Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 68, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

SECOND-ORDER BOUNDARY-VALUE PROBLEMS WITH VARIABLE EXPONENTS

GIUSEPPINA D'AGUÌ

Abstract. In this article, we study ordinary differential equations with $p(x)$-Laplacian and subject to small perturbations of nonhomogeneous Neumann conditions. We establish the existence of an unbounded sequence of weak solutions by using variational methods.

1. Introduction

In this article, we consider the following boundary value problem involving an ordinary differential equation with $p(x)$-Laplacian operator, and nonhomogeneous Neumann conditions:

$$\begin{gathered}
-(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u(x)|^{p(x)-2}u(x) = \lambda f(x,u) \quad \text{in }]0,1[\\
|u'(0)|^{p(0)-2}u'(0) = -\mu g(u(0)), \\
|u'(1)|^{p(1)-2}u'(1) = \mu h(u(1)).
\end{gathered} \tag{1.1}$$

Here $p \in C([0,1],\mathbb{R})$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, (that is $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$, $t \to f(x,t)$ is continuous for almost every $x \in [0,1]$), $g, h : \mathbb{R} \to \mathbb{R}$ are nonnegative continuous functions, λ and μ are real parameters with $\lambda > 0$ and $\mu \geq 0$, $\alpha \in L^\infty([0,1])$, with $\operatorname{ess\,inf}_{[0,1]} \alpha > 0$.

The necessary framework for the study of problems involving the $p(x)$-Laplacian operator is represented by the functions spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. The basic properties of such spaces can be found in [10, 13], and for a complete overview on this subject we refer to [7, 16].

Differential problems with nonstandard $p(x)$-growth have been studied by many authors, see for instance [5, 6, 8, 9, 15, 17, 19] and the references therein.

When $p(x) = p$ is constant, (1.1) reduces to the ordinary p-Laplacian problem

$$\begin{gathered}
-(|u'(x)|^{p-2}u'(x))' + \alpha(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in }]0,1[\\
|u'(0)|^{p-2}u'(0) = -\mu g(u(0)), \\
|u'(1)|^{p-2}u'(1) = \mu h(u(1)).
\end{gathered} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34B15, 34L30.
Key words and phrases. Neumann problem; $p(x)$-Laplacian; variable exponent Sobolev spaces.

Submitted December 17, 2013. Published March 7, 2014.

Some results concerning such a problem, when $h \equiv g$, can be found in [4] (see for instance Theorem 4.1), where the authors obtain infinitely many solutions for a class of variational-hemivariational inequality by using the nonsmooth analysis.

In [11], the authors obtain one solution for weighed $p(x)$-Laplacian ordinary system, generalizing some results obtained by Hartman [12] and Mawhin [14] which studied, respectively, the constant cases $p(x) = 2$ and $p(x) = p$.

Zhang [21], via Leray-Schauder degree, obtained sufficient conditions for the existence of one solution for a weighted $p(x)$-Laplacian system boundary value problem.

By using minimax methods, in [20], the authors study the periodic solutions for a class of systems with nonstandard $p(x)$-growth.

In the present paper, under an appropriate oscillating behaviour of the primitive of the nonlinearity and a suitable growth at infinity of the primitives of g and h, the existence of infinitely many weak solutions for (1.1), is obtained, for all λ belonging to a precise interval and provided μ small enough (Theorem 3.1). We refer also to [1, 2] and the references therein for arguments closely related to our results. Here, as a particular case, we point out the following result on the existence of infinitely many solutions to problem $(P_{\lambda,\mu})$, when $\alpha(x) = 1$ for all $x \in [0, 1]$.

Theorem 1.1. *Let $p \in C([0,1],\mathbb{R})$ such that $1 < p^- := \min_{x\in[0,1]} p(x) \leq p^+ := \max_{x\in[0,1]} p(x)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^{p^-}} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^{p^+}} = +\infty.$$

Then, for each $g : \mathbb{R} \to \mathbb{R}$ and for each $h : \mathbb{R} \to \mathbb{R}$ nonnegative continuous functions such that

$$\lim_{\xi\to+\infty} \frac{g(\xi)}{\xi^{p^- -1}} = \lim_{\xi\to+\infty} \frac{h(\xi)}{\xi^{p^- -1}} = 0,$$

the problem

$$\begin{gathered} -(|u'(x)|^{p(x)-2}u'(x))' + |u|^{p(x)-2}u = f(u) \quad \textit{in }]0,1[\\ |u'(0)|^{p(0)-2}u'(0) = -g(u(0)), \\ |u'(1)|^{p(1)-2}u'(1) = h(u(1)) \end{gathered}$$

admits infinitely many distinct pairwise nonnegative weak solutions.

It is worth mentioning that in the study of existence of infinitely many solutions for the $p(x)$-Laplacian, symmetric assumptions (see [19]) or change sign hypothesis on the nonlinearity (see [5]) are requested, while, in our main result such conditions are not required (see also Remark 3.4). In particular, here, we can study problems with positive nonlinearity (see Example 3.3).

This paper is arranged as follows. In Section 2, some definitions and results on variable exponent Lebesgue and Sobolev spaces are collected. In particular, in Proposition 2.1, an appropriate embedding constant of the space $W^{1,p(x)}([0,1])$ into $C^0([0,1])$ is estimated. Moreover, the abstract critical points theorem (Theorem 2.3) is recalled. Finally, in Section 3, our main result is established, then some particular case and some example are presented.

2. Variable exponent Lebesgue and Sobolev space

Here and in the sequel, we assume that $p \in C([0,1], \mathbb{R})$ satisfies the condition

$$1 < p^- := \min_{x \in [0,1]} p(x) \le p^+ := \max_{x \in [0,1]} p(x). \tag{2.1}$$

The variable exponent Lebesgue spaces are defined as follows

$$L^{p(x)}([0,1]) = \big\{u : [0,1] \to \mathbb{R} : u \text{ is measurable and } \int_0^1 |u|^{p(x)} dx < +\infty \big\}.$$

On $L^{p(x)}([0,1])$, we consider the norm

$$\|u\|_{L^{p(x)}([0,1])} := \inf \big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \le 1 \big\}.$$

Let X be the generalized Lebesgue-Sobolev space $W^{1,p(x)}([0,1])$ defined by

$$W^{1,p(x)}([0,1]) := \big\{u : u \in L^{p(x)}([0,1]), u' \in L^{p(x)}([0,1])\big\},$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}([0,1])} := \|u\|_{L^{p(x)}([0,1])} + \||u|\|_{L^{p(x)}([0,1])}. \tag{2.2}$$

It is well known (see [10]) that, in view of (2.1), both $L^{p(x)}([0,1])$ and $W^{1,p(x)}([0,1])$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces. Moreover, since $\alpha \in L^\infty([0,1])$, with $\alpha_- := \operatorname{ess\,inf}_{x \in [0,1]} \alpha(x) > 0$ is assumed, the norm

$$\|u\|_\alpha := \inf \big\{\sigma > 0 : \int_0^1 \Big(|\frac{u'(x)}{\sigma}|^{p(x)} + \alpha(x) |\frac{u(x)}{\sigma}|^{p(x)} \Big) dx \le 1 \big\},$$

on $W^{1,p(x)}([0,1])$ is equivalent to that introduced in (2.2).

Next, we give an estimate on the embedding constant m of $W^{1,p(x)}([0,1])$ with norm $\| \cdot \|_\alpha$ in $C^0([0,1])$.

Proposition 2.1. *For all $u \in W^{1,p(x)}([0,1])$, one has*

$$\|u\|_{C^0([0,1]} \le m \|u\|_\alpha, \tag{2.3}$$

where

$$m = \begin{cases} 2\Big[\frac{1}{\alpha_-^{\frac{p^+}{p^-(1-p^+)}} + 1}\Big]^{1/p^+} + \Big[1 - \frac{1}{\alpha_-^{\frac{p^+}{p^-(1-p^+)}} + 1}\Big]^{1/p^+} \frac{2}{\alpha_-^{1/p^-}} & \text{if } \alpha_- < 1 \\ 2\Big[\frac{1}{\alpha_-^{\frac{1}{1-p^+}} + 1}\Big]^{1/p^+} + \Big[1 - \frac{1}{\alpha_-^{\frac{1}{(1-p^+)}} + 1}\Big]^{1/p^+} \frac{2}{\alpha_-^{1/p^+}} & \text{if } \alpha_- \ge 1. \end{cases}$$

Proof. First we observe that

$$|u(t)| \le \int_0^1 |u'(t)| dt + \int_0^1 |u(t)| dt, \quad \forall u \in W^{1,p(x)}[(0,1)].$$

Moreover, taking into account Hölder inequality in variable exponent Lebesgue space (see, for instance, [7, Lemma 3.2.20]), one has

$$\|u\|_{L^1[0,1]} \le 2\|u\|_{L^{p(x)}[0,1]},$$
$$\|u'\|_{L^1[0,1]} \le 2\|u'\|_{L^{p(x)}[0,1]}.$$

Therefore,

$$\|u(t)\|_{C^0([0,1])} \le 2\|u\|_{W^{p(x)}[0,1]}, \quad \forall u \in W^{1,p(x)}[(0,1)]. \tag{2.4}$$

In the variable exponent Sobolev space, we consider the equivalent norm

$$\begin{aligned}\|u\|_\alpha &:= \inf\big\{\lambda > 0 : \int_0^1 \Big(|\frac{u'(x)}{\lambda}|^{p(x)} + \alpha(x)|\frac{u(x)}{\lambda}|^{p(x)}\Big)dx \le 1\big\}\\ &= \inf\{\lambda > 0 : \rho_\alpha(\frac{u}{\lambda}) \le 1\}.\end{aligned}$$

From definition of $\|u\|_\alpha$ one has

$$\begin{aligned}1 &\ge \rho_\alpha\big(\frac{u}{\|u\|_\alpha}\big)\\ &= \int_0^1 \Big(|\frac{u'(x)}{\|u\|_\alpha}|^{p(x)} + \alpha(x)|\frac{u(x)}{\|u\|_\alpha}|^{p(x)}\Big)dx\\ &\ge \int_0^1 \Big(|\frac{u'(x)}{\|u\|_\alpha}|^{p(x)} + \alpha_-|\frac{u(x)}{\|u\|_\alpha}|^{p(x)}\Big)dx.\end{aligned}$$

Now we suppose that $\alpha_- < 1$, one has

$$1 \ge \int_0^1 \Big(|\frac{u'(x)}{\|u\|_\alpha}|^{p(x)} + |\frac{u(x)}{(\frac{1}{\alpha^-})^{1/p^-}\|u\|_\alpha}|^{p(x)}\Big)dx.$$

This leads to

$$\int_0^1 |\frac{u'(x)}{\|u\|_\alpha}|^{p(x)}dx = k \le 1 \tag{2.5}$$

$$\int_0^1 |\frac{u(x)}{\big(\frac{1}{\alpha^-}\big)^{1/p^-}\|u\|_a}|^{p(x)}dx = 1 - k \le 1. \tag{2.6}$$

From (2.5) and (2.6), dividing by respectively by k and $1-k$, we obtain

$$\begin{aligned}\||u'|\|_{L^{p(x)}} &\le k^{1/p^+}\|u\|_\alpha,\\ \||u|\|_{L^{p(x)}} &\le \frac{(1-k)^{1/p^+}}{\alpha_-^{1/p^-}}\|u\|_\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}\||u|\|_{W^{p(x)}} &\le k^{1/p^+}\|u\|_\alpha + \frac{(1-k)^{1/p^+}}{\alpha_-^{1/p^-}}\|u\|_\alpha = \Big(k^{1/p^+} + \frac{(1-k)^{1/p^+}}{\alpha_-^{1/p^-}}\Big)\|u\|_\alpha\\ &\le \Big\{\Big[\frac{1}{\alpha_-^{\frac{p^+}{p^-(1-p^+)}}+1}\Big]^{1/p^+} + \Big[1 - \frac{1}{\alpha_-^{\frac{p^+}{p^-(1-p^+)}}+1}\Big]^{1/p^+}\frac{1}{\alpha_-^{1/p^-}}\Big\}\|u\|_\alpha.\end{aligned}$$

In a similar way, we work when $\alpha_- \ge 1$ and we obtain

$$\||u|\|_{W^{p(x)}} \le \Big\{\frac{1}{\big(\alpha_-^{\frac{1}{1-p^+}}+1\big)^{1/p^+}} + \big[1 - \frac{1}{\alpha_-^{\frac{1}{(1-p^+)}}+1}\big]^{1/p^+}\frac{1}{\alpha^{1/p^+}}\Big\}\|u\|_\alpha.$$

Now, taking also into account (2.4), we claim the thesis. □

Remark 2.2. It is worth mentioning that if $\alpha_- \ge 1$, the constant m does not exceed 2. Instead, when $\alpha_- < 1$, m depend on α_- and in particular is less than $2(1 + \frac{1}{\alpha_-})$.

In the sequel, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g, h : \mathbb{R} \to \mathbb{R}$ are two nonnegative continuous functions, and λ and μ are real parameters. We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if:

(1) $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
(2) $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;
(3) for every $s > 0$ there is a function $l_s \in L^1([0,1])$ such that
$$\sup_{|\xi| \le s} |f(x,\xi)| \le l_s(x)$$
for a.e. $x \in [0,1]$.

Put
$$\begin{gathered} F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R}, \\ G(t) = \int_0^t g(\xi)d\xi \quad \text{for all } t \in \mathbb{R}, \\ H(t) = \int_0^t h(\xi)d\xi \quad \text{for all } t \in \mathbb{R}. \end{gathered}$$

We recall that $u : [0,1] \to \mathbb{R}$ is a weak solution of problem (1.1) if $u \in W^{1,p(x)}([0,1])$ satisfies the condition
$$\begin{aligned} &\int_0^1 |u'(x)|^{p(x)-2}u'(x)v'(x)dx + \int_0^1 \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)dx \\ &- \lambda \int_0^1 f(x,u(x))v(x)dx - \mu[g(u(0))v(0) + h(u(1))v(1)] = 0, \end{aligned}$$
for all $v \in W^{1,p(x)}([0,1])$.

To prove our main theorem, we use critical point theory and in particular [3, Theorem 2.1], that we recall here.

Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ is a (strongly) continuous, coercive, sequentially weakly lower semicontinuous and Gâteaux differentiable function, $\Psi : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous and Gâteaux differentiable function. For every $r > \inf_X \Phi$, put
$$\begin{gathered} \varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \\ \overline{\gamma} := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r). \end{gathered}$$

Theorem 2.3. *Under the above assumptions of X, Φ and Ψ, the following alternatives hold:*

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X.*
(b) *if $\overline{\gamma} < +\infty$ then, for each $\lambda \in]0, \frac{1}{\overline{\gamma}}[$, the following alternative holds: either the functional $\Phi - \lambda\Psi$ has a global minimum, or there exists a sequence $\{u_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*
(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either there exists a global minimum of Φ which is a local minimum of $\Phi - \lambda\Psi$, or there exists a sequence $\{u_n\}$ of pairwise distinct critical points (local*

minima) of $\Phi-\lambda\Psi$, with $\lim_{n\to+\infty}\Phi(u_n)=\inf_X\Phi$, which weakly converges to a global minimum of Φ.

3. Main Result

In this section, we establish an existence result of infinitely many solutions to problem (1.1). Put

$$A:=\liminf_{\xi\to+\infty}\frac{\int_0^1\max_{|t|<\xi}F(x,t)dx}{\xi^{p^-}},\quad B:=\limsup_{\xi\to+\infty}\frac{\int_0^1 F(x,\xi)dx}{\xi^{p^+}},$$

and

$$\lambda_1=\frac{\|\alpha\|_1}{p^-B},\quad \lambda_2=\frac{1}{p^+m^{p^-}A},\tag{3.1}$$

where $\|\alpha\|_1$ is the usual norm in $L^1(\Omega)$ and m is given by Proposition 2.1.

Theorem 3.1. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ an L^1-Carathéodory function. Assume that*

$$\liminf_{\xi\to+\infty}\frac{\int_0^1\max_{|t|<\xi}F(x,t)dx}{\xi^{p^-}}<\frac{p^-}{p^+m^{p^-}\|\alpha\|_1}\limsup_{\xi\to+\infty}\frac{\int_0^1 F(x,\xi)dx}{\xi^{p^+}}.$$

Then, for each $\lambda\in\,]\lambda_1,\lambda_2[$, for each $g:\mathbb{R}\to\mathbb{R}$ and for each $h:\mathbb{R}\to\mathbb{R}$ nonnegative continuous functions such that

$$G_\infty=\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^{p^-}}<+\infty,\quad H_\infty=\limsup_{\xi\to+\infty}\frac{H(\xi)}{\xi^{p^-}}<+\infty,$$

and for each $\mu\in[0,\delta[$, with

$$\delta=\frac{1-m^{p^-}p^+\lambda A}{m^{p^-}p^+[G_\infty+H_\infty]},$$

problem (1.1) *admits a sequence of weak solutions which is unbounded in the space $W^{1,p(x)}([0,1])$.*

Proof. Our aim is to apply Theorem 2.3. To this end, fix λ,μ, g and h satisfying our assumptions. Let X be the Sobolev space $W^{1,p(x)}([0,1])$. For any $u\in X$, set

$$\Phi(u):=\int_0^1\frac{1}{p(x)}\Big(|u'|^{p(x)}+\alpha(x)|u|^{p(x)}\Big)dx,$$
$$\Psi(u):=\int_0^1 F(x,u(x))dx+\frac{\mu}{\lambda}[G(u(0))+H(u(1))].$$

It is well known that they satisfy all regularity assumptions requested in Theorem 2.3 and that the critical points in X of the functional $I_\lambda=\Phi-\lambda\Psi$ are precisely the weak solutions of problem (1.1). Let $\{c_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty}c_n=+\infty$, and

$$\lim_{n\to+\infty}\frac{\int_0^1\max_{|t|<c_n}F(x,t)dx}{c_n^{p^-}}=A.$$

Put $r_n=\frac{1}{p^+}\frac{c_n^{p^-}}{m^{p^-}}$, for each $n\in\mathbb{N}$ and $\Phi(v)<r_n$, then, owing to [5, Proposition 2.2], one has

$$\|v\|_\alpha\le\max\{(p^+r_n)^{\frac{1}{p^+}},(p^+r_n)^{\frac{1}{p^-}}\}=\frac{c_n}{m},$$

and so, by (2.3),

$$\max_{x\in[0,1]}|v(x)|\le m\|v\|_\alpha\le c_n.$$

Therefore, one has

$$\begin{aligned}
\varphi(r_n)&\le\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v)}{r_n}\\
&\le\frac{\int_0^1\max_{|t|\le c_n}F(x,t)dx+\frac{\mu}{\lambda}\max_{|t|\le c_n}[G(t)+H(t)]}{\frac{1}{p^+}\frac{c_n^{p^-}}{m^{p^-}}}\\
&\le p^+m^{p^-}\frac{\int_0^1\max_{|t|\le c_n}F(x,t)dx+\frac{\mu}{\lambda}[G(c_n)+H(c_n)]}{c_n^{p^-}},\quad\text{for all } n\in\mathbb{N}.
\end{aligned}$$

Then

$$\overline{\gamma}\le\liminf_{n\to+\infty}\varphi(r_n)\le p^+m^{p^-}A+\frac{\mu}{\lambda}p^+m^{p^-}[G_\infty+H_\infty]<+\infty.$$

Now, let $\{\eta_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty}\eta_n=+\infty$, and

$$\lim_{n\to+\infty}\frac{\int_0^1F(x,\eta_n)dx}{\eta_n^{p^+}}=B. \tag{3.2}$$

For each $n\in\mathbb{N}$, put $w_n(x)=\eta_n$, for all $x\in[0,1]$. Clearly $w_n(x)\in W^{1,p(x)}([0,1])$ for each $n\in\mathbb{N}$. Hence, we have

$$\begin{aligned}
\Phi(w_n)&=\int_0^1\frac{1}{p(x)}\Big(|w_n'|^{p(x)}+\alpha(x)|w_n|^{p(x)}\Big)dx\\
&=\int_0^1\frac{1}{p(x)}\alpha(x)\eta_n^{p(x)}dx\\
&\le\int_0^1\frac{1}{p^-}\alpha(x)\eta_n^{p^+}dx=\frac{\eta_n^{p^+}}{p^-}\|\alpha\|_1.
\end{aligned}$$

Now, for each $n\in\mathbb{N}$, one has

$$\begin{aligned}
\Psi(w_n)&=\int_0^1F(x,w_n(x))dx+\frac{\mu}{\lambda}[G(w_n)+H(w_n)]\\
&=\int_0^1F(x,\eta_n)dx+\frac{\mu}{\lambda}[G(\eta_n)+H(\eta_n)],
\end{aligned}$$

and so

$$\begin{aligned}
I_\lambda(w_n)&=\Phi(w_n)-\lambda\Psi(w_n)\\
&\le\frac{\eta_n^{p^+}}{p^-}\|\alpha\|_1-\lambda\Big[\int_0^1F(x,\eta_n)dx+\frac{\mu}{\lambda}[G(\eta_n)+H(\eta_n)]\Big].
\end{aligned}$$

Now, consider the following cases.

If $B<+\infty$, we let $\epsilon\in]0,B-\frac{\|\alpha\|_1}{\lambda p^-}[$. From (3.2), there exists ν_ϵ such that

$$\int_0^1F(x,\eta_n)dx>(B-\epsilon)\eta_n^{p^+},\quad\text{for all } n>\nu_\epsilon,$$

and so

$$I_\lambda(w_n)<\frac{\eta_n^{p^+}}{p^-}\|\alpha\|_1-\lambda\big[(B-\epsilon)\eta_n^{p^+}+\frac{\mu}{\lambda}[G(\eta_n)+H(\eta_n)]\big]$$

$$= \eta_n^{p^+}\big[\frac{\|\alpha\|_1}{p^-} - \lambda(B-\epsilon)\big] - \mu[G(\eta_n)+H(\eta_n)].$$

Since $\frac{\|\alpha\|_1}{p^-} - \lambda(B-\epsilon) < 0$, one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

If $B=+\infty$, fix $M > \frac{\|\alpha\|_1}{\lambda p^-}$. From (3.2), there exists ν_M such that

$$\int_0^1 F(x,\eta_n)dx > M\eta_n^{p^+}, \quad \text{for all } n > \nu_M;$$

moreover

$$\begin{aligned} I_\lambda(w_n) &< \frac{\eta_n^{p^+}}{p^-}\|\alpha\|_1 - \lambda[M\eta_n^{p^+} + \frac{\mu}{\lambda}[G(\eta_n)+H(\eta_n)]] \\ &= \eta_n^{p^+}\big(\frac{\|\alpha\|_1}{p^-} - \lambda M\big) - \mu[G(\eta_n)+H(\eta_n)]. \end{aligned}$$

Since $\frac{\|\alpha\|_1}{p^-} - \lambda M < 0$, this leads to

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

Taking into account that

$$]\frac{\|\alpha\|_1}{p^- B}, \frac{1}{p^+ m^{p^-} A}[\subseteq]0, \frac{1}{\gamma}[,$$

and that I_λ does not possess a global minimum, from part (b) of Theorem 2.3, there exists an unbounded sequence $\{u_n\}$ of critical points, and our conclusion is achieved. $\square$

As an immediate consequence, here we present an existence result for the homogeneous Neumann problem

$$\begin{gathered} -(|u'(x)|^{p(x)-2}u'(x))' + \alpha(x)|u|^{p(x)-2}u = \lambda f(x,u) \quad \text{in }]0,1[\\ u'(0) = u'(1) = 0. \end{gathered} \tag{3.3}$$

Theorem 3.2. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ an L^1-Carathéodory function. Assume that*

$$\liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|t|<\xi} F(x,t)dx}{\xi^{p^-}} < \frac{p^-}{p^+ m^{p^-}\|\alpha\|_1} \limsup_{\xi\to+\infty} \frac{\int_0^1 F(x,\xi)dx}{\xi^{p^+}}.$$

Then, for each $\lambda \in]\lambda_1,\lambda_2[$, where λ_1 and λ_2 are given in (3.1)*, problem* (3.3) *admits a sequence of weak solutions which is unbounded in $W^{1,p(x)}([0,1])$.*

Example 3.3. Let $p\in C([0,1])$ satisfying (2.1) and with $p^-\geq 2$, and let $\{b_n\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$ be the sequences defined as follows $b_1 = 2$, $b_{n+1} = (b_n)^{2(p^++1)}$ and $a_n = (b_n)^{2p^+}$ for all $n\in\mathbb{N}$. Moreover let $f:\mathbb{R}\to\mathbb{R}$ be a positive continuous function defined by

$$f(t) = \begin{cases} 2^{(p^++1)}\sqrt{1-(1-t)^2}+1 & t\in[0,2], \\ (a_n - (b_n)^{p^++1})\sqrt{1-(a_n-1-t)^2}+1 & t\in \cup_{n=1}^{+\infty}[a_n-2,a_n], \\ ((b_{n+1})^{p^++1}-a_n)\sqrt{1-(b_{n+1}-1-t)^2}+1 & t\in\cup_{n=1}^{+\infty}[b_{n+1}-2,b_{n+1}], \\ 1 & \text{otherwise.} \end{cases}$$

Put $F(\xi)=\int_0^\xi f(t)dt$ for all $\xi\in\mathbb{R}$. In particular, one has $F(a_n)=a_n\frac{\pi}{2}+a_n$ for all $n\in\mathbb{N}$ and $F(b_n)=(b_n)^{p^++1}\frac{\pi}{2}+b_n$ for all $n\in\mathbb{N}$. Hence,

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^{p^-}}=\lim_{n\to+\infty}\frac{F(a_n)}{a_n^{p^-}}=0,$$

and

$$\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^{p^+}}=\lim_{n\to+\infty}\frac{F(b_n)}{b_n^{p^+}}=+\infty.$$

Then, owing to Theorem 3.1, the problem

$$\begin{gathered}-|u'|^{p(x)-2}u'+|u|^{p(x)-2}u=f(u)\quad\text{in }]0,1[\\ |u'(0)|^{p(0)-2}u'(0)=-\frac{1}{1+(u(0))^2},\\ |u'(1)|^{p(1)-2}u'(1)=u(1)\arctan u(1),\end{gathered}$$

admits infinitely many weak solutions.

Remark 3.4. In [19] the existence of infinitely many solutions to problem (1.1) when $\alpha(x)=1$, is proved. Two of key assumptions of [19, Theorem 4.8] are

$$f(x,-u)=-f(x,u),\quad\text{for all } x\in[0,1],\ u\in\mathbb{R}. \tag{3.4}$$

$$g(-u)=-g(u),\quad\text{for all } u\in\mathbb{R}. \tag{3.5}$$

Clearly, [19, Theorem 4.8] cannot be applied to the problem of Example 3.3, since, there, the nonlinearity f and the function g are not symmetric for which (3.4) and (3.5) are not satisfied.

Acknowledgements. The author was partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) – project "Analisi non lineare e problemi ellittici".

References

[1] G. Bonanno, G. D'Aguì; *On the Neumann problem for elliptic equations involving the p-Laplacian*, J. Math. Anal. Appl. **358** (2009) 223–228.

[2] G. Bonanno, G. D'Aguì; *Multiplicity results for a perturbed elliptic Neumann problem*, Abstr. Appl. Anal. **2010** (2010) 1-10.

[3] G. Bonanno, G. Molica Bisci; *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities,* Bound. Value Probl. 2009 (2009) 1–20.

[4] G. Bonanno, D. Motreanu, P. Winkert; *Variational-hemivariational inequalities with small perturbations of nonhomogeneous Neumann boundary conditions*, J. Math. Anal. Appl. 381 (2011) no. 2, 627–637.

[5] F. Cammaroto, A. Chinnì, B. Di Bella; *Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian*, Nonlinear Anal. **71** (2009) 4486–4492.

[6] G. D'Aguì, A. Sciammetta; *Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions*, Nonlinear Anal., Volume 75, Issue 14, (2012), 5612–5619.

[7] L. Diening, P. Harjulehto, P. Hästö, M. Ružička; *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, **2017** Springer-Verlag, Heidelberg (2011).

[8] X. Fan, S. G. Deng; *Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations*, Nonlinear Anal. **67** (2007) 3064–3075.

[9] X. Fan, S. G. Deng; *Multiplicity of positive solutions for a class of inhomogeneous Neumann problems involving the $p(x)$-Laplacian*, NoDEA Nonlinear Differential Equations Appl. **16** (2009) 255-271.

[10] X. L. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001) 424–446.

[11] X. L. Fan, H. Q. Wu, F. Z. Wang; *Hartman-type results for $p(t)$-Laplacian systems*, Nonlinear Anal. **52** (2003) 585–594.

[12] P. Hartman; *On boundary value problems for systems of ordinary nonlinear second order differential equations*, Trans. Amer. Math. Soc. **96** (1960) 493–509.

[13] O. Kováčik, J. Rákosník; *On the spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. **41** (1991) 592–618.

[14] J. Mawhin; *Some boundary value problems for Hartman-type perturbations of the ordinary vector p-Laplacian,* Nonlinear Anal. **40** (2000) 497–503.

[15] M. Mihăilescu; *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplacian operator*, Nonlinear Anal. **67** (2007) 1419–1425.

[16] J. Musielak; *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics **1034**, Springer, Berlin (1983).

[17] C. Qian, Z. Shen, M. Yang; *Existence of solutions for $p(x)$-Laplacian nonhomogeneous Neumann problems with indefinite weight*, Nonlinear Anal. **11** (2010) 446–458.

[18] M. Ružička; *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin 2000.

[19] J. Yao; *Solutions for Neumann boundary value problems involving the $p(x)$-Laplacian operators*, Nonlinear Anal. **68** (2008) 1271–1283.

[20] X. J. Wang, R. Yuan; *Existence of periodic solutions for $p(t)$-Laplacian systems*, Nonlinear Anal. **70** (2009) 866–880.

[21] Q. Zhang; *Existence of solutions for weighted $p(r)$-Laplacian system boundary value problems*, J. Math. Anal. Appl. **327** (2007) 127–141.

Giuseppina D'Aguì

Department of Civil, Information Technology, Construction Environmental Engineering and Applied Mathematics University of Messina, 98166 - Messina, Italy

E-mail address: `dagui@unime.it`