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GROWTH OF SOLUTIONS TO SECOND-ORDER COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence of non-trivial subnormal solutions for second-order linear differential equations. We show that under certain conditions some differential equations do not have subnormal solutions, also that the hyper-order of every solution equals one.

1. Introduction

In this article, we use standard notation from the value distribution theory of meromorphic functions (see [8, 12]). In addition, we denote the order of growth of $f(z)$ by $\sigma(f)$. The hyper-order of $f(z)$ is defined by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Consider the second order homogeneous linear periodic differential equation

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{1.1}$$

where $P(z)$ and $Q(z)$ are polynomials in z and not both constants. It is well known that every solution f of (1.1) is entire.

For be a meromorphic function f, define

$$\sigma_e(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{r} \tag{1.2}$$

to be the e-type order of f. If $f \not\equiv 0$ is a solution of (1.1) satisfying $\sigma_e(f) = 0$, then we say that f is a nontrivial subnormal solution of (1.1).

Wittich [10], Gundersen and Steinbart [7], Xiao [11] etc. have investigated the subnormal solution of (1.1), and obtained good results. In 2007, Chen and Shon [3] studied the existence of subnormal solutions of the general equation

$$f'' + \big(P_1(e^z) + P_2(e^{-z})\big)f' + \big(Q_1(e^z) + Q_2(e^{-z})\big)f = 0, \tag{1.3}$$

and obtained the following results.

Theorem 1.1. *Let $P_j(z)$, $Q_j(z)$ $(j = 1, 2)$ be the polynomials in z. If*

$$\deg Q_1 > \deg P_1 \quad or \quad \deg Q_2 > \deg P_2 \tag{1.4}$$

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then (1.3) *has no nontrivial subnormal solution, and every solution of* (1.3) *satisfies* $\sigma_2(f) = 1$.

Theorem 1.2. *Let* $P_j(z)$, $Q_j(z)$ $(j = 1, 2)$ *be the polynomials in* z. *If*

$$\deg Q_1 < \deg P_1 \quad \text{and} \quad \deg Q_2 < \deg P_2 \tag{1.5}$$

and $Q_1 + Q_2 \not\equiv 0$, *then* (1.3) *has no nontrivial subnormal solution, and every solution of* (1.3) *satisfies* $\sigma_2(f) = 1$.

Question. What can we said when $\deg P_1 = \deg Q_1$ and $\deg P_2 = \deg Q_2$ for (1.3)? We will prove the following theorem.

Theorem 1.3. . *Let*

$$\begin{gathered} P_1(z) = a_n z^n + \dots + a_1 z + a_0, \\ Q_1(z) = b_n z^n + \dots + b_1 z + b_0, \\ P_2(z) = c_m z^m + \dots + c_1 z + c_0, \\ Q_2(z) = d_m z^m + \dots + d_1 z + d_0, \end{gathered}$$

where a_i, b_i $(i = 0, \dots, n)$, c_j, d_j $(j = 0, \dots, m)$ *are constants,* $a_n b_n c_m d_m \neq 0$. *Suppose that* $a_n d_m = c_m b_n$ *and any one of the following three hypotheses holds:*

(i) *there exists* i *satisfying* $\left(-\frac{b_n}{a_n}\right) a_i + b_i \neq 0$, $0 < i < n$; *(ii) there exists* j *satisfying* $\left(-\frac{b_n}{a_n}\right) c_j + d_j \neq 0$, $0 < j < m$;

(iii)
$$\left(-\frac{b_n}{a_n}\right)^2 + \left(-\frac{b_n}{a_n}\right)(a_0 + c_0) + b_0 + d_0 \neq 0.$$

Then (1.3) *has no non-trivial subnormal solution, and every non-trivial solution* f *satisfies* $\sigma_2(f) = 1$.

We remark that the equation

$$f'' + (e^{2z} + e^{-z} + 1)f' + (2e^{2z} + 2e^{-z} - 2)f = 0$$

has a subnormal solution $f_0 = e^{-2z}$. Here $n = 2$, $m = 1$, $a_2 = 1$, $b_2 = 2$, $a_1 = b_1 = 0$, $c_1 = 1$, $d_1 = 2$, $a_0 + c_0 = 1$, $b_0 + d_0 = -2$, $(-\frac{b_2}{a_2}) \cdot a_1 + b_1 = 0$, and $(-\frac{b_2}{a_2})^2 + (-\frac{b_2}{a_2})(a_0 + c_0) + b_0 + d_0 = 0$. This shows that the restrictions (i)–(iii) in Theorem 1.3 are sharp.

Another problem we want to consider in this paper is what condition will guarantee the more general form

$$f'' + \left(P_1(e^{\alpha z}) + P_2(e^{-\alpha z})\right)f' + \left(Q_1(e^{\beta z}) + Q_2(e^{-\beta z})\right)f = 0, \tag{1.6}$$

where $P(z), Q(z)$ are polynomials in z, α, β are complex constants, does not have a non-trivial subnormal solution? We will prove the following theorems.

Theorem 1.4. *Let*

$$\begin{gathered} P_1(z) = a_{1m_1} z^{m_1} + \dots + a_{11} z + a_{10}, \\ P_2(z) = a_{2m_2} z^{m_2} + \dots + a_{21} z + a_{20}, \\ Q_1(z) = b_{1n_1} z^{n_1} + \dots + b_{11} z + b_{10}, \\ Q_2(z) = b_{2n_2} z^{n_2} + \dots + b_{21} z + b_{20}, \end{gathered}$$

where $m_k \geq 1$, $n_k \geq 1\,(k = 1, 2)$ *are integers,* a_{1i_1} $(i_1 = 0, 1, \dots, m_1)$, a_{2i_2} $(i_2 = 0, 1, \dots, m_2)$, b_{1j_1} $(j_1 = 0, 1, \dots, n_1)$, b_{2j_2} $(j_2 = 0, 1, \dots, n_2)$, α *and* β *are complex* *constants, $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2} \neq 0$, $\alpha\beta \neq 0$. Suppose $m_1\alpha = c_1n_1\beta$ $(0 < c_1 < 1)$ or $m_2\alpha = c_2n_2\beta$ $(0 < c_2 < 1)$. Then (1.6) has no non-trivial subnormal solution and every non-trivial solution f satisfies $\sigma_2(f) = 1$.*

Theorem 1.5. *Let*

$$\begin{gathered}
P_1(z) = a_{1m_1}z^{m_1} + \dots + a_{11}z + a_{10},\\
P_2(z) = a_{2m_2}z^{m_2} + \dots + a_{21}z + a_{20},\\
Q_1(z) = b_{1n_1}z^{n_1} + \dots + b_{11}z + b_{10},\\
Q_2(z) = b_{2n_2}z^{n_2} + \dots + b_{21}z + b_{20},
\end{gathered}$$

where $m_k \geq 1$, $n_k \geq 1$ $(k = 1,2)$ are integers, a_{1i_1} $(i_1 = 0,1,\dots,m_1)$, a_{2i_2} $(i_2 = 0,1,\dots,m_2)$, b_{1j_1} $(j_1 = 0,1,\dots,n_1)$, b_{2j_2} $(j_2 = 0,1,\dots,n_2)$, α and β are complex constants, $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2} \neq 0$, $\alpha\beta \neq 0$. Suppose $m_1\alpha = c_1n_1\beta$ $(c_1 > 1)$ and $m_2\alpha = c_2n_2\beta$ $(c_2 > 1)$. Then (1.6) has no non-trivial subnormal solution and every non-trivial solution f satisfies $\sigma_2(f) = 1$.

Note that a subnormal solution $f_0 = e^{-z} + 1$ satisfies the equation

$$f'' - [e^{3z} + e^{2z} + e^{-z}]f' - [e^{2z} + e^{-z}]f = 0.$$

Here $\alpha = \frac{1}{2}$, $\beta = 1/3$, $m_1 = 6$, $m_2 = 2$, $n_1 = 6$, $n_2 = 3$, $m_1\alpha = \frac{3}{2}n_1\beta$ and $m_2\alpha = n_2\beta$. This shows that the restrictions that $m_1\alpha = c_1n_1\beta$ $(c_1 > 1)$ and $m_2\alpha = c_2n_2\beta\,(c_2 > 1)$ can not be omitted.

2. Some lemmas

Let $P(z) = (a + ib)z^n + \dots$ be a polynomial with degree $n \geq 1$. and $z = re^{i\theta}$. We will we denote $\delta(P,\theta) = a\cos(n\theta) - b\sin(n\theta)$.

Lemma 2.1 ([8])**.** *Let $P(z) = a_nz^n + a_{n-1}z^{n-1} + \dots + a_0$ be a polynomial with $a_n \neq 0$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r = |z| > r_0$ we have the inequalities*

$$(1-\varepsilon)|a_n|r^n \leq |P(z)| \leq (1+\varepsilon)|a_n|r^n\,.$$

Lemma 2.2 ([8])**.** *Let $g : (0,+\infty) \to \mathbb{R}$ and $h : (0,+\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ holds for all $r > r_0$.*

Lemma 2.3. [5] *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$. Let $H = \{(k_1,j_1),(k_2,j_2),\dots,(k_q,j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$, for $i = 1,2,\dots,q$. And let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0,2\pi)$ that has linear measure zero, such that if $\psi \in [0,2\pi) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_0$ and for all $(k,j) \in H$, we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.4 ([6, 9])**.** *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then, there exists an infinite sequence of points $z_n = r_ne^{i\theta}$ $(n = 1,2,\dots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\Big|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\Big| \leq \frac{1}{(k-j)!}|z_n|^{(k-j)}(1+o(1)) \quad (j = 0,\dots,k-1). \tag{2.2}$$

Lemma 2.5 ([2]). *Let $f(z)$ be an entire function with $\sigma(f) = \sigma < \infty$. Let there exists a set $E \subset [0, 2\pi)$ with linear measure zero, such that for any $\arg z = \theta_0 \in [0, 2\pi) \setminus E$, $|f(re^{i\theta_0})| \le Mr^k$ ($M = M(\theta_0) > 0$ is a constant, $k(> 0)$ is constant independent of θ_0). Then $f(z)$ is a polynomial of $\deg f \le k$.*

Lemma 2.6 ([1]). *Let A and B be entire functions of finite order. If $f(z)$ is a solution of the equation*

$$f'' + Af' + Bf = 0,$$

then $\sigma_2(f) \le \max\{\sigma(A), \sigma(B)\}$.

Lemma 2.7 ([4]). *Let $f(z)$ be an entire function of infinite order with $\sigma_2 = \alpha\,(0 \le \alpha < \infty)$, and a set $E \subset [1, \infty)$ have a finite logarithmic measure. Then, there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, $r_k \to \infty$, and such that*

(1) *if $\sigma_2(f) = \alpha$ $(0 < \alpha < \infty)$, then for any given ε_1 $(0 < \varepsilon_1 < \alpha)$,*

$$\exp\{r_k^{\alpha-\varepsilon_1}\} < \nu(r_k) < \exp\{r_k^{\alpha+\varepsilon_1}\}, \tag{2.3}$$

(2) *if $\sigma(f) = \infty$ and $\sigma_2(f) = 0$, then for any given ε_2 $(0 < \varepsilon_2 < 1/2)$, and any large $M\,(> 0)$, we have, for r_k sufficiently large,*

$$r_k^M < \nu(r_k) < \exp\{r_k^{\varepsilon_2}\}. \tag{2.4}$$

Lemma 2.8 ([5]). *Let f be a transcendental meromorphic function, and $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j $(0 \le i < j \le 2)$, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,*

$$\Big|\frac{f^{(j)}(z)}{f^{(i)}(z)}\Big| \le B\Big(\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\Big)^{j-i}. \tag{2.5}$$

Remark 2.9 ([3]). From the proof of Lemma 2.8, we can see that the exceptional set E satisfies that if a_n and b_m $(n, m = 1, 2, \dots)$ denote all zeros and poles of f, respectively, $O(a_n)$ and $O(b_m)$ denote sufficiently small neighborhoods of a_n and b_m, respectively, then

$$E = \{|z| : z \in (\cup_{n=1}^{+\infty} O(a_n)) \cup (\cup_{m=1}^{+\infty} O(b_m))\}.$$

Hence, if $f(z)$ is a transcendental entire function, and z is a point that satisfies $|f(z)|$ to be sufficiently large, then (2.5) holds.

3. Proof of Theorem 1.3

Suppose that $f(z)$ is a non-trivial subnormal solution of (1.3). Let

$$h(z) = e^{(b_n/a_n)z} f(z),$$

then $h(z)$ is a non-trivial subnormal solution of

$$\begin{aligned} &h'' + \Big(2(-\frac{b_n}{a_n}) + P_1(e^z) + P_2(e^{-z})\Big)h' \\ &+\Big((-\frac{b_n}{a_n})^2 + (-\frac{b_n}{a_n})\big(P_1(e^z) + P_2(e^{-z})\big) + Q_1(e^z) + Q_2(e^{-z})\Big)h = 0. \end{aligned}$$

Since any one of the following three hypotheses holds:

(i) there exists i satisfying $(-\frac{b_n}{a_n})a_i + b_i \ne 0$, $0 < i < n$;

(ii) there exists j satisfying $(-\frac{b_n}{a_n})c_j + d_j \ne 0$, $0 < j < m$;

(iii)
$$\Big((-\frac{b_n}{a_n})^2+(-\frac{b_n}{a_n})(a_0+c_0)+b_0+d_0\Big)\neq 0,$$
we obtain
$$(-\frac{b_n}{a_n})^2+(-\frac{b_n}{a_n})(P_1(e^z)+P_2(e^{-z}))+Q_1(e^z)+Q_2(e^{-z})\not\equiv 0. \tag{3.1}$$
From $a_nd_m=c_mb_n$, we obtain
$$\deg P_2(z)>m-1\geq \deg[(-\frac{b_n}{a_n})P_2(z)+Q_2(z)]. \tag{3.2}$$
Combining (3.1) and (3.2) with
$$\deg P_1(z)>n-1\geq \deg[(-\frac{b_n}{a_n})P_1(z)+Q_1(z)], \tag{3.3}$$
we obtain the conclusion by using Theorem 1.2.

4. Proof of Theorem 1.4

Suppose $f(\not\equiv 0)$ is a solution of (1.6), then f is an entire function. Next we will prove that f is transcendental. Since $Q_1(e^{\beta z})+Q_2(e^{-\beta z})\not\equiv 0$, we see that any nonzero constant can not be a solution of the (1.6). Now suppose that $f_0=b_nz^n+\dots+b_1z+b_0$, ($n\geq 1, b_n,\dots,b_0$ are constants, $b_n\neq 0$) is a polynomial solution of (1.6).

(1) $m_1\alpha=c_1n_1\beta\,(0<c_1<1)$. Take $z=re^{i\theta}$, such that $\delta(\beta z,\theta)=|\beta|\cos(\arg\beta+\theta)>0$, then $\delta(\alpha z,\theta)=\frac{n_1c_1}{m_1}\delta(\beta z,\theta)>0$. From (1.6) and Lemma 2.1, that for a sufficiently large r and $\varepsilon>0$, we have
$$\begin{aligned}
&(1-\varepsilon)|b_n|r^n|b_{1n_1}|e^{n_1\delta(\beta z,\theta)r}(1-o(1)) \qquad \leq |Q_1(e^{\beta z})+Q_2(e^{-\beta z})|\cdot|f_0|\\
&\leq |f_0''|+|P_1(e^{\alpha z})+P_2(e^{-\alpha z})|\cdot|f_0'|\\
&\leq |a_{1m_1}|e^{m_1\delta(\alpha z,\theta)r}n(n-1)(1+\varepsilon)|b_n|r^{n-1}(1+o(1))\\
&\leq M_1e^{m_1\cdot\frac{n_1c_1}{m_1}\delta(\beta z,\theta)r}r^{n-1}(1+o(1))\\
&\leq M_1e^{n_1c_1\delta(\beta z,\theta)r}r^{n-1}(1+o(1)),
\end{aligned} \tag{4.1}$$
where $M_1>0$ is some constant. Since $0<c_1<1$, we see that (4.1) is a contradiction.

(2) $m_2\alpha=c_2n_2\beta\,(0<c_2<1)$. Take $z=re^{i\theta}$, such that $\delta(\beta z,\theta)=|\beta|\cos(\arg\beta+\theta)<0$, then $\delta(\alpha z,\theta)=\frac{n_2c_2}{m_2}\delta(\beta z,\theta)<0$. From (1.6) and Lemma 2.1, that for a sufficiently large r and $\varepsilon>0$, we have
$$\begin{aligned}
&(1-\varepsilon)|b_n|r^n|b_{2n_2}|e^{-n_2\delta(\beta z,\theta)r}(1-o(1))\\
&\leq |Q_1(e^{\beta z})+Q_2(e^{-\beta z})|\cdot|f_0|\\
&\leq |f_0''|+|P_1(e^{\alpha z})+P_2(e^{-\alpha z})|\cdot|f_0'|\\
&\leq |a_{2m_2}|e^{-m_2\delta(\alpha z,\theta)r}n(n-1)(1+\varepsilon)|b_n|r^{n-1}(1+o(1))\\
&\leq M_2e^{-m_2\cdot\frac{n_2c_2}{m_2}\delta(\beta z,\theta)r}r^{n-1}(1+o(1))\\
&\leq M_2e^{-n_2c_2\delta(\beta z,\theta)r}r^{n-1}(1+o(1)),
\end{aligned} \tag{4.2}$$

where $M_2 > 0$ is some constant. Since $0 < c_2 < 1$, we see that (4.2) is also a contradiction. Thus we obtain that f is transcendental.

By Lemma 2.6 and $\max\{\sigma(P_1(e^{\alpha z})), \sigma(P_2(e^{-\alpha z})), \sigma(Q_1(e^{\beta z})), \sigma(Q_2(e^{-\beta z}))\} = 1$, we see that $\sigma_2(f) \leq 1$. By Lemma 2.8, we can see that there exists a subset $E \subset (1, \infty)$ having a logarithmic measure $m_l E < \infty$ and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E$, we have

$$|\frac{f^{(j)}(z)}{f(z)}| \leq B[T(2r, f)]^{j+1}, \quad j = 1, 2. \tag{4.3}$$

(1) Suppose $m_1\alpha = c_1 n_1 \beta$ $(0 < c_1 < 1)$. Take $z = re^{i\theta}$, such that $\delta(\beta z, \theta) > 0$, then $\delta(\alpha z, \theta) = \frac{n_1 c_1}{m_1}\delta(\beta z, \theta) > 0$. From (1.6), (4.3), that for a sufficiently large r and $r \notin [0,1] \cup E$, we have

$$\begin{aligned}
&(1-\varepsilon)|b_{1n_1}|e^{n_1\delta(\beta z,\theta)r}(1-o(1)) \\
&\leq |Q_1(e^{\beta z}) + Q_2(e^{-\beta z})| \\
&\leq |\frac{f''(z)}{f(z)}| + |P_1(e^{\alpha z}) + P_2(e^{-\alpha z})||\frac{f'(z)}{f(z)}| \\
&\leq B[T(2r,f)]^3 + (1+\varepsilon)|a_{1m_1}|e^{m_1\delta(\alpha z,\theta)r}B[T(2r,f)]^2(1+o(1)) \\
&\leq C[T(2r,f)]^3 e^{m_1\cdot\frac{n_1c_1}{m_1}\delta(\beta z,\theta)r}(1+o(1)) \\
&\leq C[T(2r,f)]^3 e^{n_1c_1\delta(\beta z,\theta)r}(1+o(1)).
\end{aligned} \tag{4.4}$$

Since $0 < c_1 < 1$, by lemma 2.2, (4.4), we obtain $\sigma_2(f) \geq 1$. So $\sigma_2(f) = 1$.

Next we prove that any $f(\not\equiv 0)$ is not subnormal. If f is subnormal, then for any $\varepsilon > 0$,

$$T(r, f) \leq e^{\varepsilon r}. \tag{4.5}$$

When taking $z = re^{i\theta}$, such that $\delta(\beta z, \theta) > 0$, by (4.4) and (4.5), we deduce that

$$\begin{aligned}
(1-\varepsilon)|b_{1n_1}|e^{n_1\delta(\beta z,\theta)r}(1-o(1)) &\leq C[T(2r,f)]^3 e^{n_1c_1\delta(\beta z,\theta)r}(1+o(1)) \\
&\leq Ce^{6\varepsilon r}\cdot e^{n_1c_1\delta(\beta z,\theta)r}(1+o(1)).
\end{aligned} \tag{4.6}$$

We see that (4.6) is a contradiction when $0 < \varepsilon < \frac{1}{6}n_1\delta(\beta z, \theta)(1 - c_1)$. Hence (1.6) has no non-trivial subnormal solution and every solution f satisfies $\sigma_2(f) = 1$.

(2) Suppose $m_2\alpha = c_2 n_2 \beta$ $(0 < c_2 < 1)$. Take $z = re^{i\theta}$, such that $\delta(\beta z, \theta) < 0$, then $\delta(\alpha z, \theta) = \frac{n_2 c_2}{m_2}\delta(\beta z, \theta) < 0$. Using the similar method as in the proof of (1), we obtain the conclusion.

5. Proof of Theorem 1.5

Suppose that $f(\not\equiv 0)$ is a solution of (1.6), then f is an entire function. Next we will prove that f is transcendental. Since $Q_1(e^{\beta z}) + Q_2(e^{-\beta z}) \not\equiv 0$, we see that any nonzero constant can not be a solution of the Eq.(1.6). Now suppose that $f_0 = b_n z^n + \dots + b_1 z + b_0$, $(n \geq 1, b_n, \dots, b_0$ are constants, $b_n \neq 0)$ is a polynomial solution of (1.6).

Take $z = re^{i\theta}$, such that $\delta(\alpha z, \theta) = |\alpha|\cos(\arg\alpha + \theta) > 0$, then $\delta(\beta z, \theta) = \frac{m_1}{c_1 n_1}\delta(\alpha z, \theta) > 0$. From (1.6) and Lemma 2.1, that for a sufficiently large r and

$\varepsilon > 0$, we have

$$\begin{aligned}
&(1-\varepsilon)|b_n|nr^{n-1}|a_{1m_1}|e^{m_1\delta(\alpha z,\theta)r}(1-o(1)) \qquad \le |P_1(e^{\alpha z})+P_2(e^{-\alpha z})|\cdot|f_0'|\\
&\le |f_0''|+|Q_1(e^{\beta z})+Q_2(e^{-\beta z})|\cdot|f_0|\\
&\le |b_{1n_1}|e^{n_1\delta(\beta z,\theta)r}n(n-1)(1+\varepsilon)|b_n|r^n(1+o(1))\\
&\le Me^{n_1\cdot\frac{m_1}{c_1n_1}\delta(\alpha z,\theta)r}r^n(1+o(1))\\
&\le Me^{\frac{m_1}{c_1}\delta(\alpha z,\theta)r}r^n(1+o(1)),
\end{aligned} \tag{5.1}$$

where $M>0$ is some constant. Since $c_1>1$, we see that (5.1) is a contradiction. Thus we obtain that f is transcendental.

First step. We prove that $\sigma(f)=\infty$. We assume that $\sigma(f)=\sigma<\infty$. By Lemma 2.3, we know that for any given $\varepsilon>0$, there exists a set $E\subset[0,2\pi)$ which has linear measure zero, such that if $\psi\in[0,2\pi)\setminus E$, then there is a constant $R_0=R_0(\psi)>1$, such that for all z satisfying $\arg z=\psi$ and $|z|=r\ge R_0$, we have

$$|\frac{f''(z)}{f'(z)}|\le r^{\sigma-1+\varepsilon}. \tag{5.2}$$

Let $H=\{\theta\in[0,2\pi):\delta(\alpha z,\theta)=0\}$; then H is a finite set. Now we take a ray $\arg z=\theta\in[0,2\pi)\setminus(E\cup H)$, then $\delta(\alpha z,\theta)>0$ or $\delta(\alpha z,\theta)<0$. We divide the proof into the following two cases.

Case 1. If $\delta(\alpha z,\theta)>0$, then $\delta(\beta z,\theta)=\frac{m_1}{c_1n_1}\delta(\alpha z,\theta)>0$, $\delta(-\alpha z,\theta)<0$ and $\delta(-\beta z,\theta)<0$. We assert that $|f'(re^{i\theta})|$ is bounded on the ray $\arg z=\theta$. If $|f'(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.4, there exists a sequence of points $z_t=r_te^{i\theta}(t=1,2,\dots)$ such that as $r_t\to\infty$, $f'(z_t)\to\infty$ and

$$|\frac{f(z_t)}{f'(z_t)}|\le r_t(1+o(1)). \tag{5.3}$$

By (1.6), we obtain that

$$-[P_1(e^{\alpha z_t})+P_2(e^{-\alpha z_t})]=\frac{f''(z_t)}{f'(z_t)}+[Q_1(e^{\beta z_t})+Q_2(e^{-\beta z_t})]\cdot\frac{f(z_t)}{f'(z_t)}. \tag{5.4}$$

From $\delta(\alpha z,\theta)>0$, we have

$$|P_1(e^{\alpha z_t})+P_2(e^{-\alpha z_t})|\ge(1-\varepsilon)|a_{1m_1}|e^{m_1\delta(\alpha z_t,\theta)r_t}(1-o(1)), \tag{5.5}$$

$$|Q_1(e^{\beta z_t})+Q_2(e^{-\beta z_t})|\le Me^{n_1\delta(\beta z_t,\theta)r_t}(1+o(1)). \tag{5.6}$$

Substituting (5.2), (5.3), (5.5) and (5.6) in (5.4), we obtain

$$\begin{aligned}
&(1-\varepsilon)|a_{1m_1}|e^{m_1\delta(\alpha z_t,\theta)r_t}(1-o(1))\\
&\le r_t^{\sigma-1+\varepsilon}+Me^{n_1\delta(\beta z_t,\theta)r_t}(1+o(1))r_t(1+o(1))\\
&\le Mr_t^{\sigma+\varepsilon}e^{\frac{m_1}{c_1}\delta(\alpha z_t,\theta)r_t}(1+o(1)).
\end{aligned} \tag{5.7}$$

Since $c_1>1$, $\delta(\alpha z_t,\theta)>0$, when $r_t\to\infty$, (5.7) is a contradiction. Hence $|f'(re^{i\theta})|\le C$. So

$$|f(re^{i\theta})|\le Cr. \tag{5.8}$$

Case 2. If $\delta(\alpha z,\theta) < 0$, then $\delta(\beta z,\theta) = \frac{m_2}{c_2 n_2}\delta(\alpha z,\theta) < 0$, $\delta(-\alpha z,\theta) > 0$ and $\delta(-\beta z,\theta) > 0$. Using the similar method as above, we can obtain that

$$|f(re^{i\theta})| \le Cr. \tag{5.9}$$

Since the linear measure of $E \cup H$ is zero, by (5.8), (5.9) and Lemma 2.5, we know that $f(z)$ is a polynomial, which contradicts the assumption that $f(z)$ is transcendental. Therefore $\sigma(f) = \infty$.

Second step. We prove that (1.6) has no non-trivial subnormal solution. Now suppose that (1.6) has a non-trivial subnormal solution f_0. By the conclusion in the first step, $\sigma(f_0) = \infty$. By Lemma 2.6, we see that $\sigma_2(f_0) \le 1$. Set $\sigma_2(f_0) = \omega \le 1$. By Lemma 2.8, we see that there exists a subset $E_1 \subset (1,\infty)$ having finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$|\frac{f_0^{(j)}(z)}{f_0(z)}| \le B[T(2r, f_0)]^3, \quad (j = 1,2). \tag{5.10}$$

From the Wiman-Valiron theory, there is a set $E_2 \subset (1,\infty)$ having finite logarithmic measure, so we can choose z satisfying $|z| = r \notin E_2$ and $|f_0(z)| = M(r, f_0)$. Thus, we have

$$\frac{f_0^{(j)}(z)}{f_0(z)} = \big(\frac{\upsilon(r)}{z}\big)^j (1 + o(1)), \quad j = 1,2, \tag{5.11}$$

where $\upsilon(r)$ is the central index of $f_0(z)$.

By Lemma 2.7, we see that there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f_0(z_n)| = M(r_n, f_0)$, $\theta_n \in [0,2\pi)$, $\lim_{n\to\infty} \theta_n = \theta_0 \in [0,2\pi)$, $r_n \notin [0,1] \cup E_1 \cup E_2$, $r_n \to \infty$, and if $\omega > 0$, we see that for any given ε_1 $(0 < \varepsilon_1 < \omega)$, and for sufficiently large r_n,

$$\exp\{r_n^{\omega-\varepsilon_1}\} < \upsilon(r_n) < \exp\{r_n^{\omega+\varepsilon_1}\}, \tag{5.12}$$

and if $\omega = 0$, then by $\sigma(f_0) = \infty$ and Lemma 2.7, we see that for any given ε_2 $(0 < \varepsilon_2 < 1/2)$, and for any sufficiently large M, as r_n is sufficiently large,

$$r_n^M < \upsilon(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{5.13}$$

From (5.12) and (5.13), we obtain that

$$\upsilon(r_n) > r_n, \quad r_n \to \infty. \tag{5.14}$$

For θ_0, let $\delta = \delta(\alpha z,\theta_0) = |\alpha|\cos(\arg\alpha + \theta_0)$, then $\delta < 0$, or $\delta > 0$, or $\delta = 0$. We divide this proof into three cases.

Case 1. $\delta > 0$. By $\theta_n \to \theta_0$, we see that there is a constant $N(> 0)$ such that, as $n > N$, $\delta(\alpha z_n,\theta_n) > 0$. Since f_0 is a subnormal solution, for any given $\varepsilon(0 < \varepsilon < \frac{1}{12}(1 - \frac{1}{c_1})\delta(\alpha z_n,\theta_n))$, we have

$$[T(2r_n, f_0)]^3 \le e^{6\varepsilon r_n} \le e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z_n,\theta_n) r_n}. \tag{5.15}$$

By (5.10), (5.11), (5.15), we have

$$\begin{aligned}
\big(\frac{\upsilon(r_n)}{r_n}\big)^j (1 + o(1)) &= |\frac{f_0^{(j)}(z_n)}{f_0(z_n)}| \\
&\le B[T(2r_n, f_0)]^3 \\
&\le B e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z_n,\theta_n) r_n}, \quad j = 1,2.
\end{aligned} \tag{5.16}$$

Since $\delta(\alpha z_n, \theta_n) > 0$, from (1.6), (5.11), we obtain that

$$\begin{aligned}
&(1-\varepsilon)\frac{\upsilon(r_n)}{r_n}|a_{1m_1}|e^{m_1\delta(\alpha z_n,\theta_n)r_n}(1-o(1))\\
&\le |\frac{f_0'(z_n)}{f_0(z_n)}\big(P_1(e^{\alpha z_n})+P_2(e^{-\alpha z_n})\big)|\\
&= |\frac{f_0''(z_n)}{f_0(z_n)}+[Q_1(e^{\beta z_n})+Q_2(e^{-\beta z_n})]|\\
&\le \big(\frac{\upsilon(r_n)}{r_n}\big)^2(1+o(1))+(1+\varepsilon)|b_{1n_1}|e^{n_1\delta(\beta z_n,\theta_n)r_n}(1+o(1))\\
&\le M_1\big(\frac{\upsilon(r_n)}{r_n}\big)^2 e^{\frac{m_1}{c_1}\delta(\alpha z_n,\theta_n)r_n}(1+o(1)).
\end{aligned} \tag{5.17}$$

From (5.16) and (5.17), we can obtain

$$\begin{aligned}
&(1-\varepsilon)|a_{1m_1}|e^{m_1(1-\frac{1}{c_1})\delta(\alpha z_n,\theta_n)r_n}(1-o(1))\\
&\le M_1Be^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z_n,\theta_n)r_n}(1+o(1)).
\end{aligned} \tag{5.18}$$

Since $c_1 > 1$ and $m_1 \ge 1$, we see that (5.18) is a contradiction.

Case 2. $\delta < 0$. By $\theta_n \to \theta_0$, we see that there is a constant $N(> 0)$ such that, as $n > N$, $\delta(\alpha z_n, \theta_n) < 0$. Since f_0 is a subnormal solution, for any given ε $(0 < \varepsilon < -\frac{1}{12}(1-\frac{1}{c_2})\delta(\alpha z_n, \theta_n))$, we have

$$[T(2r_n, f_0)]^3 \le e^{6\varepsilon r_n} \le e^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}. \tag{5.19}$$

By (5.10), (5.11), (5.19) we have

$$\begin{aligned}
\big(\frac{\upsilon(r_n)}{r_n}\big)^j(1+o(1)) &= |\frac{f_0^{(j)}(z_n)}{f_0(z_n)}| \le B[T(2r_n,f_0)]^3\\
&\le Be^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}, \quad j=1,2.
\end{aligned} \tag{5.20}$$

By (5.11) and (1.6), we obtain

$$\begin{aligned}
&(1-\varepsilon)\frac{\upsilon(r_n)}{r_n}|a_{2m_2}|e^{-m_2\delta(\alpha z_n,\theta_n)r_n}(1-o(1))\\
&\le |\frac{f_0'(z_n)}{f_0(z_n)}\big(P_1(e^{\alpha z_n})+P_2(e^{-\alpha z_n})\big)|\\
&= |\frac{f_0''(z_n)}{f_0(z_n)}+[Q_1(e^{\beta z_n})+Q_2(e^{-\beta z_n})]|\\
&\le \big(\frac{\upsilon(r_n)}{r_n}\big)^2(1+o(1))+(1+\varepsilon)|b_{2n_2}|e^{-n_2\delta(\beta z_n,\theta_n)r_n}(1+o(1))\\
&\le M_2\big(\frac{\upsilon(r_n)}{r_n}\big)^2 e^{-\frac{m_2}{c_2}\delta(\alpha z_n,\theta_n)r_n}(1+o(1)).
\end{aligned} \tag{5.21}$$

From (5.20) and (5.21), we can deduce that

$$\begin{aligned}
&(1-\varepsilon)|a_{2m_2}|e^{-m_2(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}(1-o(1))\\
&\le M_2Be^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z_n,\theta_n)r_n}(1+o(1)).
\end{aligned} \tag{5.22}$$

Since $c_2 > 1$ and $m_2 \ge 1$, we see that (5.22) is a contradiction.

Case 3. $\delta = 0$. Then $\theta_0 \in H = \{\theta | \theta \in [0, 2\pi), \delta(\alpha z, \theta) = 0\}$. Since $\theta_n \to \theta_0$, for any given $\varepsilon > 0$, we see that there is an integer $N\,(> 0)$, as $n > N$, $\theta_n \in [\theta_0 - \varepsilon, \theta_0 + \varepsilon]$ and $z_n = r_n e^{i\theta_n} \in \overline{\Omega} = \{z : \theta_0 - \varepsilon \le \arg z \le \theta_0 + \varepsilon\}$. By Lemma 2.8, there exists a subset $E_3 \subset (1, \infty)$ having finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$, we have

$$|\frac{f_0''(z)}{f_0'(z)}| \le B[T(2r, f_0')]^2. \tag{5.23}$$

Now we consider the growth of $f_0(re^{i\theta})$ on a ray $\arg z = \theta \in \overline{\Omega} \setminus \{\theta_0\}$. Denote $\Omega_1 = [\theta_0 - \varepsilon, \theta_0)$, $\Omega_2 = (\theta_0, \theta_0 + \varepsilon]$. We can easily see that when $\theta_1 \in \Omega_1, \theta_2 \in \Omega_2$, then $\delta(\alpha z, \theta_1) \cdot \delta(\alpha z, \theta_2) < 0$. Without loss of generality, we suppose that $\delta(\alpha z, \theta) > 0\,(\theta \in \Omega_1)$ and $\delta(\alpha z, \theta) < 0\,(\theta \in \Omega_2)$.

Since when $\theta \in \Omega_1$, $\delta(\alpha z, \theta) > 0$. Recall f_0 is subnormal, then for any given ε $(0 < \varepsilon < \frac{1}{8}(1 - \frac{1}{c_1})\delta(\alpha z, \theta))$,

$$[T(2r, f_0')]^2 \le e^{4\varepsilon r} \le e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha z, \theta) r}. \tag{5.24}$$

We assert that $|f_0'(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f_0'(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 2.4, there exists a sequence $\{y_j = R_j e^{i\theta}\}$ such that $R_j \to \infty$, $f_0'(y_j) \to \infty$ and

$$|\frac{f_0(y_j)}{f_0'(y_j)}| \le R_j(1 + o(1)). \tag{5.25}$$

By (5.23), (5.24), we see that for sufficiently large j,

$$|\frac{f_0''(y_j)}{f_0'(y_j)}| \le B[T(2R_j, f_0')]^2 \le Be^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha y_j, \theta) R_j}. \tag{5.26}$$

By (1.6), we deduce that

$$\begin{aligned} &(1-\varepsilon)|a_{1m_1}| e^{m_1 \delta(\alpha y_j, \theta) R_j}(1 - o(1)) \\ &\le | - \big(P_1(e^{\alpha y_j}) + P_2(e^{-\alpha y_j})\big) | \\ &\le |\frac{f_0''(y_j)}{f_0'(y_j)}| + |Q_1(e^{\beta y_j}) + Q_2(e^{-\beta y_j})| \cdot |\frac{f_0(y_j)}{f_0'(y_j)}| \\ &\le C_1 e^{\frac{1}{2}(1-\frac{1}{c_1})\delta(\alpha y_j, \theta) R_j} e^{n_1 \delta(\beta y_j, \theta) R_j} R_j(1 + o(1)) \\ &\le C_1 e^{[\frac{1}{2}(1-\frac{1}{c_1}) + \frac{m_1}{c_1}]\delta(\alpha y_j, \theta) R_j} R_j(1 + o(1)). \end{aligned} \tag{5.27}$$

Since $\delta(\alpha y_j, \theta) > 0$, $c_1 > 1$, we know that when $R_j \to \infty$, (5.27) is a contradiction. Hence

$$|f_0(re^{i\theta})| \le Cr, \tag{5.28}$$

on the ray $\arg z = \theta \in \Omega_1$.

When $\theta \in \Omega_2$, $\delta(\alpha z, \theta) < 0$. Recall f_0 is subnormal, then for any given ε $(0 < \varepsilon < -\frac{1}{8}(1 - \frac{1}{c_2})\delta(\alpha z, \theta))$,

$$[T(2r, f_0')]^2 \le e^{4\varepsilon r} \le e^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha z, \theta) r}. \tag{5.29}$$

We assert that $|f_0'(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f_0'(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, using the similar proof as above, we can obtain that

$$\begin{aligned}&(1-\varepsilon)|a_{2m_2}|e^{-m_2(1-\frac{1}{c_2})\delta(\alpha y_j,\theta)R_j}(1-o(1))\\ &\le C_2 e^{-\frac{1}{2}(1-\frac{1}{c_2})\delta(\alpha y_j,\theta)R_j}R_j(1+o(1))\end{aligned} \tag{5.30}$$

Since $\delta(\alpha y_j,\theta)<0$ and $c_2>1$, we know that when $R_j\to\infty$, (5.30) is a contradiction. Hence

$$|f_0(re^{i\theta})|\le Cr, \tag{5.31}$$

on the ray $\arg z=\theta\in\Omega_2$. By (5.28), (5.31), we see that $|f_0(re^{i\theta})|$ satisfies

$$|f_0(re^{i\theta})|\le Cr, \tag{5.32}$$

on the ray $\arg z=\theta\in\overline{\Omega}\setminus\{\theta_0\}$. However, since f_0 is transcendental and $\{z_n=r_ne^{i\theta_n}\}$ satisfies $|f_0(z_n)|=M(r_n,f_0)$, we see that for any large $N(>2)$, as n is sufficiently large,

$$|f_0(z_n)|=|f_0(r_ne^{i\theta_n})|\ge r_n^N. \tag{5.33}$$

Since $z_n\in\overline{\Omega}$, by (5.32), (5.33), we see that for sufficiently,large n,

$$\theta_n=\theta_0.$$

Thus for sufficiently large n, $\delta(\alpha z_n,\theta_n)=0$ and

$$|P_1(e^{\alpha z_n})+P_2(e^{-\alpha z_n})|\le C,\quad |Q_1(e^{\beta z_n})+Q_2(e^{-\beta z_n})|\le C. \tag{5.34}$$

By (1.6), (5.11), we obtain that

$$\begin{aligned}&-\Big(\frac{\upsilon(r_n)}{z_n}\Big)^2(1+o(1))\\ &=\big(P_1(e^{\alpha z_n})+P_2(e^{-\alpha z_n})\big)\Big(\frac{\upsilon(r_n)}{z_n}\Big)(1+o(1))+[Q_1(e^{\beta z_n})+Q_2(e^{-\beta z_n})].\end{aligned} \tag{5.35}$$

By (5.34), (5.35) and (5.14) we obtain that

$$\upsilon(r_n)\le 2Cr_n, \tag{5.36}$$

by (5.12) (or (5.13)), we see that (5.36) is a contradiction. Hence (1.6) has no non-trivial subnormal solution.

Third step. We prove that all solutions of (1.6) satisfies $\sigma_2(f)=1$. If there is a solution f_1 satisfying $\sigma_2(f_1)<1$, then $\sigma_e(f_1)=0$, that is to say f_1 is subnormal, but this contradicts the conclusion in step 2. Hence $\sigma_2(f)=1$. This completes the proof of Theorem 1.5.

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