} is bounded in $C^{0,1}([0,T])$ by $C = C(k)$. Then, there exists a subsequence $\{\mathbf{f}_{n_j}\}$ that converges to some $\mathbf{f} \in X$ as $n_j \to \infty$. Thus, $\{\mathbf{f}_n\}$ is pre-compact in X.*

Proof. By the Ascoli-Arzela theorem, there exists a subsequence $\{\mathbf{f}_{n_1}\}$ such that the sequence of the first components $f_{n_1 1}$ converges in $C([0,T])$. Then, taking a subsequence, one can obtain $\{n_2\}$ a subsequence of $\{n_1\}$ such that both the first and second components of $\mathbf{f}_{n_2}$ converge. Continuing in this way one obtains a sequence of subsequences, each a subsequence of the previous one such that $\mathbf{f}_{n_j}$ has the first j components converging to functions in $C([0,T])$. Therefore, the diagonal subsequence has the property that it has every component converging to a function in $C([0,T])$. The resulting function is $\mathbf{f} \in \prod_k C([0,T])$. □

Now, for $m \in \mathbb{N}$ and $\phi \in V'$, define $l_m(t) \equiv \max(0, t - (1/m))$ and $\psi_{m,\phi} : L^{p'}([0,T];V') \to C([0,T])$ by

$$\psi_{m,\phi} u(t) \equiv \int_0^T \langle m\phi \mathcal{X}_{[l_m(t),t]}(s), u(s)\rangle_V ds = m \int_{l_m(t)}^t \langle \phi, u(s)\rangle_V ds.$$

Here, $\mathcal{X}_{[l_m(t),t]}(\cdot)$ is the characteristic function of the interval $[l_m(t),t]$ and $\langle \cdot,\cdot\rangle_V$ is the duality pairing between V and V'.

Let $\mathcal{D} = \{\phi_r\}_{r=1}^{\infty}$ denote a countable dense subset of V. Then, the pairs $(m,\phi) \in \mathbb{N} \times \mathcal{D}$ form a countable set, and let (m_k, ϕ_{r_k}) denote an enumeration of these pairs. To simplify the notation, we set

$$f_k(u)(t) \equiv \psi_{m_k,\phi_{r_k}}(u)(t) = m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u(s)\rangle_V ds.$$

For fixed $\omega \notin N$ and k, the functions $\{t \to f_k(u_j(\cdot,\omega))(t)\}_j$ are uniformly bounded and equicontinuous because they are in $C^{0,1}([0,T])$. Indeed, we have

$$|f_k(u_j(\cdot,\omega))(t)| = \left| m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u_j(s,\omega)\rangle_V ds \right| \le C(\omega) \|\phi_{r_k}\|_V,$$

and for $t \le t'$,

$$\begin{aligned} &|f_k(u_j(\cdot,\omega))(t) - f_k(u_j(\cdot,\omega))(t')| \\ &\le \Big| m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u_j(s,\omega)\rangle_V ds - m_k \int_{l_{m_k}(t')}^{t'} \langle \phi_{r_k}, u_j(s,\omega)\rangle_V \, ds \Big| \\ &\le 2m_k |t' - t| C(\omega) \|\phi_{r_k}\|_{V'}. \end{aligned}$$

By Lemma 2.3, the set of functions $\{\mathbf{f}(u_j(\cdot,\omega))\}_{j=n}^{\infty}$ is pre-compact in the space $X = \prod_k C([0,T])$. We now define a set valued map $\Gamma^n : \Omega \to X$ by

$$\Gamma^n(\omega) \equiv \overline{\cup_{j\ge n}\{\mathbf{f}(u_j(\cdot,\omega))\}},$$

where the closure is taken in X. Then, $\Gamma^n(\omega)$ is the closure of a pre-compact set in X and so $\Gamma^n(\omega)$ is compact in X. From the definition, a function $\mathbf{f}$ is in $\Gamma^n(\omega)$ if and only if $d(\mathbf{f}, \mathbf{f}(w_l)) \to 0$ as $l \to \infty$, where each w_l is one of the $u_j(\cdot,\omega)$ for $j \ge n$. In the topology on X, this happens if and only if for every k,

$$f_k(t) = \lim_{l\to\infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, w_l(s,\omega)\rangle_V ds,$$

where the limit is the uniform limit in t.

Lemma 2.4. *The mapping $\omega \to \Gamma^n(\omega)$ is an $\mathcal{F}$ measurable set-valued map with values in X. If σ is a measurable selection, then for each t, $\omega \to \sigma(t,\omega)$ is $\mathcal{F}$ measurable and $(t,\omega) \to \sigma(t,\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}$ measurable.*

We note that if σ is a measurable selection then $\sigma(\omega) \in \Gamma^n(\omega)$, so $\sigma = \sigma(\cdot,\omega)$ is a continuous function. To have σ measurable means that $\sigma_k^{-1}(\text{open}) \in \mathcal{F}$, where the open set is in $C([0,T])$.

Proof of Lemma 2.4. Let O be a basic open set in X so that $O = \prod_{k=1}^{\infty} O_k$, where O_k is a proper open set of $C([0,T])$ only for $k \in \{k_1, \cdots, k_r\}$, while in the rest of the components the open set is the whole space $C([0,T])$. We need to show that

$$\Gamma^{n-}(O) \equiv \{\omega : \Gamma^n(\omega) \cap O \ne \emptyset\} \in \mathcal{F}.$$

Now, $\Gamma^{n-}(O) = \cap_{i=1}^{r}\{\omega : \Gamma^n(\omega)_{k_i} \cap O_{k_i} \ne \emptyset\}$, so we consider whether

$$\{\omega : \Gamma^n(\omega)_{k_i} \cap O_{k_i} \ne \emptyset\} \in \mathcal{F}. \tag{2.2}$$

From the definition of $\Gamma^n(\omega)$, this is equivalent to the condition that $f_{k_i}(u_j(\cdot,\omega)) = (\mathbf{f}(u_j(\cdot,\omega)))_{k_i} \in O_{k_i}$ for some $j \ge n$, and so the set in (2.2) is of the form

$$\cup_{j=n}^{\infty}\{\omega : (\mathbf{f}(u_j(\cdot,\omega)))_{k_i} \in O_{k_i}\}.$$

Now $\omega \to (\mathbf{f}(u_j(\cdot,\omega)))_{k_i}$ is $\mathcal{F}$ measurable into $C([0,T])$ and so the above set is in $\mathcal{F}$. To see this, let $g \in C([0,T])$ and consider the inverse image of the ball with radius r and center g,

$$B(g,r) = \{\omega : \|(\mathbf{f}(u_j(\cdot,\omega)))_{k_i} - g\|_{C([0,T])} < r\}.$$

By continuity considerations,

$$\|(\mathbf{f}(u_j(\cdot,\omega)))_{k_i} - g\|_{C([0,T])} = \sup_{t\in\mathbb{Q}\cap[0,T]} |(\mathbf{f}(u_j(t,\omega)))_{k_i} - g(t)|,$$

which is the supremum over countably many $\mathcal{F}$ measurable functions and so it is $\mathcal{F}$ measurable. Since every open set is the countable union of such balls, the $\mathcal{F}$

measurability follows. Hence, $\Gamma^{n-}(O)$ is $\mathcal{F}$ measurable whenever O is a basic open set.

Now, X is a separable metric space and so every open set is a countable union of these basic sets. Let $U \subseteq X$ be open with $U = \cup_{l=1}^{\infty} O_l$ where O_l is such a basic open set. Then,

$$\Gamma^{n-}(U) = \cup_{l=1}^{\infty} \Gamma^{n-}(O_l) \in \mathcal{F}.$$

The existence of a measurable selection follows from the standard theory of measurable multi-functions [3] or [9, Vol. 1, Page 141]. If σ is one of these measurable selections, the evaluation at t is $\mathcal{F}$ measurable. Thus, $\omega \to \sigma(t,\omega)$ is $\mathcal{F}$ measurable with values in $\mathbb{R}^\infty$. Also, $t \to \sigma(t,\omega)$ is continuous, and so it follows that in fact σ is product measurable as claimed. □

Definition 2.5. Let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty} \Gamma^n(\omega)$.

Lemma 2.6. Γ *is a nonempty* $\mathcal{F}$ *measurable set-valued function with values in compact subsets of* X*. There exists a measurable selection* γ *such that* $(t,\omega) \to \gamma(t,\omega)$ *is* $\mathcal{P}$ *measurable. Also, for each* ω*, there exists a subsequence,* $u_{n(\omega)}(\cdot,\omega)$ *such that for each* k*,*

$$\gamma_k(t,\omega) = \lim_{n(\omega)\to\infty} \mathbf{f}(u_{n(\omega)}(t,\omega))_k = \lim_{n(\omega)\to\infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u_{n(\omega)}(s,\omega)\rangle_V ds.$$

Proof. From the definition of $\Gamma(\omega) = \cap_{n=1}^{\infty} \Gamma^n(\omega)$ it follows that $\omega \to \Gamma(\omega)$ is a compact set-valued map in X and is nonempty because each $\Gamma^n(\omega)$ is nonempty and compact, and the $\Gamma^n(\omega)$ are nested. We next show that $\omega \to \Gamma(\omega)$ is $\mathcal{F}$ measurable. Indeed, each Γ^n is compact valued and $\mathcal{F}$ measurable so, if F is closed,

$$\Gamma(\omega) \cap F = \cap_{n=1}^{\infty} \Gamma^n(\omega) \cap F,$$

and the left-hand side is not empty iff each $\Gamma^n(\omega) \cap F \neq \emptyset$. Thus, for F closed,

$$\{\omega : \Gamma(\omega) \cap F \neq \emptyset\} = \cap_n \{\omega : \Gamma^n(\omega) \cap F \neq \emptyset\},$$

and so

$$\Gamma^{-}(F) = \cap_n \Gamma^{n-}(F) \in \mathcal{F}.$$

The last claim follows from the theory of multi-functions, see, e.g., [3, 9]. The fact that $\Gamma^n(\omega)$ is compact implies that strong measurability and measurability coincide, [9, Vol. 1, p.143]. Thus, Γ^n is measurable and $\Gamma^{n-}(U) \in \mathcal{F}$, for U open, implies $\Gamma^{n-}(F) \in \mathcal{F}$ for F closed. Thus, $\omega \to \Gamma(\omega)$ is nonempty compact valued in X and strongly $\mathcal{F}$ measurable.

Standard theory, [9, Vol. 1, pp 141-2], also guarantees the existence of an $\mathcal{F}$ measurable selection $\omega \to \gamma(\omega)$ with $\gamma(\omega) \in \Gamma(\omega)$, for each ω, and also that $t \to \gamma_k(t,\omega)$ (the kth component of γ) is continuous. Next, we consider the product measurability of γ_k. We know that $\omega \to \gamma_k(\omega)$ is $\mathcal{F}$ measurable into $C([0,T])$ and since pointwise evaluation is continuous, $\omega \to \gamma_k(t,\omega)$ is $\mathcal{F}$ measurable. (Indeed, a continuous function of a measurable function is measurable.) Then, since $t \to \gamma_k(t,\omega)$ is continuous, it follows that γ_k is a $\mathcal{P}$ measurable real valued function and that γ is a $\mathcal{P}$ measurable $\mathbb{R}^\infty$ valued function. Since $\gamma(\omega) \in \Gamma(\omega)$, it follows that for each $n, \gamma(\omega) \in \Gamma^n(\omega)$. Hence, there exists $j_n \geq n$ such that for each ω,

$$d(\mathbf{f}(u_{j_n}(\cdot,\omega)), \gamma(\omega)) < 2^{-n}.$$

Therefore, for a suitable subsequence $\{u_{n(\omega)}(\cdot,\omega)\}$, we have

$$\gamma(\omega) = \lim_{n(\omega)\to\infty} \mathbf{f}(u_{n(\omega)}(\cdot,\omega)).$$

for each ω. In particular, for each k and for each t, we have

$$\gamma_k(t,\omega) = \lim_{n(\omega)\to\infty} \mathbf{f}(u_{n(\omega)}(t,\omega))_k = \lim_{n(\omega)\to\infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u_{n(\omega)}(s,\omega)\rangle_V ds, \quad (2.3)$$

□

Note that it is not clear that $(t,\omega) \to \mathbf{f}(u_{n(\omega)}(t,\omega))$ is $\mathcal{P}$ measurable, although $(t,\omega) \to \gamma(t,\omega)$ is $\mathcal{P}$ measurable.

We have now all the ingredients needed to prove the theorem.

Proof of Theorem 2.2. By assumption, there exists a subsequence, still denoted by $n(\omega)$, such that, in addition to (2.3), the weak limit $\lim_{n(\omega)\to\infty} u_{n(\omega)}(\cdot,\omega) = u(\cdot,\omega)$ exists in $L^{p'}([0,T];V')$ such that $t \to u(t,\omega)$ is weakly continuous into V'. Then, (2.3) also holds for this further subsequence and in addition,

$$m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u(s,\omega)\rangle_V ds = \lim_{n(\omega)\to\infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u_{n(\omega)}(s,\omega)\rangle_V ds = \gamma_k(t,\omega).$$

Let $\phi \in \mathcal{D}$ be given, then there exists a subsequence, denoted by k, such that $m_k \to \infty$ and $\phi_{r_k} = \phi$. (Recall that (m_k, ϕ_{r_k}) denotes an enumeration of the pairs $(m,\phi) \in \mathbb{N} \times \mathcal{D}$.) Then, passing to the limit and using the assumed continuity of $s \to u(s,\omega)$, the left-hand side of this equality converges to $\langle \phi, u(s,\omega)\rangle_V$ and so the right-hand side, $\gamma_k(t,\omega)$, must also converge and for each ω. Since the right-hand side is a product measurable function of (t,ω), it follows that the pointwise limit is also product measurable. Hence, $(t,\omega) \to \langle \phi, u(t,\omega)\rangle_V$ is product measurable for each $\phi \in \mathcal{D}$. Since $\mathcal{D}$ is a dense set, it follows that $(t,\omega) \to \langle \phi, u(t,\omega)\rangle_V$ is $\mathcal{P}$ measurable for all $\phi \in V$ and so by the Pettis theorem, [27], $(t,\omega) \to u(t,\omega)$ is $\mathcal{P}$ measurable into V'. This completes the proof. □

Actually, one can say more about the measurability of the approximating sequence and in fact, we can obtain one for which $\omega \to u_{n(\omega)}(t,\omega)$ is also $\mathcal{F}$ measurable.

Lemma 2.7. *Suppose that $u_{n(\omega)} \to u$ weakly in $L^{p'}([0,T];V')$, where u is product measurable, and $\{u_{n(\omega)}\}$ is a subsequence of $\{u_n\}$, such that there exists a set of measure zero $N \subseteq \Omega$ and*

$$\sup_{t\in[0,T]} \|u_n(t,\omega)\|_{V'} < C(\omega), \quad \text{for } \omega \notin N.$$

Then, there exists a subsequence of $\{u_n\}$, denoted as $\{u_{k(\omega)}\}$, such that $u_{k(\omega)} \to u$ weakly in $L^{p'}([0,T];V')$, $\omega \to k(\omega)$ is $\mathcal{F}$ measurable, and $\omega \to u_{k(\omega)}(t,\omega)$ is also $\mathcal{F}$ measurable, for each $\omega \notin N$.

We introduce the notation

$$\mathcal{V} \equiv L^p([0,T];V), \quad \mathcal{V}' \equiv L^{p'}([0,T];V').$$

Proof. Assume that $f, g \in \mathcal{V}'$ and let $\{\phi_k\}$ be a countable dense subset of $\mathcal{V}$. Then, a bounded set in $\mathcal{V}'$ with the weak topology can be considered a complete metric space using the metric

$$d(f,g) \equiv \sum_{j=1}^{\infty} 2^{-j} \frac{|\langle \phi_k, f-g\rangle_{\mathcal{V}}|}{1+|\langle \phi_k, f-g\rangle_{\mathcal{V}}|}.$$

Now, let $k(\omega)$ be the first index of $\{u_n\}$ that is at least as large as k and such that

$$d(u_{k(\omega)}, u) \le 2^{-k}.$$

Such an index exists because there exists a convergent sequence $u_{n(\omega)}$ that converges weakly to u. In fact,

$$\{\omega : k(\omega) = l\} = \{\omega : d(u_l, u) \le 2^{-k}\} \cap \cap_{j=1}^{l-1} \{\omega : d(u_j, u) > 2^{-k}\}.$$

Since u is product measurable and each u_l is also product measurable, these are all measurable sets with respect to $\mathcal{F}$ and so $\omega \to k(\omega)$ is $\mathcal{F}$ measurable. Now, we have that $u_{k(\omega)} \to u$ weakly in $L^{p'}([0,T]; V')$, for each ω, and each function is $\mathcal{F}$ measurable because

$$u_{k(\omega)}(t,\omega) = \sum_{j=1}^{\infty} \mathcal{X}_{[k(\omega)=j]} u_j(t,\omega),$$

and every term in the sum is $\mathcal{F}$ measurable. □

Finally, when all the functions have values in a separable Hilbert space H, the same arguments yield the following theorem noting that the norms in (2.1) and (2.4) are different.

Theorem 2.8. *Let H be a real separable Hilbert space. Let the functions $t \to u_n(t,\omega)$, for $n \in \mathbb{N}$, be in $L^2([0,T]; H)$ and $(t,\omega) \to u_n(t,\omega)$ be $\mathcal{B}([0,T]) \times \mathcal{F} \equiv \mathcal{P}$ measurable into H. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then for all n,*

$$\sup_{t\in[0,T]} |u_n(t,\omega)|_H \le C(\omega). \tag{2.4}$$

Further, suppose that for each $\omega \notin N$, each subsequence of $\{u_n\}$ has a subsequence that converges weakly in $L^2([0,T]; H)$ to $u(\cdot,\omega) \in L^2([0,T]; H)$ such that $t \to u(t,\omega)$ is weakly continuous into H. Then, there exists a product measurable function u such that $t \to u(t,\omega)$ is weakly continuous into H. Moreover, there exists, for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot,\omega) \to u(\cdot,\omega)$ weakly in $L^2([0,T]; H)$.

3. Measurability for delay-differential equations

In this section we use our main theorem to establish a Peano-type existence theorem that provides a solution of the differential equation that retains its product measurability. In particular, this result applies to general second order ordinary differential equations with one delay. It is an interesting example of the above theory and will be used in the Section 4 to show the convergence of the Galerkin method. Moreover, although the material on filtrations is not needed below, we include it because it is of interest and will be used in the future. We note that a *filtration* on $[0,T]$ consists of a family of σ-algebras with $\mathcal{F}_t$ for each $t \in [0,T]$ such that for $s < t, \mathcal{F}_s \le \mathcal{F}_t$. In applications to stochastic integration, $\mathcal{F}_t$ is often chosen as $\sigma(W(s) : s \le t)$, the smallest σ-algebra for which each one of the $W(s)$ is measurable, where $t \to W(t)$ is a Wiener process.

We recall that $\mathcal{P} \equiv \mathcal{B}([0,T]) \times \mathcal{F}$, Ω is the sample space, and $M(\omega)$ and $C(\omega)$ represent constants that depend only on the problem data and ω.

Our first result deals with the case when $\mathbf{N}$ and $\mathbf{f}$ are bounded functions. We assume that for fixed $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^d$,

$$(t,\omega) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega) \tag{3.1}$$

is product measurable and that $(t,\mathbf{u},\mathbf{v},\mathbf{w}) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$ is continuous. We make this assumption so that if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are each product measurable functions, then so is

$$(t,\omega) \to \mathbf{N}(t,\mathbf{u}(t,\omega),\mathbf{v}(t,\omega),\mathbf{w}(t,\omega),\omega).$$

This follows by using approximations with simple functions.

Our result is as follows.

Theorem 3.1. *Suppose that the function $\mathbf{N} : [0,T] \times \mathbb{R}^{3d} \times \Omega \to \mathbb{R}^d$ is such that for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, $t \in [0,T]$ and $\omega \in \Omega$ the mapping $(t,\omega) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$ is product measurable. Also, suppose that the mapping $(t,\mathbf{u},\mathbf{v},\mathbf{w}) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$ is continuous and that*

$$|\mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)| \le M(\omega), \tag{3.2}$$

uniformly in $(t,\mathbf{u},\mathbf{v},\mathbf{w})$. Let $\mathbf{f}$ be $\mathcal{P}$ measurable and $\mathbf{f}(\cdot,\omega) \in L^2([0,T];\mathbb{R}^d)$.

Then, for $h \ge 0$, there exists a $\mathcal{P}$ measurable solution $\mathbf{u}$ to the integral equation

$$\mathbf{u}(t,\omega) + \int_0^t \mathbf{N}(s,\mathbf{u}(s,\omega),\mathbf{u}(s-h,\omega),\mathbf{w}(s,\omega),\omega)ds = \mathbf{u}_0(\omega) + \int_0^t \mathbf{f}(s,\omega)ds. \tag{3.3}$$

Here, $\mathbf{u}_0$ has values in $\mathbb{R}^d$ and is $\mathcal{F}$ measurable, $\mathbf{u}(s-h,\omega) = \mathbf{u}_0(\omega)$ if $s-h<0$ and for $\mathbf{w}_0$ a given $\mathcal{F}$ measurable function,

$$\mathbf{w}(t,\omega) \equiv \mathbf{w}_0(\omega) + \int_0^t \mathbf{u}(s,\omega)ds.$$

Proof. The proof is based on the use of the delay operator τ_δ defined as follows. For $\delta > 0$, we let

$$\tau_\delta \mathbf{u}(s) \equiv \begin{cases} \mathbf{u}(s-\delta) & \text{if } s > \delta, \\ \mathbf{0} & \text{if } s-\delta \le 0. \end{cases}$$

Now, let $\mathbf{u}_n$ be the solution of the equation

$$\begin{aligned} &\mathbf{u}_n(t,\omega) + \int_0^t \mathbf{N}(s,\tau_{1/n}\mathbf{u}_n(s,\omega),\mathbf{u}_n(s-h,\omega),\tau_{1/n}\mathbf{w}_n(s,\omega),\omega)ds \\ &= \mathbf{u}_0(\omega) + \int_0^t \mathbf{f}(s,\omega)ds. \end{aligned}$$

It follows that $(t,\omega) \to \mathbf{u}_n(t,\omega)$ is $\mathcal{P}$ measurable. The assumptions on $\mathbf{N}$ guarantee that for a fixed ω the family of functions $\{\mathbf{u}_n(\cdot,\omega)\}$ is uniformly bounded, indeed,

$$\sup_{t\in[0,T]} |\mathbf{u}_n(t,\omega)| \le |\mathbf{u}_0(\omega)| + \int_0^T M(\omega)ds + \int_0^T |\mathbf{f}(s,\omega)|ds \equiv C(\omega).$$

It is also equicontinuous since for $s < t$,

$$\begin{aligned} |\mathbf{u}_n(t,\omega) - \mathbf{u}_n(s,\omega)| \le &\int_s^t |\mathbf{N}(r,\tau_{1/n}\mathbf{u}_n(r,\omega),\mathbf{u}_n(r-h,\omega),\tau_{1/n}\mathbf{w}_n(r,\omega),\omega)|dr \\ &+ \int_s^t |\mathbf{f}(r,\omega)|dr \end{aligned}$$

$$\le C(\omega, \mathbf{f})|t-s|^{1/2}.$$

Therefore, by the Ascoli-Arzela theorem, for each ω there exist a subsequence $\tilde{n}(\omega)$, which depends on ω, and a function $\tilde{\mathbf{u}}(t,\omega)$ such that

$$\mathbf{u}_{\tilde{n}(\omega)}(t,\omega) \to \tilde{\mathbf{u}}(t,\omega) \text{ uniformly in } C([0,T];\mathbb{R}^d).$$

This verifies that the assumptions of Theorem 2.8 hold. It follows that there exists a function $\bar{\mathbf{u}}$ that is product measurable and a subsequence $\{\mathbf{u}_{n(\omega)}\}$, for each ω, such that

$$\lim_{n(\omega)\to\infty} \mathbf{u}_{n(\omega)}(\cdot,\omega) = \bar{\mathbf{u}}(\cdot,\omega) \text{ weakly in } L^2([0,T];\mathbb{R}^d)$$

and that $t \to \bar{\mathbf{u}}(t,\omega)$ is continuous, since weak continuity is the same as continuity in $\mathbb{R}^d$. The same argument given above applied to the $\mathbf{u}_{n(\omega)}$, for a fixed ω, yields a further subsequence, denoted as $\{\mathbf{u}_{\bar{n}(\omega)}(\cdot,\omega)\}$ which converges uniformly to a function $\mathbf{u}(\cdot,\omega)$ on $[0,T]$. So $\bar{\mathbf{u}}(t,\omega) = \mathbf{u}(t,\omega)$ in $L^2([0,T];\mathbb{R}^d)$. Since both of these functions are continuous in t, they must be equal for all t. Hence, $(t,\omega) \to \mathbf{u}(t,\omega)$ is product measurable. Passing to the limit in the equation solved by $\{\mathbf{u}_{\bar{n}(\omega)}(\cdot,\omega)\}$ and using the dominated convergence theorem, we obtain

$$\mathbf{u}(t,\omega) - \mathbf{u}_0(\omega) + \int_0^t \mathbf{N}(s, \mathbf{u}(s,\omega), \mathbf{u}(s-h,\omega), \mathbf{w}(s,\omega), \omega)ds = \int_0^t \mathbf{f}(s,\omega)ds.$$

Thus $t \to \mathbf{u}(t,\omega)$ is a product measurable solution of the integral equation. □

The theorem provides the existence of the approximate solutions needed in the next theorem in which the assumption that the integrand is bounded is replaced with an appropriate estimate. However, first we mention the following elementary lower-bound inequality that is used below.

Lemma 3.2. *Assume that* $\mathbf{w}(t) = \mathbf{w}_0(\omega) + \int_0^t \mathbf{u}(s,\omega)ds$, *define* $\mathbf{v}$ *as*

$$\mathbf{v}(t) = \begin{cases} \mathbf{u}(t-h) & \text{if } t \ge h, \\ \mathbf{u}_0 & \text{otherwise,} \end{cases}$$

and that the following estimate holds true,

$$(\mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega),\mathbf{u}) \ge -C(t,\omega) - \mu(|\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2).$$

Then,

$$\int_0^t (\mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega),\mathbf{u})ds \ge -C\Big(C(\omega) + \int_0^t |\mathbf{u}|^2 ds\Big),$$

for some constant C *depending on the initial data but not on* $\mathbf{u}$.

Proof. We have

$$\begin{aligned} \int_0^t |\mathbf{u}(s-h)|^2 ds &= \int_0^h |\mathbf{u}_0|^2 ds + \int_h^t |\mathbf{u}(s-h)|^2 ds \\ &= |\mathbf{u}_0|^2 h + \int_0^{t-h} |\mathbf{u}(s)|^2 ds \\ &\le |\mathbf{u}_0|^2 h + \int_0^t |\mathbf{u}(s)|^2 ds, \end{aligned}$$

when $t \geq h$ and when $s < h$, the integral is dominated by

$$|\mathbf{u}_0|^2 t \leq |\mathbf{u}_0|^2 h \leq |\mathbf{u}_0|^2 h + \int_0^t |\mathbf{u}(s)|^2 ds.$$

Next, using the usual inequalities yields

$$\begin{aligned}
\int_0^t |\mathbf{w}(s)|^2 ds &\leq \int_0^t \big|\mathbf{w}_0 + \int_0^s \mathbf{u}(r)dr\big|^2 ds \\
&\leq \int_0^t \Big(|\mathbf{w}_0|^2 + 2|\mathbf{w}_0|\big|\int_0^s \mathbf{u}(r)dr\big| + \big|\int_0^s \mathbf{u}(r)dr\big|^2\Big) ds \\
&\leq T|\mathbf{w}_0|^2 + T|\mathbf{w}_0|^2 + \int_0^t \big|\int_0^s \mathbf{u}(r)dr\big|^2 ds + \int_0^t \big|\int_0^s \mathbf{u}(r)dr\big|^2 ds \\
&\leq 2T|\mathbf{w}_0|^2 + 2\int_0^t \Big(\int_0^s |\mathbf{u}(r)|dr\Big)^2 ds \\
&\leq 2T|\mathbf{w}_0|^2 + 2\int_0^t s\int_0^s |\mathbf{u}(r)|^2 dr ds \\
&\leq 2T|\mathbf{w}_0|^2 + 2T^2\int_0^t |\mathbf{u}(r)|^2 dr.
\end{aligned}$$

These estimates lead directly to the claimed result. □

We now state a more general result in which $\mathbf{N}$ is only bounded from below, which is the main result in this section.

Theorem 3.3. *Suppose that the function* $\mathbf{N} : [0,T] \times \mathbb{R}^{3d} \times \Omega \to \mathbb{R}^d$ *is such that for* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, t \in [0,T]$ *and* $\omega \in \Omega$ *the mapping* $(t,\omega) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$ *is product measurable. Also, suppose*

$$(t,\mathbf{u},\mathbf{v},\mathbf{w}) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$$

is continuous, and there are a nonnegative function $C(\cdot,\omega) \in L^1([0,T])$ *and a positive constant* μ *such that*

$$(\mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega),\mathbf{u}) \geq -C(t,\omega) - \mu(|\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2). \tag{3.4}$$

Moreover, let $\mathbf{f}$ *be product measurable and* $\mathbf{f}(\cdot,\omega) \in L^2([0,T];\mathbb{R}^d)$.

Then, for each $h \geq 0$, *there exists a product measurable solution* $\mathbf{u}$ *to the integral equation*

$$\mathbf{u}(t,\omega) + \int_0^t \mathbf{N}(s,\mathbf{u}(s,\omega),\mathbf{u}(s-h,\omega),\mathbf{w}(s,\omega),\omega)ds = \mathbf{u}_0(\omega) + \int_0^t \mathbf{f}(s,\omega)ds, \tag{3.5}$$

where $\mathbf{u}_0$ *has values in* $\mathbb{R}^d$ *and is* $\mathcal{F}$ *measurable. Here,* $\mathbf{u}(s-h,\omega) \equiv \mathbf{u}_0(\omega)$ *for all* $s - h \leq 0$ *and for* $\mathbf{w}_0$ *a given* $\mathcal{F}$ *measurable function,*

$$\mathbf{w}(t,\omega) \equiv \mathbf{w}_0(\omega) + \int_0^t \mathbf{u}(s,\omega)ds$$

Proof. The idea of the proof is to bound the variables, so that $\mathbf{N}$ is bounded, use Theorem 3.1 to obtain product measure solutions, and pass to the limit when the variables are allowed be be unbounded.

Let P_m denote the projection onto the closed ball $\overline{B(\mathbf{0}, 9^m)} \subset \mathbb{R}^d$. Then, it follows from Theorem 3.1 that there exists a product measurable solution $\mathbf{u}_m$ of the integral equation

$$\begin{aligned}&\mathbf{u}_m(t,\omega) + \int_0^t \mathbf{N}(s, P_m\mathbf{u}_m(s,\omega), P_m\mathbf{u}_m(s-h,\omega), P_m\mathbf{w}_m(s,\omega),\omega)ds\\&= \mathbf{u}_0(\omega) + s\int_0^t \mathbf{f}(s,\omega)ds.\end{aligned}$$

Next, we define a stopping time

$$\tau_m(\omega) \equiv \inf\big\{t \in [0,T] : |\mathbf{u}_m(t,\omega)|^2 + |\mathbf{w}_m(t,\omega)|^2 > 2^m\big\},$$

where we use the convention that $\inf\{\emptyset\} = T$. Localizing with the stopping time,

$$\begin{aligned}&\mathbf{u}_m^{\tau_m}(t,\omega) + \int_0^t \mathcal{X}_{[0,\tau_m]}\mathbf{N}(s, \mathbf{u}_m^{\tau_m}(s,\omega), \mathbf{u}_m^{\tau_m}(s-h,\omega), \mathbf{w}_m^{\tau_m}(s,\omega),\omega)ds\\&= \mathbf{u}_0(\omega) + \int_0^t \mathcal{X}_{[0,\tau_m]}\mathbf{f}(s,\omega)ds.\end{aligned}$$

Note that the stopping time allowed to eliminate the projection operator in the equation. Then, we obtain

$$\begin{aligned}&\frac{1}{2}|\mathbf{u}_m^{\tau_m}(t,\omega)|^2\\&+ \int_0^t \Big(\mathcal{X}_{[0,\tau_m]}\mathbf{N}\big(s, \mathbf{u}_m^{\tau_m}(s,\omega), \mathbf{u}_m^{\tau_m}(s-h,\omega), \mathbf{w}_m^{\tau_m}(s,\omega),\omega\big), \mathbf{u}_m^{\tau_m}(s,\omega)\Big)ds\\&= \frac{1}{2}|\mathbf{u}_0(\omega)|^2 + \int_0^t \mathcal{X}_{[0,\tau_m]}(\mathbf{f}(s,\omega), \mathbf{u}_m^{\tau_m}(s,\omega))ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}|\mathbf{u}_m^{\tau_m}(t,\omega)|^2 &\le \int_0^t \Big(\mu\Big(|\mathbf{u}_m^{\tau_m}(s,\omega)|^2 + |\mathbf{u}_m^{\tau_m}(s-h,\omega)|^2 + |\mathbf{w}_m^{\tau_m}(s,\omega)|^2\Big)\\&\quad + C(s,\omega) + \frac{1}{2}|\mathbf{f}(s,\omega)|^2\Big)ds + \frac{1}{2}\int_0^t |\mathbf{u}_m^{\tau_m}(s,\omega)|^2 ds + \frac{1}{2}|\mathbf{u}_0(\omega)|^2.\end{aligned}$$

We note that

$$|\mathbf{u}_0|^2 h + \int_0^t |\mathbf{u}_n^{\tau_n}(s)|^2 ds \ge \int_0^t |\mathbf{u}_n^{\tau_n}(s-h,\omega)|^2 ds,$$

and

$$\begin{aligned}\int_0^t |\mathbf{w}_n^{\tau_n}(s,\omega)|^2 ds &= \int_0^t \Big|\mathbf{w}_0 + \int_0^s \mathcal{X}_{[0,\tau_n]}\mathbf{u}_n(r)dr\Big|^2 ds\\&= \int_0^t \Big|\mathbf{w}_0 + \int_0^s \mathcal{X}_{[0,\tau_n]}\mathbf{u}_n^{\tau_n}(r)dr\Big|^2 ds\\&\le C(\mathbf{w}_0(\omega)) + CT\int_0^t |\mathbf{u}_n^{\tau_n}|^2 ds.\end{aligned}$$

Using now the Gronwall inequality yields

$$\begin{aligned}|\mathbf{u}_m^{\tau_m}(t,\omega)|^2 &\le C(\mathbf{u}_0(\omega), \mathbf{w}_0(\omega), \mu, \|C(\cdot,\omega)\|_{L^1([0,T];\mathbb{R}^d)}, T, \|\mathbf{f}(\cdot,\omega)\|_{L^2([0,T];\mathbb{R}^d)})\\&= C(\omega).\end{aligned}$$

Thus, it follows from the definition of the stoping time that for a.e. $\omega, \tau_m = T$ for all m large enough, say for $m \geq M(\omega)$ where $C(\omega) \leq 2^{M(\omega)}$. Next, we define the functions

$$\mathbf{y}_n(t,\omega) \equiv \mathbf{u}_n^{\tau_n}(t,\omega),$$

which are product measurable and satisfy

$$\begin{aligned}&\mathbf{y}_n(t,\omega) + \int_0^t \mathcal{X}_{[0,\tau_n]}\mathbf{N}\Big(s, \mathbf{y}_n(s,\omega), \mathbf{y}_n(s-h,\omega), \mathbf{w}_0(\omega) + \int_0^s \mathbf{y}_n(r,\omega)dr, \omega\Big)ds \\ &= \mathbf{u}_0(\omega) + \int_0^t \mathcal{X}_{[0,\tau_n]}\mathbf{f}(s,\omega)ds.\end{aligned}$$

So each function is also continuous in t. Since $\tau_n = T$ for large enough n, it follows that

$$\begin{aligned}&\mathbf{y}_n(t,\omega) + \int_0^t \mathbf{N}\Big(s, \mathbf{y}_n(s,\omega), \mathbf{y}_n(s-h,\omega), \mathbf{w}_0(\omega) + \int_0^s \mathbf{y}_n(r,\omega)dr, \omega\Big)ds \\ &= \mathbf{u}_0(\omega) + \int_0^t \mathbf{f}(s,\omega)ds.\end{aligned}$$

Also, these functions satisfy the inequality

$$\sup_{t\in[0,T]} |\mathbf{y}_n(t,\omega)|^2 \leq C(\omega) \leq 2^{M(\omega)} < 9^{M(\omega)}, \tag{3.6}$$

where the constants on the right-hand side do not depend on n. Thus, for fixed ω, we can regard $\mathbf{N}$ as bounded and the same reasoning used in Theorem 3.1 involving the Ascoli-Arzela theorem implies that every subsequence has a further subsequence that converges to a solution of the integral equation for that ω. Hence, it is continuous into $\mathbb{R}^d$. It follows from the measurable selection theorem, Theorem 2.2, that there exists a product measurable function $\mathbf{u}$ that is continuous in t such that $\mathbf{u}(\cdot,\omega) = \lim_{n(\omega)\to\infty} \mathbf{y}_{n(\omega)}(\cdot,\omega)$ in $L^2([0,T];\mathbb{R}^d)$. By the reasoning above, there is a further subsequence, denoted the same way, for which $\lim_{n\to\infty} \mathbf{y}_{n(\omega)}$ in $C([0,T];\ \mathbb{R}^d)$ solves the integral equation for a fixed ω. Thus $\mathbf{u}$ is a product measurable solution to the integral equation (3.5) as claimed. □

We made use of estimate (3.4) in the proof of this theorem. However, all that is really needed is the following simpler condition.

Corollary 3.4. *Suppose that the function $\mathbf{N} : [0,T] \times \mathbb{R}^{3d} \times \Omega \to \mathbb{R}^d$ is such that for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, t \in [0,T]$ and $\omega \in \Omega$ the mapping $(t,\omega) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$ is product measurable. Also, suppose that $(t,\mathbf{u},\mathbf{v},\mathbf{w}) \to \mathbf{N}(t,\mathbf{u},\mathbf{v},\mathbf{w},\omega)$ is continuous. Moreover, for each ω, for each solution $\mathbf{u}(\cdot,\omega)$ of the integral equation*

$$\mathbf{u}(t,\omega) + \int_0^t \mathbf{N}(s, \mathbf{u}(s,\omega), \mathbf{u}(s-h,\omega), \mathbf{w}(s,\omega), \omega)ds = \mathbf{u}_0(\omega) + \int_0^t \mathbf{f}(s,\omega)ds, \tag{3.7}$$

there exists an estimate of the form

$$\sup_{t\in[0,T]} |\mathbf{u}(t,\omega)| \leq C(\omega) < \infty. \tag{3.8}$$

Moreover, let $\mathbf{f}$ be product measurable and $\mathbf{f}(\cdot,\omega) \in L^1([0,T];\mathbb{R}^d)$; $\mathbf{u}_0$ has values in $\mathbb{R}^d$ and is $\mathcal{F}$ measurable and $\mathbf{u}(s-h,\omega) \equiv \mathbf{u}_0(\omega)$ whenever $s-h \leq 0$; and

$$\mathbf{w}(t,\omega) \equiv \mathbf{w}_0(\omega) + \int_0^t \mathbf{u}(s,\omega)ds,$$

where $\mathbf{w}_0$ *is a given* $\mathcal{F}$ *measurable function.*

Then, for $h \geq 0$*, there exists a product measurable solution* $\mathbf{u}$ *of the integral equation* (3.5)*.*

We note that the same conclusions apply when there is no dependence of the integrand on $\mathbf{u}(s-h,\omega)$, that is, there are no delays, or on $\mathbf{w}(s,\omega)$.

4. A contact problem with friction

We apply our theoretical result, Theorem 1, to an important problem of dynamic contact with friction between a viscoelastic body and a deformable foundation in which the coefficient of friction depends on the relative slip speed. The problem without randomness was studied in [15], and the novelty here is that the gap between the body and the foundation is assumed to be a random variable and so is the foundation's velocity. These two changes make the model much more realistic since in engineering applications both can be determined only up to relatively large tolerances.

The setting of the problem is depicted in Figure 1. A viscoelastic body occupies the domain $U \subseteq \mathbb{R}^d$ (where $d = 2, 3$ in applications) that is a bounded open subset with Lipschitz boundary $\Gamma = \partial U$. The boundary Γ consists of three parts: Γ_D where a Dirichlet data is prescribed, Γ_N where a Neumann condition holds, and the potential contact surface with the foundation Γ_C. We denote by $\mathbf{n}$ the outer unit normal to U on Γ. Moreover, when the foundation is planar, we assume that it moves with velocity $\mathbf{v}^*$. We also let $d = 3$ as the 2D case is somewhat simpler. Finally, in this section μ denotes the friction coefficient, and not the Lebesgue measure.

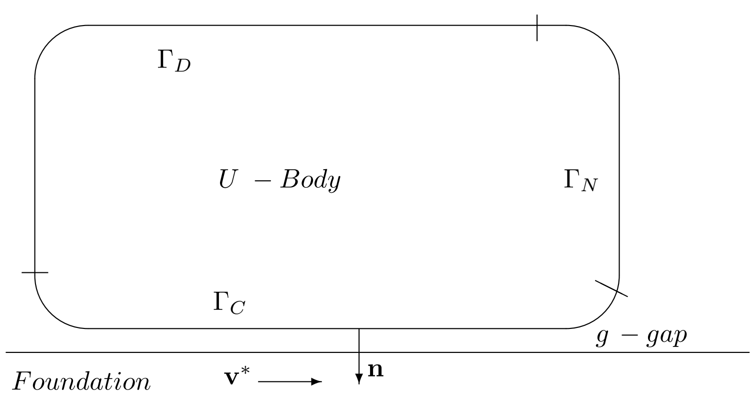

Figure 1. Γ_C is the contact surface and g is the gap

We denote by $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ the displacement vector for $\mathbf{x} \in \overline{U}$ and $t \in [0, T]$, by $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ the linearized strain tensor, and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the stress tensor; here and below $i, j, k, l = 1, 2, 3$. A dot above a symbol denotes the partial time derivative, while an index following a coma indicates partial derivative with respect to the indicated spatial variable, i.e. $u_{i,j} = \partial u_i/\partial x_j$. Moreover, summation over an index that appears twice is implied.

We assume that the material is linearly viscoelastic with short-term memory, with constitutive relation

$$\boldsymbol{\sigma} = A\boldsymbol{\varepsilon}(\mathbf{u}) + C\boldsymbol{\varepsilon}(\dot{\mathbf{u}}),$$

where $A = (a_{ijkl})$ is the elasticity tensor and $C = (c_{ijkl})$ the viscosity tensor, both described in more detail below. The body is being acted upon by the force density $\rho\mathbf{f}$, and for the sake of simplicity we rescale the variables so that the material density is $\rho = 1$. On Γ_D the body is clamped so that $\mathbf{u} = 0$, and a prescribed traction $\mathbf{t}$ acts on Γ_N.

The dynamic equations of motion and the initial and boundary conditions are as follows.

$$\ddot{\mathbf{u}} = \operatorname{Div} \boldsymbol{\sigma}(\mathbf{u}, \dot{\mathbf{u}}) + \mathbf{f}, \quad (t, \mathbf{x}) \in (0, T) \times U, \tag{4.1}$$

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x} \in U, \tag{4.2}$$

$$\mathbf{u}(t, \mathbf{x}) = 0, \quad (t, \mathbf{x}) \in (0, T) \times \Gamma_D; \quad \boldsymbol{\sigma}(t, \mathbf{x}) \cdot \mathbf{n} = \mathbf{t}, \quad (t, \mathbf{x}) \in (0, T) \times \Gamma_N. \tag{4.3}$$

Here, $\mathbf{n}$ is the outer unit normal to U on Γ.

Next, we describe the contact conditions on Γ_C. To that end we need the normal and tangential components and parts of the vectors on the surface, so we let

$$u_n = \mathbf{u} \cdot \mathbf{n}, \quad \mathbf{u}_\tau = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n},$$
$$\sigma_n = \sigma_{ij} n_j n_i, \quad \sigma_{\tau i} = \sigma_{ij} n_j - \sigma_n n_i,$$

written more simply, $\sigma_n = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}$ and $\sigma_\tau = \boldsymbol{\sigma}\mathbf{n} - \sigma_n \mathbf{n}$.

We assume that contact is described by the normal compliance condition (see, e.g., [20, 10, 11, 12, 16, 24, 25] and the references therein) and the friction process by an appropriately modified condition of the Coulomb-type, thus, for $\mathbf{x} \in \Gamma_C$, we assume

$$\sigma_n = -p(u_n - g), \tag{4.4}$$

where $p(\cdot)$ is the normal compliance function, assumed to be nonnegative and to vanish when there is no contact, i.e., when the normal displacement is less than the gap, $u_n \le g$. The friction process is in the tangential direction and there is relative motion only when the tangential traction reaches the threshold of the *friction bound* denoted by $F\mu$, where μ is the friction coefficient, described below. We refer the reader to [15, 25] for further details. The friction condition is

$$|\sigma_\tau| \le F(u_n - g)\mu(|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|), \tag{4.5}$$

$$|\sigma_\tau| = \mu(|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|)F(u_n - g) \quad \text{implies} \quad \dot{\mathbf{u}}_\tau - \mathbf{v}^* = -\lambda\sigma_\tau. \tag{4.6}$$

where $\lambda \ge 0$. Here, $\mathbf{v}^*$ is the velocity of the foundation, which is known, and the friction coefficient μ depends on the relative slip-rate, and is assumed to be a bounded positive function having a bounded continuous derivative. It is reasonable to assume that μ depends on $\mathbf{x} \in \Gamma_C$, related to the pointwise roughness of the contact surface, however, we do not make this dependence explicit for the sake of simpler notation.

The function g represents the gap between the contact surface Γ_C and a foundation along the direction $\mathbf{n}$. One of the novel ingredients in this paper is that we allow the gap to be random, which better describes real contact processes. Moreover, part of the novelty is that we do not need to make any assumption on the sample space. It is often the case that it is assumed to be the unit interval or the real line but no such assumption is needed here. Indeed, the exact form of the sample space does not enter the arguments. Therefore, in each application, one may specify the appropriate sample space Ω freely. Therefore, we do not specify Ω below. Thus,

$$g = g(t, \mathbf{x}, \omega),$$

where $\omega \in \Omega$ and we assume that $(t, \mathbf{x}, \omega) \to g(\mathbf{x}, \omega)$ is $\mathcal{B}([0,T]\times\Gamma_C)\times\mathcal{F}$ measurable, where $\mathcal{B}([0,T]\times\Gamma_C)$ denotes the Borel sets of $[0,T]\times\Gamma_C$. We assume that the gap is nonnegative (we do not consider 'shrink-fit' cases) and bounded, so

$$0 \le g(t, \mathbf{x}, \omega) \le l_* < \infty,$$

for all $(t, \mathbf{x}, \omega)$ and some l_*. Additional novelty in this work is that the motion of the foundation $\mathbf{v}^*$ is assumed to be a stochastic process

$$\mathbf{v}^* = \mathbf{v}^*(t, \mathbf{x}, \omega),$$

and is $\mathcal{B}([0,T]\times\Gamma_C)\times\mathcal{F}$ measurable. We also assume that $\mathbf{v}^*(t,\ \mathbf{x}, \omega)$ is uniformly bounded, and to simplify the notation, we suppress the dependence on $t, \mathbf{x}$ and ω.

The normal compliance contact condition (4.4) says that σ_n the normal component of the traction density on Γ_C is dependent on the normal interpenetration of the body's surface asperities into those of the foundation surface. Conditions (4.5) and (4.6) model friction. They say that no sliding takes place until $|\sigma_\tau|$ reaches the friction bound $F(u_n - g)\mu(0)$ and when this occurs, the tangential force density has a direction opposite to the relative tangential velocity (4.6). The dependence of the friction coefficient on the magnitude of the slip velocity, $|\dot{\mathbf{u}}_\tau - \mathbf{v}^*|$ is important and well documented (see, e.g., [25] and the references therein) and so it has been included.

The two new features in this model are that the gap and the foundation's velocity are random variables for each $\mathbf{x} \in \Gamma_C$. Our aim is to show the measurability of the solutions. Thus, for a fixed ω, we have a friction problem that has been studied in the literature, and it is the measurability which is of interest here.

We assume the following on the functions p and F. Both p and F are increasing and

$$\delta^2 r - K \le p(r) \le K(1+r), \quad r \ge 0,\ p(r) = 0,\ r \le 0, \tag{4.7}$$

$$F(r) \le K(1+r),\ r \ge 0, \quad F(r) = 0,\ r \le 0, \tag{4.8}$$

$$|\mu(r_1) - \mu(r_2)| \le Lip(\mu)|r_1 - r_2|, \quad \|\mu\|_\infty \le C, \tag{4.9}$$

and for $\psi = F, p$, and $r_1, r_2 \ge 0$,

$$|\psi(r_1) - \psi(r_2)| \le K|r_1 - r_2|. \tag{4.10}$$

One could consider more general growth conditions than these (see [12]), but we keep this part simple to emphasize the new stochastic features.

The stress tensor is given by

$$\sigma_{ij} = A_{ijkl} u_{k,l} + C_{ijkl} \dot{u}_{k,l}, \tag{4.11}$$

where A and C are in $L^\infty(U)$ and for $B = A$ or C, we have the following symmetries.

$$B_{ijkl} = B_{ijlk}\,, \quad B_{jikl} = B_{ijkl}\,, \quad B_{ijkl} = B_{klij}\,, \tag{4.12}$$

and we also assume that

$$B_{ijkl} H_{ij} H_{kl} \ge \varepsilon H_{rs} H_{rs} \tag{4.13}$$

for all symmetric H_{ij}.

In the rest of this section, V is a closed subspace of $(H^1(U))^3$ containing the space of test functions $(C_0^\infty(U))^3$; $\rightharpoonup$ denotes weak convergence in the case of a reflexive Banach space and weak$*$ convergence for a few examples of dual spaces that are not reflexive, while $\rightarrow$ means strong convergence; γ denotes the trace map from $W^{1,2}(U)$ into $L^2(\Gamma)$; H denotes $(L^2(U))^3$ and we always identify H and H' to write

$$V \subseteq H = H' \subseteq V',$$

so that (V, H, V') is a Gelfand triple. The duality pairing of V and V' is denoted by $\langle \cdot, \cdot \rangle_V$. We also define

$$\mathcal{V} = L^2(0,T;V), \quad \mathcal{H} = L^2(0,T,H), \quad \mathcal{V}' = L^2(0,T;V').$$

We refer to [1, 18] for standard notation and properties of Sobolev Spaces.

4.1. **The Abstract Problem.** In this subsection we derive an abstract formulation of the problem that allows us to use various tools and results from the theory of evolution equations. However, first, we recall two theorems about compact sets of functions found in Lions [19] and Simon [26], respectively, that we need below. These theorems apply for a fixed $\omega \in \Omega$.

Theorem 4.1. *Assume that the sets W, U and Y are such that $W \subseteq U \subseteq Y$, and the inclusion map of W into U is compact and the inclusion map of U into Y is continuous. Let $p \geq 1$, $q > 1$, and define*

$$S = \{\mathbf{u} \in L^p(0,T;W) : \mathbf{u}' \in L^q(0,T;Y) \text{ and } \|\mathbf{u}\|_{L^p(0,T;W)} + \|\mathbf{u}'\|_{L^q(0,T;Y)} < R\}.$$

Then, S is pre-compact in $L^p(0,T;U)$.

Theorem 4.2. *Let W, U and Y, and p, q, be as in Theorem 4.1 and let*

$$S = \{\mathbf{u} : \|\mathbf{u}(t)\|_W + \|\mathbf{u}'\|_{L^q(0,T;Y)} \leq R \quad \text{for } t \in [0,T]\}.$$

Then, S is pre-compact in $C(0,T;U)$.

Now, we obtain an abstract formulation of the problem (4.1)–(4.6). We begin by defining the operators $M, A : V \to V'$ by

$$\langle M\mathbf{u}, \mathbf{v}\rangle_V = \int_U C_{ijkl}\mathbf{u}_{k,l}\mathbf{v}_{i,j}dx, \tag{4.14}$$

$$\langle A\mathbf{u}, \mathbf{v}\rangle_V = \int_U A_{ijkl}\mathbf{u}_{k,l}\mathbf{v}_{i,j}dx. \tag{4.15}$$

Also, let the operator $\mathbf{v} \to P(\mathbf{u})$, mapping $\mathcal{V}$ into $\mathcal{V}'$, be given by

$$\langle P(\mathbf{u}), \mathbf{w}\rangle_V = \int_0^T \int_{\Gamma_C} p(u_n - g)w_n \, dSdt, \tag{4.16}$$

where dS is surface measure on Γ and

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds, \tag{4.17}$$

for $\mathbf{u}_0 \in V$. We note that P depends on $\mathbf{u}_0$ but we suppress this in favor of simpler notation. Let

$$\gamma_\tau^* : L^2\big(0,T;L^2(\Gamma_C)^3\big) \to \mathcal{V}',$$

be defined as

$$\langle \gamma_\tau^*\xi, \mathbf{w}\rangle_V \equiv \int_0^T \int_{\Gamma_C} \xi \cdot \mathbf{w}_\tau \, dSdt.$$

Finally, we assume that $\mathbf{f}(\cdot,\omega) \in L^2(0,T;V')$ and it includes the body force $\hat{\mathbf{f}}$ in U and the traction $\mathbf{t}$ on Γ_N.

The abstract form of problem (4.1)–(4.6), is as follows.

Problem $\mathbf{P}_{\rm abst}$. Find $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ and $\xi \in L^2(0,T;L^2(\Gamma_C)^3)$ such that

$$\mathbf{v}' + M\mathbf{v} + A\mathbf{u} + P\mathbf{u} + \gamma_\tau^* \xi = \mathbf{f} \quad \text{in } \mathcal{V}', \tag{4.18}$$

with $\mathbf{v}(0,\omega) = \mathbf{v}_0(\omega) \in H$, and

$$\mathbf{u}(t,\omega) = \mathbf{u}_0(\omega) + \int_0^t \mathbf{v}(s,\omega) ds, \quad \mathbf{u}_0(\omega) \in V, \tag{4.19}$$

and for all $\mathbf{w} \in \mathcal{V}$,

$$\langle \gamma_\tau^* \xi, \mathbf{w} \rangle_V \le \int_0^T \int_{\Gamma_C} F(u_n - g) \mu(|\mathbf{v}_\tau - \mathbf{v}^*|) \cdot [|\mathbf{v}_\tau - \mathbf{v}^* + \mathbf{w}_\tau| - |\mathbf{v}_\tau - \mathbf{v}^*|] \, dS \, dt. \tag{4.20}$$

We note that if $\mathbf{v}$ solves the abstract problem $\mathbf{P}_{\rm abst}$, then $\mathbf{u}$ is a weak solution of (4.1)–(4.6). As usual, other variational and stable boundary conditions can be incorporated by the appropriate choice of V and $\mathbf{f}(\cdot,\omega) \in L^2(0,T;V')$. The following is the main result for the cases with continuous friction coefficient.

Theorem 4.3. *Let $\mathbf{u}_0(\omega) \in V, \mathbf{v}_0(\omega) \in H$, for each $\omega \in \Omega$, these functions being $\mathcal{F}$ measurable. Assume that $\mathbf{f}(\cdot,\omega) \in \mathcal{V}'$, and the gap $(t,\omega) \to g(t,\omega)$ and the sliding velocity $(t,\omega) \to \mathbf{v}^*(t,\omega)$ are $\mathcal{B}([0,T]) \times \mathcal{F}$ measurable and bounded. Then, there exists a solution $(\mathbf{u}, \mathbf{v})$ to the problem (4.18)–(4.20) for each ω. This solution $(t,\omega) \to (\mathbf{u}(t,\omega), \mathbf{v}(t,\omega))$ is measurable into V, H and V'. If, in addition, the friction coefficient μ is Lipschitz continuous, then the solution is unique.*

To carry out the proofs of existence and uniqueness, we note that both M and A are coercive, nonnegative, and symmetric. That is, for two constants $\delta > 0$, $\lambda \ge 0$ they satisfy the following conditions

$$\langle B\mathbf{u}, \mathbf{u} \rangle \ge \delta^2 \|\mathbf{u}\|_W^2 - \lambda |\mathbf{u}|_H^2, \quad \langle B\mathbf{u}, \mathbf{u} \rangle \ge 0, \quad \langle B\mathbf{u}, \mathbf{v} \rangle = \langle B\mathbf{v}, \mathbf{u} \rangle, \tag{4.21}$$

for $B = M$ or A. Indeed, (4.21) is a consequence of (4.11)–(4.13) and Korn's inequality [23].

4.2. **An Approximate Problem.** To establish the theorem, we use a sequence of approximate problems that we solve using the Galerkin method. To that end, we first regularize the friction condition, which has a subgradient form. We approximate the norm function $\gamma(\mathbf{r}) = |\mathbf{r}|$ with the function

$$\Psi_\varepsilon(\mathbf{r}) = \sqrt{|\mathbf{r}|^2 + \varepsilon},$$

which is convex, Lipschitz continuous, and has bounded derivative, and it converges uniformly to $\gamma(\mathbf{r}) = |\mathbf{r}|$ on $\mathbb{R}$ as $\varepsilon \to 0$, moreover,

$$|\Psi_\varepsilon(\mathbf{x}) - \Psi_\varepsilon(\mathbf{y})| \le |\mathbf{x} - \mathbf{y}|, \quad |\Psi_\varepsilon'(\mathbf{t})| \le 1.$$

Furthermore, Ψ_ε' is Lipschitz continuous with Lipschitz constant $C/\sqrt{\varepsilon}$, where Ψ_ε' denotes the gradient or Frechet derivative of the scalar valued function.

The approximate problem to which we apply the Galerkin method is obtained by replacing the friction condition (4.20) with its regularization, and is as follows.

Problem $\mathbf{P}_\varepsilon$. Find $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ such that

$$\mathbf{v}' + M\mathbf{v} + A\mathbf{u} + P\mathbf{u} + \gamma_\tau^* F(u_n - g(\omega)) \mu(|\mathbf{v}_\tau - \mathbf{v}^*(\omega)|) \Psi_\varepsilon'(\mathbf{v}_\tau - \mathbf{v}^*(\omega)) = \mathbf{f} \tag{4.22}$$

in $\mathcal{V}'$, with $\mathbf{v}(0) = \mathbf{v}_0 \in H$, where

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds, \quad \mathbf{u}_0 \in V. \tag{4.23}$$

Here, we indicate that the gap and the velocity of the foundation are random variables depending on $\omega \in \Omega$, which is fixed, and the approximate friction operator is defined for $\mathbf{w} \in V$ in the following manner,

$$\begin{aligned}&\langle \gamma_\tau^* F(u_n - g(\omega))\mu(|\mathbf{v}_\tau - \mathbf{v}^*(\omega)|)\Psi_\varepsilon'(\mathbf{v}_\tau - \mathbf{v}^*(\omega)), \mathbf{w}\rangle \\ &= \int_{\Gamma_C} F(u_n - g(\omega))\mu(|\mathbf{v}_\tau - \mathbf{v}^*(\omega)|)\Psi_\varepsilon'(\mathbf{v}_\tau - \mathbf{v}^*(\omega)) \cdot \mathbf{w}_\tau \, dS.\end{aligned}$$

Let R denote the Riesz map from V to V' defined by $\langle R\mathbf{u}, \mathbf{v}\rangle_V = (\mathbf{u}, \mathbf{v})_H$. Then, $R^{-1} : H \to V$ is a compact and self-adjoint operator and so there exists a complete orthonormal basis $\{\mathbf{e}_k\}$ for H, such that $\{\mathbf{e}_k\} \subseteq V$ and

$$R\mathbf{e}_k = \lambda_k \mathbf{e}_k,$$

where $\lambda_k \to \infty$. Let $V_n = \operatorname{span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then, $\cup_n V_n$ is dense in H and is also dense in V, and $\{\mathbf{e}_k\}$ is an orthogonal set in V. Indeed, we have

$$0 = (\mathbf{e}_k, \mathbf{e}_l)_H = \frac{1}{\lambda_k}(R\mathbf{e}_k, \mathbf{e}_l)_H = \frac{1}{\lambda_k}\langle R\mathbf{e}_k, \mathbf{e}_l\rangle = \frac{1}{\lambda_k}(\mathbf{e}_l, \mathbf{e}_k)_V, \quad k \neq l.$$

Next, to show that $\cup_n V_n$ is dense in V, assume that this is not so, then there exists $f \in V'$, $f \neq 0$, such that $\cup_n V_n$ is in $\ker(f)$. But $f = R\mathbf{u}$, for some $\mathbf{u}$, and so

$$0 = \langle R\mathbf{u}, \mathbf{e}_k\rangle = \langle R\mathbf{e}_k, \mathbf{u}\rangle_V = \lambda_k(\mathbf{e}_k, \mathbf{u})_H,$$

for all $\mathbf{e}_k$ and so $\mathbf{u} = \mathbf{0}$ by the density of $\cup_n V_n$ in H, and hence $R\mathbf{u} = 0 = f$ after all, a contradiction.

Now, we apply the Galerkin method to Problem $\mathbf{P}_\varepsilon$. Let

$$\mathbf{v}_k(t,\omega) = \sum_{j=1}^k x_j(t,\omega)\mathbf{e}_j, \quad \mathbf{u}_k(t,\omega) = \mathbf{u}_0 + \int_0^t \mathbf{v}_k(s,\omega)ds,$$

and let $\mathbf{v}_k$ be the solution to the following integral equation, for each ω and $j \leq k$. We now suppress the dependence on ω to simplify the notation, unless it is needed.

$$\begin{aligned}&\Big\langle \mathbf{v}_k(t) - \mathbf{v}_{0k} + \int_0^t M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k \\ &+ \gamma_\tau^* F(u_{kn} - g)\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\Psi_\varepsilon'(\mathbf{v}_{k\tau} - \mathbf{v}^*), \mathbf{e}_j \Big\rangle \\ &= \int_0^t \langle \mathbf{f}, \mathbf{e}_j\rangle ds.\end{aligned} \tag{4.24}$$

Here, $\mathbf{v}_{0k} \to \mathbf{v}_0 \in H$ and the equation holds for each $\mathbf{e}_j$ for each $j \leq k$. Then, this integral equation reduces to a system of ordinary differential equations for the vector $\mathbf{x}(t,\omega)$ whose j^{th} component is $x_j(t,\omega)$ mentioned above. We will obtain existence and measurability of $\mathbf{x}$ from Theorem 3.3.

We differentiate, multiply by x_j, add and then integrate and after some manipulations we obtain various terms that need to be estimated. For the friction term we have,

$$\int_0^t \int_{\Gamma_C} F(u_{kn} - g)\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\Psi_\varepsilon'(\mathbf{v}_{k\tau} - \mathbf{v}^*) \cdot \mathbf{v}_{k\tau} \, dS \, ds$$

$$
\begin{aligned}
&= \int_0^t \int_{\Gamma_C} F(u_{kn}-g)\mu(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\Psi'_\varepsilon(\mathbf{v}_{k\tau}-\mathbf{v}^*)\cdot(\mathbf{v}_{k\tau}-\mathbf{v}^*)\,dS\,ds \\
&\quad + \int_0^t \int_{\Gamma_C} F(u_{kn}-g)\mu(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\Psi'_\varepsilon(\mathbf{v}_{k\tau}-\mathbf{v}^*)\cdot\mathbf{v}^*\,dS\,ds.
\end{aligned}
$$

The first term on the right-hand side is nonnegative and the second term is bounded below by an expression of the form

$$
\begin{aligned}
-C\int_0^t \int_{\Gamma_C} (1+|u_{kn}|)|\mathbf{v}^*|\,dS\,ds &\geq -C\int_0^t \|\mathbf{u}_k\|_W \|\mathbf{v}^*\|_{L^2(\Gamma_C)^3} ds - C \\
&\geq -C\int_0^t \|\mathbf{u}_k\|_W - C,
\end{aligned}
$$

where C is independent of ε,ω and k.

Here, the space W embeds compactly into V and the trace map from W to $L^2(\Gamma_C)^3$ is continuous. We note that the use of the space W is not essential here, however, below we do need this intermediate space. To estimate the term with P, one uses the linear growth condition of P in (4.7).

It follows from equivalence of norms in finite dimensional spaces, the assumed estimates on M, A, and P, and standard manipulations depending on compact embeddings, that there exists an estimate suitable to apply Theorem 3.3 to obtain the existence of a solution such that $(t,\omega)\to\mathbf{x}(t,\omega)$ is measurable into $\mathbb{R}^k$ which implies that $(t,\omega)\to\mathbf{v}_k(t,\omega)$ is product measurable into V and H. This yields the measurable Galerkin approximation of a solution.

Also, the estimates and compact embedding results for Sobolev spaces lead to the inequality

$$
|\mathbf{v}_k(t)|_H^2 + \int_0^T \|\mathbf{v}_k\|_V^2 ds + \|\mathbf{u}_k(t)\|_V^2 \leq C, \tag{4.25}
$$

where the constant C does not depend on ε or k.

Next, we need to estimate the time derivative in $\mathcal{V}'$. The integral equation implies that for all $\mathbf{w}\in V_k$,

$$
\begin{aligned}
&\langle \mathbf{v}'_k(t),\mathbf{w}\rangle_V \\
&+ \langle M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k + \gamma_\tau^* F(u_{kn}-g)\mu(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\Psi'_\varepsilon(\mathbf{v}_{k\tau}-\mathbf{v}^*),\mathbf{w}\rangle_V \\
&= \langle \mathbf{f},\mathbf{w}\rangle,
\end{aligned} \tag{4.26}
$$

where the dependence on t and ω is suppressed. In terms of inner products in V this reduces to

$$
\begin{aligned}
&(R^{-1}\mathbf{v}'_k(t),\mathbf{w})_V \\
&+ \Big(R^{-1}(M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k + \gamma_\tau^* F(u_n-g)\mu(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\Psi'_\varepsilon(\mathbf{v}_{k\tau}-\mathbf{v}^*)),\mathbf{w}\Big)_V \\
&= (R^{-1}\mathbf{f},\mathbf{w})_V.
\end{aligned}
$$

In terms of the orthogonal projection in V onto V_k, denoted by P_k, this takes the form

$$
\begin{aligned}
&(R^{-1}\mathbf{v}'_k(t),P_k\mathbf{w})_V \\
&+ (R^{-1}\Big(M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k + \gamma_\tau^* F(u_n-g)\mu(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\Psi'_\varepsilon(\mathbf{v}_{k\tau}-\mathbf{v}^*)\Big),P_k\mathbf{w})_V \\
&= (R^{-1}\mathbf{f},P_k\mathbf{w})_V,
\end{aligned}
$$

for all $\mathbf{w} \in V$. Now $\mathbf{v}_k'(t) \in V_k$ and so the first term can be simplified and we can write

$$
\begin{aligned}
&(R^{-1}\mathbf{v}_k'(t), \mathbf{w})_V \\
&+ (R^{-1}\Big(M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k + \gamma_\tau^* F(u_n - g)\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\Psi_\varepsilon'(\mathbf{v}_{k\tau} - \mathbf{v}^*)\Big), P_k\mathbf{w})_V \\
&= (R^{-1}\mathbf{f}, P_k\mathbf{w})_V,
\end{aligned}
$$

for all $\mathbf{w} \in V$. Then it follows that for all $\mathbf{w} \in V$,

$$
\begin{aligned}
&(R^{-1}\mathbf{v}_k'(t), \mathbf{w})_V \\
&+ (P_k R^{-1}\Big(M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k + \gamma_\tau^* F(u_n - g)\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\Psi_\varepsilon'(\mathbf{v}_{k\tau} - \mathbf{v}^*)\Big), \mathbf{w})_V \\
&= (P_k R^{-1}\mathbf{f}, \mathbf{w})_V.
\end{aligned}
$$

Thus, in V we have

$$
\begin{aligned}
&R^{-1}\mathbf{v}_k'(t) + P_k R^{-1}\big(M\mathbf{v}_k + A\mathbf{u}_k + P\mathbf{u}_k + \gamma_\tau^* F(u_n - g)\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\Psi_\varepsilon'(\mathbf{v}_{k\tau} - \mathbf{v}^*)\big) \\
&= P_k R^{-1}\mathbf{f}.
\end{aligned}
$$

and R^{-1} preserves norms while P_k decreases them. Hence, the estimate (4.25) implies that $\|\mathbf{v}_k'\|_{\mathcal{V}'}$ is also bounded independently of ε and k. Then, summarizing the above estimates and restoring ω, yields

$$
|\mathbf{v}_k(t,\omega)|_H + \|\mathbf{v}_k(\cdot,\omega)\|_{\mathcal{V}} + \|\mathbf{v}_k'(\cdot,\omega)\|_{\mathcal{V}'} + \|\mathbf{u}_k(t,\omega)\|_V \le C, \tag{4.27}
$$

where C is a constant that does not depend on ε and k. Also, integrating (4.26), leads to

$$
\begin{aligned}
&i_k^*\Big(\mathbf{v}_k(t) - \mathbf{v}_{0k} + \int_0^t M\mathbf{v}_k ds + \int_0^t A\mathbf{u}_k ds + \int_0^t P\mathbf{u}_k\, ds \\
&+ \int_0^t \gamma_\tau^* F(u_{kn} - g)\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\Psi_\varepsilon'(\mathbf{v}_{k\tau} - \mathbf{v}^*) ds\Big) \\
&= i_k^* \int_0^t \mathbf{f} ds,
\end{aligned} \tag{4.28}
$$

where i_k^* is the dual map to the inclusion map $i_k : V_k \to V$.

Let W be an intermediate space introduced above such that

$$
V \subseteq W \subseteq H, \quad V \text{ dense in } W,
$$

where the embedding is compact and the trace map onto $L^2(U)$ is continuous. Using Theorems 4.1 and 4.2, it follows that for each fixed $\omega \in \Omega$, the following convergences hold true for suitable subsequences, still denoted as $\{\mathbf{v}_k\}$, which may depend on ω. We note that the compactness of the embedding of V into W and consequently into H implies the compactness of the embedding of $H = H'$ into W'. Then using the estimates and these theorems, as $k \to \infty$, we obtain

$$
\mathbf{v}_k \rightharpoonup \mathbf{v} \quad \text{in } \mathcal{V}, \tag{4.29}
$$

$$
\mathbf{v}_k' \rightharpoonup \mathbf{v}' \quad \text{in } \mathcal{V}', \tag{4.30}
$$

$$
\mathbf{v}_k \to \mathbf{v} \quad \text{strongly in } C([0,T], W'), \tag{4.31}
$$

$$
\mathbf{v}_k \to \mathbf{v} \quad \text{strongly in } L^2([0,T]; W), \tag{4.32}
$$

$$
\mathbf{v}_k(t) \to \mathbf{v}(t) \quad \text{in } W \text{ for a.a. } t, \tag{4.33}
$$

$$
\mathbf{u}_k \to \mathbf{u} \quad \text{strongly in } C([0,T]; W), \tag{4.34}
$$

$$A\mathbf{u}_k \rightharpoonup A\mathbf{u} \quad \text{in } \mathcal{V}', \tag{4.35}$$
$$M\mathbf{v}_k \rightharpoonup M\mathbf{v} \quad \text{in } \mathcal{V}'. \tag{4.36}$$

It follows from these convergences and the density of $\cup_n V_n$ in V that by passing to a limit and using the dominated convergence theorem and the strong convergences above in the nonlinear terms, we obtain the following equation that holds in V',

$$\begin{aligned} &\mathbf{v}(t) + \int_0^t M\mathbf{v} ds + \int_0^t A\mathbf{u} ds + \int_0^t P\mathbf{u} ds \\ &+ \int_0^t \gamma_\tau^* F(u_n - g(\omega))\mu(|\mathbf{v}_\tau - \mathbf{v}^*(\omega)|)\Psi_\varepsilon'(\mathbf{v}_\tau - \mathbf{v}^*(\omega)) ds \\ &= \mathbf{v}_0 + \int_0^t \mathbf{f} ds. \end{aligned} \tag{4.37}$$

Thus, $t \to \mathbf{v}(t,\omega)$ is continuous into V'. This fact together with estimate (4.27) imply that the conditions of Theorem 2.2 are satisfied. It follows that there is a function $\bar{\mathbf{v}}$ which is product measurable into V' and weakly continuous in t and such that for each ω there is a subsequence $\mathbf{v}_{k(\omega)}$ such that $\mathbf{v}_{k(\omega)}(\cdot,\omega) \rightharpoonup \bar{\mathbf{v}}(\cdot,\omega)$ in $\mathcal{V}'$. By repeating the above argument, for each ω we obtain that there exists a further subsequence, still denoted as $\mathbf{v}_{k(\omega)}$, that converges in $\mathcal{V}'$ to $\mathbf{v}(\cdot,\omega)$, which is a solution (4.37) that is continuous into V'. Hence, $\bar{\mathbf{v}}(\cdot,\omega) = \mathbf{v}(\cdot,\omega)$, and since these functions are both weakly continuous into V' they must be identical. Therefore, there is a product measurable solution $\mathbf{v}$ to each regularized problem.

It remains to pass to the regularization limit $\varepsilon \to 0$. We let $\varepsilon = 1/k$ and denote the product measurable solution of (4.37) by $\mathbf{v}_k$ and note that estimate (4.25) holds true for $\mathbf{v}_k$. Then, we obtain a subsequence, still denoted as $\mathbf{v}_k$, that has the same convergences as in (4.29)–(4.36). Thus, we obtain these convergences along with the fact that $\mathbf{v}_k$ is product measurable and for each ω it is a solution of the problem

$$\begin{aligned} &\mathbf{v}_k(t) + \int_0^t M\mathbf{v}_k\, ds + \int_0^t A\mathbf{u}_k\, ds + \int_0^t P\mathbf{u}_k\, ds \\ &+ \int_0^t \gamma_\tau^* F(u_{kn} - g(\omega))\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*(\omega)|)\Psi_{1/k}'(\mathbf{v}_{k\tau} - \mathbf{v}^*(\omega)) ds \\ &= \mathbf{v}_0 + \int_0^t \mathbf{f} ds. \end{aligned} \tag{4.38}$$

Next, in addition to (4.29)–(4.36), we have

$$\Psi_{1/k}'(\mathbf{v}_{k\tau} - \mathbf{v}^*) \rightharpoonup \xi \quad \text{in } L^\infty([0,T]; L^\infty(\Gamma_C)^3),$$

and moreover,

$$\Psi_{1/k}'(\mathbf{v}_{k\tau} - \mathbf{v}^*) \cdot \mathbf{w}_\tau \le \Psi_{1/k}(\mathbf{v}_{k\tau} - \mathbf{v}^* + \mathbf{w}_\tau) - \Psi_{1/k}(\mathbf{v}_{k\tau} - \mathbf{v}^*).$$

Therefore, passing to the limit as $k \to \infty$; using the strong convergence in the space $L^2([0,T];W)$, of $\mathbf{v}_{k\tau}$ to $\mathbf{v}_\tau$; the uniform convergence of $\Psi_{1/k}$ to $\|\cdot\|$; the pointwise convergence in W; and the dominated convergence theorem, we obtain that for $\mathbf{w} \in \mathcal{V}$,

$$\int_0^t \int_{\Gamma_C} F(u_{kn} - g(\omega))\mu(|\mathbf{v}_{k\tau} - \mathbf{v}^*(\omega)|)\Psi_{1/k}'(\mathbf{v}_{k\tau} - \mathbf{v}^*(\omega)) \cdot \mathbf{w}_\tau\, dS\, ds$$

$$\to \int_0^t \int_{\Gamma_C} F(u_n - g(\omega))\mu(|\mathbf{v}_\tau - \mathbf{v}^*(\omega)|)\xi \cdot \mathbf{w}_\tau \, dS \, ds,$$

where

$$\int_0^t \int_{\Gamma_C} \xi \cdot \mathbf{w}_\tau \, dS \, ds \le \int_0^t \int_{\Gamma_C} (|\mathbf{v}_{k\tau} - \mathbf{v}^* + \mathbf{w}_\tau| - |\mathbf{v}_{k\tau} - \mathbf{v}^*|) \, dS \, ds. \tag{4.39}$$

Then, passing to the limit in the integral equation (4.38), we obtain that for each ω, $\mathbf{v}$ is a solution of the integral equation

$$\begin{aligned} \mathbf{v}(t) &+ \int_0^t M\mathbf{v} \, ds + \int_0^t A\mathbf{u} \, ds + \int_0^t P\mathbf{u} \, ds \\ &+ \int_0^t \gamma_\tau^* F(u_n - g(\omega))\mu(|\mathbf{v}_\tau - \mathbf{v}^*(\omega)|)\xi \, ds \\ &= \mathbf{v}_0 + \int_0^t \mathbf{f} \, ds, \end{aligned} \tag{4.40}$$

where ξ satisfies the inequality (4.39). In particular, $\mathbf{v}$ is continuous into V' and now, the conclusion of the measurable selection theorem applies and yields the existence of a measurable solution to (4.40) for each ω. Taking a weak derivative, it follows that we have obtained a product measurable solution to the system (4.18)–(4.20). This completes the existence part of the proof of Theorem 4.3.

We note that when the friction coefficient $\mu(\cdot)$ is Lipschitz continuous, one can show that for each ω the solution of the integral equation (4.38) is unique, although this it is not an obvious statement, see [15]. This follows from standard procedures involving Gronwall's inequality and the various necessary estimates. Therefore, it is possible to obtain the product measurability by using more elementary methods.

We also note that it allows one to include a stochastic integral of the form $\int_0^t \Phi dW$. In this case one must consider a filtration and obtain solutions that are adapted to the filtration.

In the next section we consider the case of discontinuous friction coefficient and in this case it is not clear whether there is uniqueness, but we still obtain a measurable solution.

4.3. **Discontinuous coefficient of friction.** In this section we consider the case when the coefficient of friction is a discontinuous function of the slip speed, which was studied by us in [15]. This is the case described in elementary physics and engineering courses, as well as in a host of engineering publications on friction, which assumes, based on experimental data, that the coefficient of sliding or dynamic friction is less than the coefficient of static friction. Additional information can be found in [25] and the many references therein. Therefore, we assume the friction coefficient function μ has a jump discontinuity at 0, becoming smaller when the slip speed is positive. The graph of the multi-function friction coefficient μ is depicted in Figure 2 in blue.

We assume the friction coefficient is a set-valued function $\mu = \mu(r)$ that consists of a Lipschitz continuous function μ_s and the segment connecting the static friction coefficient μ_0 and the value $\mu_s(0)$ on the vertical axis, Figure 2.

To study the frictional contact problem with discontinuous friction coefficient, we regularize the coefficient, obtain a measurable solution to each regularized problem as above, and then pass to the limit. To that end, we approximate by the

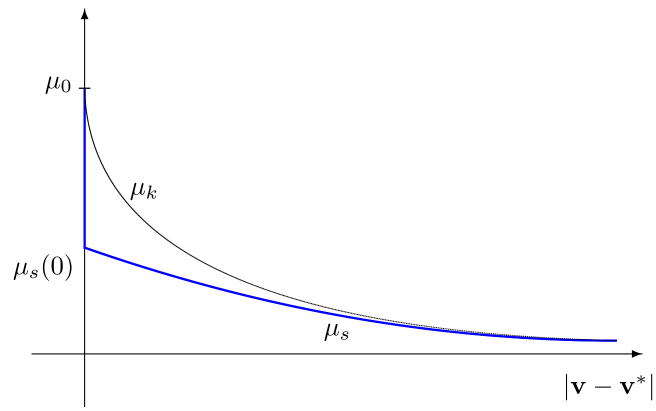

FIGURE 2. Graph of μ and μ_k vs. the slip-rate $|\mathbf{v}-\mathbf{v}^*|$.

multi-function μ with the sequence of functions μ_k, Figure 2, which are Lipschitz continuous and converge uniformly to μ on every interval of the form $[\delta,\infty)$ for $\delta>0$. It follows from Theorem 4.3 that for each k there exists a unique measurable solution of the integral equation

$$\begin{aligned}\mathbf{v}_k(t)+\int_0^t M\mathbf{v}_k\,ds+\int_0^t A\mathbf{u}_k\,ds+\int_0^t P\mathbf{u}_k\,ds\\ +\int_0^t \gamma_\tau^* F(u_{kn}-g(\omega))\mu_k(|\mathbf{v}_{k\tau}-\mathbf{v}^*(\omega)|)\xi_k ds=\mathbf{v}_0+\int_0^t \mathbf{f}\,ds,\end{aligned}\tag{4.41}$$

where

$$\int_0^t\int_{\Gamma_C}\xi_k\cdot\mathbf{w}_\tau\,dS\,ds\le\int_0^t\int_{\Gamma_C}|\mathbf{v}_{k\tau}-\mathbf{v}^*+\mathbf{w}_\tau|-|\mathbf{v}_{k\tau}-\mathbf{v}^*|\,dS\,ds.\tag{4.42}$$

Let $\gamma(\mathbf{r})=|\mathbf{r}|$, then it follows from (4.42) that for ω off a set of measure zero $\xi_k\in\partial\gamma(\mathbf{v}_{k\tau}-\mathbf{v}^*)$ a.e. t for each k. Thus,

$$\begin{aligned}&\int_0^t\int_{\Gamma_C}F(u_{kn}-g)\mu_k(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\xi_k\cdot\mathbf{w}_\tau\,dS\,ds\\ &\le\int_0^t\int_{\Gamma_C}F(u_{kn}-g)\mu_k(|\mathbf{v}_{k\tau}-\mathbf{v}^*|)(|\mathbf{v}_{k\tau}-\mathbf{v}^*+\mathbf{w}_\tau|-|\mathbf{v}_{k\tau}-\mathbf{v}^*|)\,dS\,ds.\end{aligned}$$

Now, for each ω, the estimate (4.25) holds true, thus,

$$|\mathbf{v}_k(t)|_H^2+\int_0^T\|\mathbf{v}_k\|_V^2ds+\|\mathbf{u}_k(t)\|_V^2\le C,$$

where C does not depend on k. Since ξ_k is bounded, it follows from (4.41) and this estimate, that $\mathbf{v}_k'$ is bounded in $\mathcal{V}'$. So,

$$|\mathbf{v}_k(t)|_H^2+\int_0^T\|\mathbf{v}_k\|_V^2ds+\|\mathbf{u}_k(t)\|_V^2+\|\mathbf{v}_k'\|_{\mathcal{V}'}\le C.\tag{4.43}$$

As was noted above, the constant C is independent of k. Now, for fixed ω, there exists a subsequence, still denoted as $\{\mathbf{v}_k\}$ such that the convergences obtained in

(4.29)–((4.36) hold. Thus, as $k \to \infty$,

$$\mathbf{v}_k \rightharpoonup \mathbf{v} \quad \text{in } \mathcal{V}, \tag{4.44}$$

$$\mathbf{v}_k' \rightharpoonup \mathbf{v}' \quad \text{in } \mathcal{V}', \tag{4.45}$$

$$\mathbf{v}_k \to \mathbf{v} \quad \text{strongly in } C([0,T],W'), \tag{4.46}$$

$$\mathbf{v}_k \to \mathbf{v} \quad \text{strongly in } L^2([0,T];W), \tag{4.47}$$

$$\mathbf{v}_k(t) \to \mathbf{v}(t) \quad \text{in } W \text{ for a. a. } t, \tag{4.48}$$

$$\mathbf{u}_k \to \mathbf{u} \quad \text{strongly in } C([0,T];W), \tag{4.49}$$

$$A\mathbf{u}_k \rightharpoonup A\mathbf{u} \quad \text{in } \mathcal{V}', \tag{4.50}$$

$$M\mathbf{v}_k \rightharpoonup M\mathbf{v} \quad \text{in } \mathcal{V}'. \tag{4.51}$$

We note that more can be said if a further subsequence is taken. Indeed,

$$m(t : \|\mathbf{v}_k(t) - \mathbf{v}(t)\|_W \ge \lambda) < \frac{1}{\lambda}\int_0^T \|\mathbf{v}_k - \mathbf{v}\|_W^2 ds,$$

and so there exists a subsequence, still denoted by $\{\mathbf{v}_k\}$, such that

$$m(t : \|\mathbf{v}_k(t) - \mathbf{v}(t)\|_W \ge 2^{-k}) < 2^{-k}.$$

The Borel-Cantelli lemma implies that there exists a set of measure zero $\mathcal{N}$ such that for t not in this set,

$$\|\mathbf{v}_k(t) - \mathbf{v}(t)\|_W < 2^{-k},$$

for all k sufficiently large. Thus, for all k large enough,

$$\|\mathbf{v}_{k\tau}(t) - \mathbf{v}_\tau(t)\|_{L^2(\Gamma_C)} < \frac{C}{2^k}.$$

It now follows from the usual proof of the completeness of the space L^2 that for $t \notin \mathcal{N}$,

$$\mathbf{v}_{k\tau}(t,x) \to \mathbf{v}_\tau(t,x) \quad \text{a.e. } x. \tag{4.52}$$

Passing to a further subsequence, if necessary, we may also assume that

$$\mu_k(|\mathbf{v}_{k\tau} - \mathbf{v}^*|) \rightharpoonup \hat{\mu} \;\text{ weak} * \text{ in } L^\infty([0,T];L^\infty(\Gamma_C)),$$
$$\mu_k(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\xi_k \rightharpoonup \Sigma \;\text{ weak} * \text{ in } L^\infty([0,T],L^\infty(\Gamma_C)^3).$$

Next, for a given $\mathbf{w} \in \mathcal{V}$, we consider only those (t,x) for which convergence takes place in (4.52), and denote the set as

$$S_0 \equiv \{(t,x) \notin \mathcal{M} : |\mathbf{v}_\tau(t,x) - \mathbf{v}^*| = 0\},$$

where $\mathcal{M}$ is the subset of $([0,T] \times \Gamma_C$ where convergence does not take place. Then, from the description of the μ_k, for k large enough, $\mu_k(|\mathbf{v}_{k\tau}(t,x) - \mathbf{v}^*|) \in [\mu_s(0) - \varepsilon, \mu_0]$. Let B be the set of all those $(t,x) \in S_0$ for which $\hat{\mu}(t,x) > \mu_0$ and suppose it has positive measure. Then, since S is the surface measure on Γ_C and $(m \times S)(B) > 0$, it follows from the above weak convergence that

$$\begin{aligned}
\mu_0(m\times S)(B) &= \int_0^T\int_{\Gamma_C} \mu_0\mathcal{X}_B(t,x)\,dS\,dt \\
&\ge \int_0^T\int_{\Gamma_C} \lim_{k\to\infty}\mu_k(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\mathcal{X}_B(t,x)\,dS\,dt \\
&= \lim_{k\to\infty}\int_0^T\int_{\Gamma_C} \mu_k(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\mathcal{X}_B(t,x)\,dS\,dt
\end{aligned}$$

$$= \int_0^T \int_{\Gamma_C} \hat{\mu}(t,x)\mathcal{X}_B(t,x)\, dS\, dt > \mu_0(m \times S)(B),$$

which is a contradiction. Similarly, if we assume that B consists of those $(t,x) \in S_0$ for which $\hat{\mu}(t,x) < \mu_s(0) - \varepsilon$, one obtains a contradiction unless $(m \times S)(B) = 0$. It follows that for a.e. $(t,x) \in S_0$,

$$\hat{\mu}(t,x) \in [\mu_s(0) - \varepsilon, \mu_0].$$

Since ε is arbitrary, it follows that $\hat{\mu}(t,x) \in [\mu_s(0), \mu_0]$ for a.e. (t,x). Now, let

$$S_+ \equiv \{(t,x) \notin \mathcal{M} : |\mathbf{v}_\tau(t,x) - \mathbf{v}^*| > 0\}.$$

Then, by the convergence (4.52), for a.e. (t,x),

$$\mu_k(|\mathbf{v}_{k\tau}(t,x) - \mathbf{v}^*|) \to \mu(|\mathbf{v}_\tau(t,x) - \mathbf{v}^*|),$$

and so similar arguments show that $\hat{\mu}(t,x) = \mu(|\mathbf{v}_\tau(t,x) - \mathbf{v}^*|)$ for these (t,x) as well. Thus $\hat{\mu}$ is in the graph of $\mu(|\mathbf{v}_\tau(t,x) - \mathbf{v}^*|)$ off a set of measure zero. Now, consider the friction term,

$$\begin{aligned}
&\int_0^T \int_{\Gamma_C} F(u_{kn} - g)\mu_k(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)\xi_k \cdot \mathbf{w}_\tau\, dS\, dt \\
&\leq \int_0^T \int_{\Gamma_C} F(u_{kn} - g)\mu_k(|\mathbf{v}_{k\tau} - \mathbf{v}^*|)(|\mathbf{v}_{k\tau} - \mathbf{v}^* + \mathbf{w}_\tau| - |\mathbf{v}_{k\tau} - \mathbf{v}^*|)\, dS\, dt.
\end{aligned}$$

Then the weak convergence and the strong convergence above imply

$$\begin{aligned}
&\int_0^T \int_{\Gamma_C} F(u_n - g)\Sigma \cdot \mathbf{w}_\tau\, dS\, dt \\
&\leq \int_0^T \int_{\Gamma_C} F(u_n - g)\hat{\mu}(|\mathbf{v}_{k\tau} - \mathbf{v}^* + \mathbf{w}_\tau| - |\mathbf{v}_{k\tau} - \mathbf{v}^*|)\, dS\, dt.
\end{aligned}$$

Here $\hat{\mu}(t,x) \in \mu(|\mathbf{v}_\tau(t,x) - \mathbf{v}^*|)$. Define $\xi \equiv \Sigma/\hat{\mu}$, which is well defined since $\hat{\mu} \neq 0$. Then, the above expression takes the form

$$\begin{aligned}
&\int_0^T \int_{\Gamma_C} F(u_n - g)\hat{\mu}(\frac{\Sigma}{\hat{\mu}}) \cdot \mathbf{w}_\tau\, dS\, dt \\
&= \int_0^T \int_{\Gamma_C} F(u_n - g)\hat{\mu}\xi \cdot \mathbf{w}_\tau\, dS\, dt \\
&\leq \int_0^T \int_{\Gamma_C} F(u_n - g)\hat{\mu}(|\mathbf{v}_{k\tau} - \mathbf{v}^* + \mathbf{w}_\tau| - |\mathbf{v}_{k\tau} - \mathbf{v}^*|)\, dS\, dt,
\end{aligned}$$

where $\hat{\mu}(t,x) \in \mu(|\mathbf{v}_\tau(t,x) - \mathbf{v}^*|)$.

We now return to the approximate integral equation (4.41). The strong convergence (4.49) is sufficient to pass to the limit in the term involving P. Therefore, collecting the above results establishes the following Proposition.

Proposition 4.4. *For fixed ω, there exist four functions $(\mathbf{v}, \mathbf{u}, \hat{\mu}, \xi)$ such that $\mathbf{v} \in \mathcal{V}$, $\mathbf{u} \in C([0,T], V)$, $\mathbf{v}' \in \mathcal{V}'$*

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds, \tag{4.53}$$

$$\hat{\mu} \in \mu(|\mathbf{v}_\tau(t,x) - \mathbf{v}^*|) \quad a.e.\ (t,x).$$

For all $\mathbf{w} \in \mathcal{V}$,

$$\begin{aligned}&\int_0^T \int_{\Gamma_C} F(u_n - g)\hat{\mu}\xi \cdot \mathbf{w}_\tau \, dS\, dt\\&\le \int_0^T \int_{\Gamma_C} F(u_n - g)\hat{\mu}(|\mathbf{v}_{k\tau} - \mathbf{v}^* + \mathbf{w}_\tau| - |\mathbf{v}_{k\tau} - \mathbf{v}^*|)\, dS\, dt,\end{aligned}$$

and

$$\begin{aligned}&\mathbf{v}(t) + \int_0^t M\mathbf{v} ds + \int_0^t A\mathbf{u}\, ds + \int_0^t P\mathbf{u}\, ds + \int_0^t \gamma_\tau^* F(u_n - g)\hat{\mu}\xi ds\\&= \mathbf{v}_0 + \int_0^t \mathbf{f}\, ds.\end{aligned} \tag{4.54}$$

Finally, the remaining issue is to show the existence of a measurable solution. However, this follows in the same way as above from the measurable selection theorem, Theorem 2.2. The reasoning as above shows that for a fixed ω every sequence has a subsequence that converges to a solution of the integral equation (4.53)–(4.54) that is continuous into V', this continuity follows directly from the integral equation (4.54). Moreover, one can obtain the estimate (4.43) for all of the sequence $\mathbf{v}_k$ for ω off a set of measure zero. Therefore, Theorem 2.2 asserts that there is a function $\mathbf{v}(\cdot,\omega)$ in $\mathcal{V}'$ that is product measurable into V' that is also weakly continuous in t into V', and there is a subsequence $\mathbf{v}_{k(\omega)}(\cdot,\omega)$ that converges weakly to $\mathbf{v}(\cdot,\omega)$ in $\mathcal{V}'$. We note that although $\mathbf{v}$ has values in V, it is only known to be continuous into V', which has a weaker norm. Then, it follows from the above argument that a further subsequence converges to a solution of the integral equation, and since both are weakly continuous into V', this solution to the integral equation equals this measurable function $\mathbf{v}$ for all t, and for each ω off a set of measure zero. Thus, there is a measurable solution to the stochastic friction problem with discontinuous friction coefficient, too. This result is summarized in the following theorem.

Theorem 4.5. *For each* $\omega \in \Omega$, *let* $\mathbf{u}_0(\omega) \in V$, $\mathbf{v}_0(\omega) \in H$, *and* $\mathbf{f}(\cdot,\omega) \in \mathcal{V}'$. *Also, assume the gap* g *and sliding velocity* $\mathbf{v}^*$ *are* $\mathcal{B}([0,T]) \times \mathcal{F}$ *measurable, and* μ *has a jump discontinuity at the origin. Then, there exists a solution* $\mathbf{v}$ *of the problem summarized in* (4.53)–(4.54) *for each* $\omega \in \Omega$. *This solution* $(t,\omega) \to \mathbf{v}(t,\omega)$ *is product measurable into* V, H *and* V'.

It only remains to check the last claim about measurability into the spaces V and H. By the density of V into H, it follows that H' is dense in V' and so a simple argument using the Pettis theorem implies that $\omega \to \mathbf{v}(t,\omega)$ is $\mathcal{F}$ measurable in both V and H.

Finally, we note that if we assume more regularity on $\mathbf{f}$, say that it is actually in $L^2([0,T] \times \Omega; V')$, then we could say that in fact, $\mathbf{v} \in L^2([0,T] \times \Omega; V)$. This is obtained by simply integrating the estimate (4.25) and being more careful about the structure of the constant on the right-hand side in this inequality. The measurability issue is obtained again from our major theorem. We have not done this because we want to emphasize that this extra assumption is not needed in order to get measurable solutions.

Acknowledgements. We would like to thank the anonymous referee for the constructive comments that helped to improve this article.

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