EXISTENCE OF SOLUTIONS TO QUASILINEAR ELLIPTIC PROBLEMS WITH NONLINEARITY AND ABSORPTION-REACTION GRADIENT TERM

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Abstract. In this article we study the quasilinear elliptic problem

$$-\Delta_p u = \pm |\nabla u|^\nu + f(x, u), \quad \text{in } \Omega,$$

$$u \geq 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $p > 1$ and $0 < \nu \leq p$. Moreover, $f$ is a nonnegative function verifying suitable hypotheses. The main goal of this work is to analyze the interaction between the gradient term and the function $f$ to obtain existence results.

1. INTRODUCTION

In this article we will discuss existence results for a class of quasilinear elliptic problems in the form

$$-\Delta_p u = \pm |\nabla u|^\nu + f(x, u), \quad \text{in } \Omega,$$

$$u > 0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is the classical $p$-Laplace operator and $0 < \nu \leq p$.

The function $f : \Omega \times [0, +\infty) \to [0, +\infty)$ is assumed to be Hölder continuous, non-decreasing, and such that

the function $t \mapsto \frac{f(x, t)}{t^{p-1}}$ is non-increasing for all $x \in \Omega$, \quad (1.2)

$$\lim_{t \to 0} \frac{f(x, t)}{t^{p-1}} = +\infty \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(x, t)}{t^{p-1}} = 0 \quad \text{uniformly for } x \in \Omega. \quad (1.3)$$

$$f(x, 0) \neq 0 \quad (1.4)$$

Notice that problems with gradient term are widely studied in the literature. We can cite the leading works of Boccardo, Gallouët, Murat and their collaborators, see for instance [7, 9] and [8] and the references therein. For some recent works related to our problem, we can cite [1, 2, 4, 21, 24, 5, 25].

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In the particular case $p = 2$, problem (1.1) is related to the Lane-Emden-Fowler and Emden-Fowler equations, treated in many papers; we particularly cite the works of Radulescu, and his collaborators [13, 14, 15] and more recently [12, 16] and the references therein. For the case without the absence of the gradient term, we refer to [18].

When the nonlinearity is considered as an absorption term we cite [11] where the authors prove the existence of solution even when $\Omega$ is of infinite measure, and in the same direction we cite [10].

The extension to the $p-$laplacian, of the previous results obtained in the case of the laplacian, especially when using a sub-supersolution method, has a major difficulty: no general comparison principle for the operator $-\Delta_p u \pm |\nabla u|^q$ exist at our knowledge, and there are only few partial results in this direction. In addition, the behavior of the operator changes when considering the cases $p < 2$ and $p > 2$. We refer the reader to [22] for a general discussion about this fact.

2. Preliminaries

The next comparison principles will be used frequently in this paper, for complete proofs of the first three ones we refer to [22] and we refer to [3] for the last one. Considering the problem

$$\begin{align*}
- \text{div}(a(x, \nabla u)) + H(x, \nabla u) &= f(x) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}$$

(2.1)

and having in mind the particular case

$$\begin{align*}
-\Delta_p u \pm |\nabla u|^q &= f(x) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

with $q \leq p$ we have the following result.

**Theorem 2.1** ([22]). *Under the hypotheses: $q > \frac{N(p-1)}{N-1}$, $1 < p \leq 2$ and

$$f = f_1(x) + \text{div}(f_2(x)) \quad \text{where } f_1 \in L^1(\Omega), \ f_2 \in (L^{p'}(\Omega))^N$$

(2.2)

$$|a(x, \xi) - a(x, \eta)|(|\xi| - |\eta|) \geq \alpha(|\xi|^p - |\eta|^p)^{\frac{p-2}{2}}|\xi - \eta|^2, \quad \alpha > 0$$

(2.3)

$$a(x,0) = 0$$

(2.4)

$$|a(x,\xi)| \leq \beta(k(x) + |\xi|^{p-1}), \quad \beta > 0, \ k(x) \in L^{p'}(\Omega)$$

(2.5)

$$|H(x,\xi) - H(x,\eta)| \leq \gamma(b(x) + |\xi|^{q-1} + |\eta|^{q-1})|\xi - \eta|, \quad \gamma > 0, \ b(x) \in L^{r}(\Omega),$$

(2.6)

where

$$1 \leq q \leq p - 1 + \frac{p}{N}, \quad r \geq \frac{N(q - (p-1))}{q-1} \quad (\text{with } r = \infty \text{ if } q = 1).$$

If $u$ and $v$ are respectively sub- and super-solution of (2.1), such as

$$(1 + |u|)^{q-1}u \in W^{1,p}_0(\Omega), \quad (1 + |v|)^{q-1}v \in W^{1,p}_0(\Omega), \quad \bar{q} = \frac{(N-p)(q - (p-1))}{p(p-q)}$$

(2.7)

then $u \leq v$ in $\Omega$. 
Theorem 2.2 ([22]). Under the hypotheses: $q < \frac{N(p-1)}{N-1}$, $2 - \frac{1}{N} < p \leq 2$, \([22]\), \([2.3]\), \([2.4]\) and \([2.5]\), such that

\[ |H(x, \xi) - H(x, \eta)| \leq \gamma (b(x) + |\xi|^{q-1} + |\eta|^{q-1})|\xi - \eta|, \quad \gamma > 0, \quad b(x) \in L^r(\Omega), \]

\[ r > \frac{N(p-1)}{N(p-1)-(N-1)}, \quad 1 \leq q < \frac{N(p-1)}{(N-1)}. \]  

If $u$ and $v$ are respectively sub- and super-solution of \((2.1)\), then $u \leq v$ in $\Omega$.

Theorem 2.3 ([22]). Under the hypotheses: $p > 2$, $q > \frac{p}{2} + \frac{(p-1)}{N-1}$, \([2.4]\), \([2.5]\), and

\[ |a(x, \xi) - a(x, \eta)|(|\xi - \eta| \geq \alpha(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi - \eta|^2, \quad \alpha > 0 \]  

\[ |H(x, \xi) - H(x, \eta)| \leq \gamma (b(x) + |\xi|^{q-1} + |\eta|^{q-1})|\xi - \eta|, \quad \gamma > 0, \]  

\[ b(x) \in L^N(\Omega) \quad \text{where} \quad 1 \leq q \leq \frac{p}{2} + \frac{p}{N}. \]  

If $u$ and $v$ are respectively sub- and super-solution of \((2.1)\), such as

\[ (1 + |u|)^{q-1}u \in W^{1,p}_0(\Omega), \quad (1 + |v|)^{q-1}v \in W^{1,p}_0(\Omega), \quad \eta = \frac{(N-p)(q-\frac{p}{2})}{p(q-1)} \]  

then $u \leq v$ in $\Omega$.

Theorem 2.4 ([3]). Assume that $1 < p$ and let $f$ be a non-negative continuous function such that $\frac{f(x,u)}{u^{s-1}}$ is decreasing for $s > 0$. Suppose that $u, v \in W^{1,p}_0(\Omega)$ are such that

\[-\Delta_p u \geq f(x, u), \quad u > 0 \text{ in } \Omega, \]

\[-\Delta_p v \leq f(x, v), \quad v > 0 \text{ in } \Omega. \]  

Then $u \geq v$ in $\Omega$.

Since we are dealing with a generalized notion of solution, we recall here the definition of entropy solutions for elliptic problems.

Definition 2.5. Let $u$ be a measurable function. We say that $u \in T^{1,p}_0(\Omega)$ if $T_k(u) \in W^{1,p}_0(\Omega)$ for all $k > 0$, where

\[ T_k(s) = \begin{cases} k \text{sgn}(s) & \text{if } |s| \geq k, \\ s & \text{if } |s| \leq k. \end{cases} \]  

Let $H \in L^1(\Omega)$. Then $u \in T^{1,p}_0(\Omega)$ is an entropy solution to the problem

\[-\Delta_p u = H \quad \text{in } \Omega, \]

\[ u|_{\partial\Omega} = 0, \]  

if for all $k > 0$ and all $v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, we have

\[ \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla (T_k(u - v))) = \int_\Omega HT_k(u - v). \]  

We refer to \([3]\) and \([17]\) for more properties of entropy solutions. It is clear that if $u$ is an entropy solution to problem \((1.1)\), then $u$ is a distributional solution to \((1.1)\).
3. The absorption case

In this section we consider the problem

\[-\Delta_p u + |\nabla u|^{\nu} = f(x, u) \quad \text{in } \Omega,\]
\[u > 0 \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega.\]  

(3.1)

**Theorem 3.1.** Assume that the assumptions on \(f\) hold. If \(0 < \nu \leq p\), then problem (3.1) has at least one entropy solution \(u \in W^{1,p}_0(\Omega)\).

**Proof.** We split the proof into several steps.

**Step 1: Construction of supersolution and subsolution.** To obtain the existence result we will use sub-supersolution argument. Let us consider the problem

\[-\Delta_p w = f(x, w) \quad \text{in } \Omega,\]
\[w > 0 \quad \text{in } \Omega,\]
\[w = 0 \quad \text{on } \partial \Omega.\]  

(3.2)

Then under the hypothesis on \(f\), problem (3.2) possesses a unique solution \(w\) which is a supersolution of (3.1). For the subsolution to problem (3.1), we consider \(u = 0\). Finally by Theorem 2.4 we reach that \(u \leq w\). To obtain the existence result we use a monotonicity argument. Since no general comparison principle is known for this kind of problems, we will consider different values of \(p\).

The following steps 2, 3 and 4 are devoted to proving the existence of solution in the singular case, namely \(p < 2\), but for different ranges of \(p\) and \(\nu\).

**Step 2: Existence result for** \(\frac{2N}{N+1} \leq p < 2\) **and** \(1 \leq \nu \leq p - 1 + \frac{p}{N}\). **In this case,** by [22] Theorems 3.1 and 3.2 we know that a comparison principle holds for the operator \(-\Delta_p u + |\nabla u|^{\nu}\) in the space \(W^{1,p}_0(\Omega)\).

Then, we define the sequence \(\{u_n\}_{n \in \mathbb{N}}\) as follows: \(u_0 = u\) and for \(n \geq 1\), \(u_n\) is the solution to problem

\[-\Delta_p u_n + |\nabla u_n|^{\nu} = f(x, u_{n-1}) \quad \text{in } \Omega,\]
\[u_n > 0 \quad \text{in } \Omega,\]
\[u_n = 0 \quad \text{on } \partial \Omega.\]  

(3.3)

We claim that the sequence \(\{u_n\}_{n \in \mathbb{N}}\) is increasing in \(n\) and for all \(n \geq 0\), \(u_n \leq w\). Notice that the last statement follows easily from Theorem 2.4. To prove the monotonicity of \(\{u_n\}_{n \in \mathbb{N}}\), we will use the comparison result obtained in [22]. It is clear that \(u_1\) solves

\[-\Delta_p u_1 + |\nabla u_1|^{\nu} = f(x, u_0)\]

By the definition of \(u_0\), we obtain that

\[-\Delta_p u_1 + |\nabla u_1|^{\nu} \geq -\Delta_p u_0 + |\nabla u_0|^{\nu}\]

Thus, by the comparison principle in [22], we reach \(u_1 \geq u_0\). Let us show that \(u_2 \geq u_1\). As above, \(u_2\) satisfies

\[-\Delta_p u_2 + |\nabla u_2|^{\nu} = f(x, u_1)\]

Since \(f\) is a nondecreasing function, it follows that

\[-\Delta_p u_2 + |\nabla u_2|^{\nu} \geq -\Delta_p u_1 + |\nabla u_1|^{\nu}\].
Hence \( u_2 \geq u_1 \). Therefore, the result follows by induction and then the claim follows.

Thus, using \( u_n \) as a test function in (3.3) and by the non decreasing property of \( f \), we obtain that \( \|u_n\|_{W_0^{1,p}(\Omega)} \leq C \). Hence we obtain the existence of \( u \in W_0^{1,p}(\Omega) \) such that \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega) \) and \( u_n \to u \) strongly in \( L^\sigma(\Omega) \) for all \( \sigma < p^* \).

Since \( u \leq u \leq w \in L^\infty(\Omega) \), it follows that \( u \in L^\infty(\Omega) \) and \( u_n \to u \) strongly in \( L^\sigma(\Omega) \) for all \( \sigma \geq 1 \).

Therefore, to have the existence result, we just have to prove that \( |\nabla u_n|^{\nu} \to |\nabla u|^{\nu} \) in \( L^1(\Omega) \). By the hypothesis on \( \nu \), we can see that \( \nu < p \), then using \( (u - u_n) \) as a test function in (3.3), it follows that

\[
\int \Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u dx - \int \Omega |\nabla u_n|^p dx + \int \Omega |\nabla u_n|^{\nu}(u - u_n)dx = \lambda \int \Omega f(x, u_{n-1})(u - u_n)dx.
\]

By the Dominated Convergence Theorem and as \( f \) is assumed to be Hölder continuous, we obtain
\[
\int \Omega f(x, u_{n-1})(u - u_n)dx = o(1).
\]

Now using Hölder inequality and the fact that \( \nu < p \), we obtain
\[
\int \Omega |\nabla u_n|^{\nu}(u - u_n)dx \leq \left( \int \Omega |\nabla u_n|^p dx \right)^{\nu/p} \left( \int (u - u_n)^{p-\nu} dx \right)^{\frac{p-\nu}{p}} = o(1).
\]

We obtain
\[
\int \Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u dx - \int \Omega |\nabla u_n|^p dx = o(1).
\]

Then, using Young inequality there results
\[
\int \Omega |\nabla u_n|^p dx = \int \Omega |\nabla u_n|^{p-2}\nabla u_n \nabla u dx + o(1)
\leq \frac{p-1}{p} \int \Omega |\nabla u_n|^p + \frac{1}{p} \int \Omega |\nabla u|^p dx + o(1).
\]

Thus,
\[
\int \Omega |\nabla u_n|^p dx \leq \int \Omega |\nabla u|^p dx + o(1).
\]

It is clear that
\[
\int \Omega |\nabla u|^p dx \leq \lim \inf \int \Omega |\nabla u_n|^p dx \leq \lim \sup \int \Omega |\nabla u_n|^p dx \leq \int \Omega |\nabla u|^p dx.
\]

Therefore, \( \|u_n\|_{W_0^{1,p}(\Omega)} \to \|u\|_{W_0^{1,p}(\Omega)} \) and then \( u_n \to u \) strongly in \( W_0^{1,p}(\Omega) \). Hence the existence result follows in this case.

**Step 3: Existence result for** \( \frac{2N}{N+1} \leq p < 2 \) **and** \( p - 1 + \frac{\nu}{N} \leq \nu \leq p \). **In this case,** to get a monotone sequence, we have to change the approximation. Since \( \frac{2N}{N+1} \leq p \) then \( \nu \geq 1 \).
For fixed $n \in \mathbb{N}^*$, we define the sequence $\{v_{n,k}\}_{k \in \mathbb{N}}$ as follow: $v_{n,0} = 1$ and for $k \geq 1$, $v_{n,k}$ is the solution to problem

$$-\Delta_p v_{k,n} + \frac{|\nabla v_{k,n}|^\nu}{1 + \frac{1}{n}|\nabla v_{k,n}|^\nu} = f(x, v_{k-1,n}) \quad \text{in } \Omega,$$

$$v_{k,n} > 0 \quad \text{in } \Omega,$$

$$v_{k,n} = 0 \quad \text{on } \partial \Omega. \quad (3.4)$$

Let us begin by proving that the sequence $\{v_{k,n}\}_{k \in \mathbb{N}}$ is increasing in $k$ and that $v_{k,n} \leq w$, for all $k \geq 0$. For simplicity, we set

$$H_n(\xi) = \frac{|\xi|^\nu}{1 + \frac{1}{n}|\xi|^\nu} \quad \text{where } \xi \in \mathbb{R}^N.$$

It is clear that $v_{1,n}$ solves

$$-\Delta_p v_{1,n} + H_n(\nabla v_{1,n}) = f(x, v_{0,n}).$$

By the definition of $v_{0,n}$, we obtain that

$$-\Delta_p v_{1,n} + H_n(\nabla v_{1,n}) \geq -\Delta_p v_{0,n} + H_n(\nabla v_{0,n}).$$

It is clear that $H_n$ satisfies the hypotheses of the comparison principle in [22]. Hence we reach $v_{1,n} \geq v_{0,n}$. In the same way, and using an induction argument, we conclude that $v_{k,n} \geq v_{k-1,n}$ for all $k \in \mathbb{N}^*$.

Now, as in the proof of the previous step, using $v_{k,n}$ as a test function in (3.4) and the hypothesis on $f$, we obtain that $\|v_{k,n}\|_{W^{1,p}_0(\Omega)} \leq C$. Thus we obtain the existence of $u_n \in W^{1,p}_0(\Omega)$ such that $v_{k,n} \rightharpoonup u_n$ weakly in $W^{1,p}_0(\Omega)$. As in the previous step, we can show that $v_{k,n} \to u$ strongly in $W^{1,p}_0(\Omega)$.

Note that by the previous computation we obtain easily that

$$v_{k,n} \geq v_{k,n+1} \quad \text{for all } k \geq 1.$$ 

Hence we conclude that $u_n$ is the minimal solution to problem

$$-\Delta_p u_n + \frac{|\nabla u_n|^\nu}{1 + \frac{1}{n}|\nabla u_n|^\nu} = f(x, u_n) \quad \text{in } \Omega,$$

$$u_n > 0 \quad \text{in } \Omega,$$

$$u_n = 0 \quad \text{on } \partial \Omega, \quad (3.5)$$

with $u_n \leq u_{n+1}$ for all $n \geq 1$. It is clear that $u \leq u_n \leq w \in L^\infty(\Omega)$. Then, as above using $u_n$ as a test function in (3.5), we reach that $\|u_n\|_{W^{1,p}_0(\Omega)} \leq C$ and thus, we obtain the existence of $u \in W^{1,p}_0(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega)$.

If $\nu < p$, then we follow the above computation to reach that $u_n \to u$ strongly in $W^{1,p}_0(\Omega)$ and the existence result holds.

If $\nu = p$, then as in Step 2, we obtain that

$$f(x, u_{n-1}) \to f(x, u) \quad \text{strongly in } L^1(\Omega).$$

We set $k_n(x) \equiv f(x, u_{n-1})$, then

$$-\Delta_p u_n + |\nabla u_n|^p = k_n(x)$$

with $k_n \to k \equiv f(x, u)$ strongly in $L^1(\Omega)$. Therefore, using the result of [23], we conclude that $u_n \to u$ strongly in $W^{1,p}_0(\Omega)$ and the result follows.
Step 4: Existence result for $\frac{2N}{N+1} \leq p < 2$ and $0 < \nu \leq 1$. In this case, we adopt a new approximation of the gradient term, namely we set

$$Q_n(\xi) = (|\xi| + \frac{1}{n})^\nu \quad \text{where } \xi \in \mathbb{R}^N.$$  

For fixed $n \in \mathbb{N}^*$, we define the sequence $\{v_{n,k}\}_{k \in \mathbb{N}}$ as follows: $v_{n,0} = u$ and for $k \geq 1$, $v_{n,k}$ is the solution to problem

$$-\Delta_p v_{k,n} + Q_n(\nabla v_{k,n}) = f(x, v_{k-1,n}) \quad \text{in } \Omega,$$

$$v_{k,n} > 0 \quad \text{in } \Omega,$$

$$v_{k,n} = 0 \quad \text{on } \partial \Omega. \quad (3.6)$$

As above we have $v_{k,n} \leq w$ for all $k \geq 0$. It is clear that $Q_n$ satisfies the condition of [22] Theorems 3.1 and 3.2].

We claim that the sequence $\{v_{k,n}\}_{k \in \mathbb{N}}$ is increasing in $k$, for all fixed $n$. To prove the claim, we observe that $v_{1,n}$ solves

$$-\Delta_p v_{1,n} + Q_n(\nabla v_{1,n}) = f(x, v_{0,n}).$$

By the definition of $v_{0,n}$, we obtain that

$$-\Delta_p v_{1,n} + Q_n(\nabla v_{1,n}) \geq -\Delta_p v_{0,n} + Q_n(\nabla v_{0,n}).$$

Hence, using again the comparison principle in [22], we reach that $v_{1,n} \geq v_{0,n}$. In the same way, using an iteration argument, we conclude that $v_{k,n} \geq v_{k-1,n}$ for all $k \in \mathbb{N}^*$ and then the claim follows.

Now for fixed $k$, we claim that $v_k \leq v_{k,n+1}$. Using the non decreasing property and the regularity of $f$ we see that the claim follows if we can prove that $v_{1,n} \leq v_{1,n+1}$.

By the definition of $v_{1,n}$ and $v_{1,n+1}$, we have

$$-\Delta_p v_{1,n} + Q_n(\nabla v_{1,n}) = -\Delta_p v_{1,n+1} + Q_n(\nabla v_{1,n+1}) \leq -\Delta_p v_{1,n+1} + Q_n(\nabla v_{1,n+1}).$$

Thus, using the comparison principle of [22], we conclude that $v_{1,n} \leq v_{1,n+1}$. The general result follows by induction.

Now, as in the previous steps, using $v_{k,n}$ as a test function in (3.6) and by the Hölder continuity of $f$, we obtain that $\|v_{k,n}\|_{W^{1,p}_0(\Omega)} \leq C$. Thus, we obtain the existence of $u_n \in W^{1,p}_0(\Omega)$ such that $v_{k,n} \rightarrow u_n$ weakly in $W^{1,p}_0(\Omega)$ as $k \rightarrow \infty$. The compactness arguments used in the first step allow us to prove that $v_{k,n} \rightarrow u_n$ strongly in $W^{1,p}_0(\Omega)$. Hence, we find that $u_n$ is the minimal solution to problem

$$-\Delta_p u_n + Q_n(\nabla u_n) = f(x, u_n) \quad \text{in } \Omega,$$

$$u_n > 0 \quad \text{in } \Omega,$$

$$u_n = 0 \quad \text{on } \partial \Omega, \quad (3.7)$$

with $u_n \leq u_{n+1}$ for all $n \geq 1$. It is clear that $u \leq u_n \leq w \in L^\infty(\Omega)$. Then, as above, using $u_n$ as a test function in (3.6), we obtain easily that $\|u_n\|_{W^{1,p}_0(\Omega)} \leq C$. Thus, we obtain the existence of $u \in W^{1,p}_0(\Omega)$ such that $u_n \rightarrow u$ weakly in $W^{1,p}_0(\Omega)$. Since $\nu < p$, we conclude that $u_n \rightarrow u$ strongly in $W^{1,p}_0(\Omega)$ as above, and the existence result follows.

Step 5: Existence result for $2 < p$ and $\nu \leq p$. To deal with the degenerate case $p > 2$, we will make a perturbation in the principal part of the operator, namely
for $\varepsilon > 0$, we consider the next approximating problems
\begin{align}
-L_{\varepsilon} u + |\nabla u|^\nu &= f(x,u) \quad \text{in } \Omega,
\quad u > 0 \quad \text{in } \Omega,
\quad u = 0 \quad \text{on } \partial \Omega,
\end{align}
where
\[-L_{\varepsilon} u = -\text{div}(\varepsilon + |\nabla u|^2)^{\frac{\nu-2}{2}} \nabla u)\).

We begin by proving that problem (3.8) has a minimal solution $u_{\varepsilon}$ at least for $\varepsilon$ small. Fixed $\varepsilon > 0$, then we define $w_{\varepsilon}$ to be the unique solution of problem
\begin{align}
-L_{\varepsilon} w_{\varepsilon} &= f(x,w_{\varepsilon}) \quad \text{in } \Omega,
\quad w_{\varepsilon} > 0 \quad \text{in } \Omega,
\quad w_{\varepsilon} = 0 \quad \text{on } \partial \Omega,
\end{align}
(see [19] for the proof of the uniqueness result). It is clear that $w_{\varepsilon}$ is a bounded supersolution to (3.8) and $\|w_{\varepsilon}\|_{L^\infty} \leq C$ for all $\varepsilon \geq 0$. The function $u = 0$ is also a subsolution of (3.8).

Now, for $\varepsilon$ fixed we define the sequence $\{v_{n,k}\}_{k \in \mathbb{N}}$ as follows: $v_{0,n} = w_{\varepsilon}$ and for $k \geq 1$, $v_{n,k}$ is the solution to problem
\begin{align}
-L_{\varepsilon} v_{k,n} + D_n(\nabla v_{k,n}) &= f(x,v_{k-1,n}) \quad \text{in } \Omega,
\quad v_{k,n} > 0 \quad \text{in } \Omega,
\quad v_{k,n} = 0 \quad \text{on } \partial \Omega,
\end{align}
where
\[D_n(\xi) = \begin{cases} 
\frac{|\xi|^\nu}{1 + |\xi|^p} & \text{if } 1 < \nu \leq p \\
\left(\frac{1}{p} \right)^\nu & \text{if } \nu \leq 1.
\end{cases}\]

It is clear that $v_{k,n} \leq w_{\varepsilon}$ for all $k \geq 0$.

We claim that the sequence $\{v_{k,n}\}_{k \in \mathbb{N}}$ is increasing in $k$ for every fixed $n$. To prove the claim, we observe that $v_{1,n}$ solves
\[-L_{\varepsilon} v_{1,n} + D_n(\nabla v_{1,n}) = f(x,v_{0,n}).\]
By the definition of $v_{0,n}$, we obtain that
\[-L_{\varepsilon} v_{1,n} + D_n(\nabla v_{1,n}) \geq -L_{\varepsilon} v_{0,n} + D_n(\nabla v_{0,n}).\]
Hence, using the comparison principle in [22] Theorem 4.1], we reach that $v_{1,n} \geq v_{0,n}$. In the same way, using an induction argument, we conclude that $v_{k,n} \geq v_{k-1,n}$ for all $k \in \mathbb{N}^*$ and then the claim follows.

Using $v_{k,n}$ as a test function in (3.10) we easily get that $\|v_{k,n}\|_{W_0^{1,p}(\Omega)} \leq C$. Thus, we obtain the existence of $u_n \in W_0^{1,p}(\Omega)$ such that $v_{k,n} \rightarrow u_n$ weakly in $W_0^{1,p}(\Omega)$. By the compactness argument used in the Step 2, we obtain that $u_{k,n} \rightarrow u_n$ strongly in $W_0^{1,p}(\Omega)$ and $u_n$ is the minimal solution to the problem
\begin{align}
-L_{\varepsilon} u_n + D_n(\nabla u_n) &= f(x,u_n) \quad \text{in } \Omega,
\quad u_n > 0 \quad \text{in } \Omega,
\quad u_n = 0 \quad \text{on } \partial \Omega.
\end{align}
Now, we pass to the limit in $n$. 
Using \( u_n \) as a test function in (3.11) and as \( f \) is assumed to be Hölder continuous, we find that \( \|u_n\|_{W^{1,p}_0(\Omega)} \leq C \). Thus, we obtain the existence of \( u_\varepsilon \in W^{1,p}_0(\Omega) \) such that \( u_n \rightharpoonup u_\varepsilon \) weakly in \( W^{1,p}_0(\Omega) \).

If \( \nu < p \), then using the compactness arguments of Step 2 and by the result of [23], we obtain that \( u_n \to u_\varepsilon \) strongly in \( W^{1,p}_0(\Omega) \). Hence it follows that \( u_\varepsilon \) is the minimal solution to problem

\[
-L_\varepsilon u_\varepsilon + |\nabla u_\varepsilon|^{\nu} = f(x, u_\varepsilon) \quad \text{in } \Omega,
\]

\[
u > 0 \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\]

(3.12)

If \( \nu = p \), then by the argument of the last part of Step 3 and using the compactness result of [23], we reach the strong convergence of \( \{u_n\}_{n \in \mathbb{N}} \) in \( W^{1,p}_0(\Omega) \). Thus, we obtain a minimal solution to (3.12) also in this case.

To finish, we just have to pass to the limit in \( \varepsilon \). Notice that, in general, the sequence \( \{u_\varepsilon\}_{\varepsilon} \) is not necessarily monotone in \( \varepsilon \). Using \( u_\varepsilon \) as a test function in (3.12) we reach that \( \|u_\varepsilon\|_{W^{1,p}_0(\Omega)} \leq C \) and then \( u_\varepsilon \rightharpoonup u \) weakly in \( W^{1,p}_0(\Omega) \). Since \( \underline{u} \leq u_\varepsilon \leq \overline{u}_\varepsilon \leq C \), then we easily get that

\[
f(x, u_\varepsilon) \to f(x, u) \text{ strongly in } L^1(\Omega).
\]

Since \( \nu < p \), then using a variation of the compactness result of [23], there results that \( u_\varepsilon \to u \) strongly in \( W^{1,p}_0(\Omega) \). Hence \( u \) solves

\[
-\Delta_p u + |\nabla u|^{\nu} = f(x, u) \quad \text{in } \Omega,
\]

\[
u > 0 \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\]

(3.13)

and the existence result follows. It is clear that \( \underline{u} \leq u \leq \overline{u} \).

4. THE REACTION CASE

In this section, we study the reaction case, namely we consider the problem

\[
-\Delta_p u = f(x, u) + |\nabla u|^{\nu} \quad \text{in } \Omega,
\]

\[
u > 0 \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

(4.1)

with \( \nu < p - 1 \). The main existence result reads as follows.

**Theorem 4.1.** Suppose that the hypotheses made on \( f \) hold. Then, problem (4.1) has at least one entropy solution.

**Proof.** As in the proof of Theorem 3.1, problem (4.1) has a subsolution \( \underline{u} = 0 \). To obtain a supersolution, we first consider problem

\[
-\Delta_p u = f(x, u) + 1 \quad \text{in } \Omega,
\]

\[
u > 0 \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega.
\]

(4.2)

By the assumptions on \( f \), we reach that problem (4.2) has a unique positive solution \( v \in C^{1,\sigma}(\Omega) \) with \( \sigma < 1 \). Then for \( C > 1 \) we have

\[
-\Delta_p(Cv) = C^{p-1} f(x, v) + C^{p-1}.
\]

By hypothesis (1.2), we obtain \( -\Delta_p(Cv) \geq f(x, Cv) + C^{p-1} \).
Since \( \nu < p-1 \), one can always choose \( C \) large enough to have \( C^{p-1} > C^\nu |\nabla v|^{\nu} + 1 \). Thus

\[-\Delta_p (Cv) \geq f(x, Cv) + |\nabla Cv|^{\nu} + 1\]

and then \( \overline{u} = Cv \) is a supersolution to problem (4.1).

To prove the existence, we follow the arguments used in the previous section. By the comparison principle in Theorem 2.4 we have that \( u \leq \overline{u} \).

**First case:** \( \frac{2N}{N+1} \leq p < 2 \) and \( \nu < p - 1 \). Since \( p < 2 \), then \( \nu < 1 \), thus as in the proof of Theorem 3.1, we obtain the existence of \( u_n \), the minimal solution to problem

\[-\Delta_p u_n = f(x, u_n) + Q_n(\nabla u_n) \quad \text{in } \Omega,\]

\[u_n > 0 \quad \text{in } \Omega,\]

\[u_n = 0 \quad \text{on } \partial \Omega,\]

where

\[Q_n(\xi) = (|\xi| + \frac{1}{n})^{\nu} \quad \text{for } \xi \in \mathbb{R}^N.\]

It is clear that \( u \leq u_n \leq \overline{u} \). Using \( u_n \) as a test function in (4.3) and by the fact that \( \nu < p - 1 \), it follows that \( \|u_n\|_{W^{1,p}_0(\Omega)} \leq C \).

Then we obtain the existence of \( u \in W^{1,p}_0(\Omega) \) such that \( u_n \rightarrow u \) weakly in \( W^{1,p}_0(\Omega) \) and the existence result follows.

**Second case:** \( 2 < p \) and \( \nu < p - 1 \). For fixed \( \varepsilon > 0 \) small, we claim that problem

\[-L_\varepsilon u = f(x, u) + |\nabla u|^{\nu} \quad \text{in } \Omega,\]

\[u > 0 \quad \text{in } \Omega,\]

\[u = 0 \quad \text{on } \partial \Omega,\]

where

\[-L_\varepsilon u = -\text{div}(\varepsilon + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u),\]

has a minimal solution \( u_\varepsilon \), at least for \( \varepsilon \) small such that \( u \leq u_\varepsilon \leq \overline{u} \).

Since \( u, \overline{u} \in C^{1,\alpha}(\overline{\Omega}) \), then for \( \varepsilon \) small we reach that \( u \) (respectively \( \overline{u} \)) is a subsolution (respectively supersolution) to (4.3).

Fix an \( \varepsilon \) small enough so that the previous statement still holds true, and define

\[D_n(\xi) = \begin{cases} \frac{|\xi|^{\nu}}{1 + \frac{\varepsilon}{n} |\xi|} & \text{if } 1 < \nu < p - 1, \\ \left(|\xi| + \frac{1}{n}\right)^\nu & \text{if } \nu \leq 1. \end{cases} \]

Let \( u_n \) be the minimal solution to problem

\[-L_\varepsilon u_n = f(x, u_n) + D_n(\nabla u_n) \quad \text{in } \Omega,\]

\[v_{k,n} > 0 \quad \text{in } \Omega,\]

\[v_{k,n} = 0 \quad \text{on } \partial \Omega,\]

Notice that \( u_n = \lim_{k \to \infty} v_{n,k} \) where the sequence \( \{v_{n,k}\}_{k \in \mathbb{N}} \) is defined as follows:

\[v_{n,0} = u \quad \text{and for } k \geq 1, \quad v_{k,n} \text{ is the solution to problem} \]

\[-L_\varepsilon v_{k,n} = f(x, v_{k-1,n}) + D_n(\nabla v_{k,n}) \quad \text{in } \Omega,\]

\[v_{k,n} > 0 \quad \text{in } \Omega,\]

\[v_{k,n} = 0 \quad \text{on } \partial \Omega.\]
Using \( u_n \) as a test function in (4.5) and as \( f \) is a nondecreasing Hölder continuous function, we reach \( \| u_n \|_{W_0^{1,p}(\Omega)} \leq C \). Thus, we obtain the existence of \( u_\varepsilon \in W_0^{1,p}(\Omega) \) such that \( u_n \rightharpoonup u_\varepsilon \) weakly in \( W_0^{1,p}(\Omega) \). By the compactness argument in Step 2 of Theorem 3.1 we obtain that \( u_n \rightarrow u_\varepsilon \) strongly in \( W_0^{1,p}(\Omega) \) and \( u_\varepsilon \) is the minimal solution to (4.4). It is clear that \( u \leq u_\varepsilon \leq \overline{u} \), and the claim follows.

The last step is to pass to the limit in \( \varepsilon \). Using \( u_\varepsilon \) as a test function in (4.4), we reach that \( \| u_\varepsilon \|_{W_0^{1,p}(\Omega)} \leq C \) and then \( u_\varepsilon \rightarrow u \) weakly in \( W_0^{1,p}(\Omega) \).

Since \( \nu < p \), a modification of the arguments used in the proof of Theorem 3.1 allows us to obtain that \( u_\varepsilon \rightarrow u \) strongly in \( W_0^{1,p}(\Omega) \). Thus \( u \) solves
\[
\begin{align*}
-\Delta_p u &= f(x, u) + |\nabla u|^\nu \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (4.6)

\[\square\]

**Remark 4.2.** Observe that the condition 1.4 imposed on \( f \) to ensure that 0 is a strict subsolution, is not necessary, indeed one can drop it, and consider as sub-solution the function introduced in [12], in [19] and in [20], defined by \( u = Mh(c\varphi_1) \) where \( M \) and \( c \) are positive constants to be chosen, \( \varphi_1 \) is the first eigenfunction of the p-laplacian and \( h \) is the solution to the differential equation
\[
h''(t) = q(h(t))g(h(t)),
\]
\[
h > 0, \quad h' > 0,
\]
\[
h(0) = h'(0) = 0.
\]

where \( q : (0, +\infty) \rightarrow (0, +\infty) \) is a non-increasing and Hölder continuous function, and \( g(s) \) behaves like \( \frac{1}{s^\beta} \), for some \( \beta > 0 \).

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**References**


[18] P. Lindqvist; On the equation $\Delta_p u + \lambda |u|^{p-2} u = 0$, Proc. Amer. Math. Soc. 109, no. 1 (1990), 157-164.
[22] A. Porretta; On the comparison principle for $p$-Laplace type operators with first order terms, Resultsand developments, Quaderni di Matematica 23, Department of Mathematics, Seconda Università di Napoli, Caserta, 2008.

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