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HALO-SHAPED BIFURCATION CURVES IN ECOLOGICAL SYSTEMS

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ABSTRACT. We examine the structure of positive steady state solutions for a diffusive population model with logistic growth and negative density dependent emigration on the boundary. In particular, this class of nonlinear boundary conditions depends on both the population density and the diffusion coefficient. Results in the one-dimensional case are established via quadrature methods. Additionally, we discuss the existence of a Halo-shaped bifurcation curve.

1. Introduction

We consider the diffusive logistic population dynamics model with nonlinear boundary conditions:

$$\begin{gathered} u_t = d\Delta u + au - bu^2, \quad x \in \Omega,\ t > 0 \\ d\alpha(x,u)\nabla u \cdot \eta + [1-\alpha(x,u)]u = 0, \quad x \in \partial\Omega, t > 0 \end{gathered} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ for $n \geq 1$, Δ is the Laplace operator, $d > 0$ is the diffusion coefficient, a, b are positive parameters, $\nabla u \cdot \eta$ is the outward normal derivative, and $\alpha(x,u) : \partial\Omega \times \mathbb{R} \to [0,1]$ is a nondecreasing C^1 function.

Spatiotemporal models have been extensively employed in population dynamics to describe the distribution and abundance of organisms living in a patch, Ω. The archetypal form of such a model is given by

$$u_t = d\Delta u + u\widetilde{f}(x,u), \quad x \in \Omega, t > 0$$

with $u(t,x)$ representing the population density and $\widetilde{f}(x,u)$ the per capita growth rate which could be influenced by the heterogenous environment. These ecological models were first studied by Skellam in his pioneering work, [27]. Similar models were analyzed prior to Skellam by authors such as Kolomogoroff et al. in [17]. One of the most classic examples is Fisher's equation where $\tilde{f}(x,u) = (1-u)$, which was first studied by in the Skellam in [27]. Reaction diffusion models have since been successfully applied to other spatiotemporal phenomena in disciplines such as physics, chemistry, and biology (see [4, 9, 21, 22, 28]).

Throughout the literature, the logistic growth rate, given by $\tilde{f}(x,u) = a(x) - b(x)u$, has been extensively used to model crowding effects (see [23]). However, a

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more general logistic type growth rate can be characterized as having a decreasing per capita growth function; i.e., $\tilde{f}(x,u)$ is decreasing with respect to u. In this paper, we consider logistic growth with $\tilde{f}(x,u) = (a - bu)$ where a, b are positive parameters.

To date, the homogeneous Dirichlet boundary condition, $u = 0$; $\partial\Omega$, Neumann boundary condition, $\frac{\partial u}{\partial \eta} = 0$; $\partial\Omega$, and linear combinations of the two aforementioned boundary conditions (known as a Robin boundary condition) have been employed almost exclusively in population models. Linear boundary conditions assume the behavior of the population on the boundary is independent of the population density itself. But, density dependent emigration rates from patches of habitat have been reported by several ecologists. Empirical studies conducted by ecologists have even shown a negative correlation between density and emigration rates, in which animals have a tendency to leave a patch when density is low and stay in the patch when it is high. This fact brings into question a commonly made assumption in ecology, that animals exhibit positive density dependent dispersal and patch emigration.

In fact, automatic use of this assumption has been cautioned by authors such as Paivinen et. al. in [24]. Negative density dependent dispersal has been reported in the black-headed gull *Larus ridibundus* (see [15]), Cassin's auklet *Ptychoramphus aleuticus* (see [25]), great it *Parus major* (see [14]), bighorn sheep, *Ovis canadensis* (see [20]), roe deer *Capreolus capreolus* (see [30, 31]), banner-tailed kangaroo rat *Dipodomys spectabilis* (see [16]), and the Glanville fritillary butterfly *Melitaea cinxi* (see [18]) among others.

Several mechanisms have been proposed in the literature as a cause of negative density dependent dispersal including, range position (in which the density of organisms decreases while moving along a gradient from the center of the species distribution range toward its edge), niche breadth (where a particular organism that has the ability to use a wider range of resources is assumed to be widespread and more abundant), density dependent habitat selection (in particular when organisms tend to occupy more habitats when density is low), and dispersal ability (especially when organisms differ in their ability to disperse which can reduce density but increase distribution) (see [24]). Notably, conspecific attraction has also been shown to induce negative density dependent dispersal by Kuussaari et al. who observed emigration of the Glanville fritillary butterfly out of low density areas and Danielson et al. who reported a tendency for individuals to be more attracted to areas with conspecifics, see [8, 18, 29].

In an effort to improve population models to account for this behavior, Cantrell and Cosner proposed the following nonlinear boundary condition which explicitly models conspecific attraction occurring on the boundary of a patch (see [4, 5, 6]),

$$d\alpha(x,u)\nabla u \cdot \eta + [1 - \alpha(x,u)]u = 0, \quad x \in \partial\Omega \tag{1.2}$$

where d is the diffusion coefficient, $\nabla u \cdot \eta$ is the outward normal derivative, and $\alpha(x,u) : \partial\Omega \times \mathbb{R} \to [0,1]$ is a nondecreasing C^1 function. Only recently has the nonlinear boundary condition (1.2) been studied in terms of population dynamics (see [2, 3, 4, 5, 6, 7]). Note that if $\alpha(x,u) \equiv 0$, then (1.2) becomes the Dirichlet boundary condition and all organisms leave the patch upon reaching the boundary. For the case when $\alpha(x,u) \equiv 1$, (1.2) becomes the Neumann boundary condition implying that all organisms remain on the boundary when reached. If $\alpha(x,u) =$

$\alpha_0 \in (0,1)$ then only a fraction of the organisms will remain on the boundary when reached.

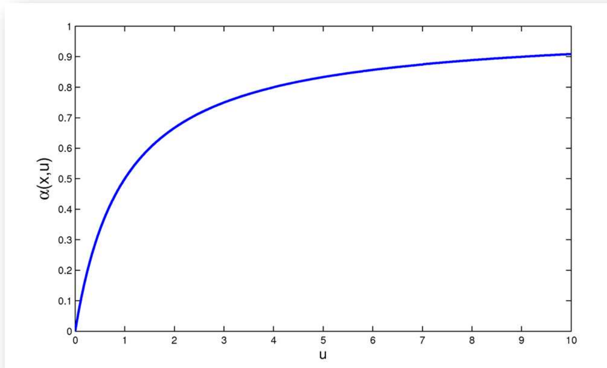

Figure 1. Typical graph of $\alpha(x,u)$

The class of $\alpha(x,u)$'s which model negative density dependent emigration on the boundary have the structure exemplified in Figure 1, where $\alpha(x,0)=0$ and $\alpha(x,u)$ is increasing to one as $u \to \infty$. With this in mind we will be interested in $\alpha(x,u)$'s of the form:

$$\alpha(x,u) = \alpha(u) := \frac{u}{u+g(u)}, \quad x \in \partial\Omega$$

where $g \in C^1([0,\infty),[\delta,\infty))$ for some $\delta > 0$, $\frac{g(u)}{u}$ tends to 0 as $u \to \infty$.

The dynamics of the population model (1.1) are completely determined by the model's steady state solutions. Thus, we are interested in obtaining the structure of positive steady state solutions of (1.1), namely we consider

$$-\Delta u = \lambda[au - bu^2], \quad x \in \Omega, \tag{1.3}$$

$$u[\frac{1}{\lambda}\nabla u \cdot \eta + g(u)] = 0, \quad x \in \partial\Omega \tag{1.4}$$

where $\lambda = 1/d$ and $d > 0$ is the diffusion coefficient. In the case when $\lambda = 1$, the authors have studied (1.3) - (1.4) with the addition of constant yield harvesting in [10, 11] for one-dimension and higher dimensions, respectively. See also [13, 12] where the authors have also considered (1.3) - (1.4) with $\lambda = 1$, strong Allee effect, and constant yield harvesting in both one-dimension and higher dimensions, respectively.

In this paper, we are interested in the case when $\lambda > 0$ is allowed to vary. Notice that the diffusion parameter will be present in the nonlinear boundary condition. In particular, we consider the case when $n = 1$, $\Omega = (0,1)$, and $g(u) \equiv 1$. Thus, we study the nonlinear boundary value problem,

$$-u'' = \lambda[au - bu^2], \quad x \in (0,1) \tag{1.5}$$

$$[-\frac{1}{\lambda}u'(0) + 1]u(0) = 0 \tag{1.6}$$

$$[\frac{1}{\lambda}u'(1)+1]u(1)=0. \tag{1.7}$$

It is clear that analyzing the positive solutions of (1.5) - (1.7) is equivalent to studying the four boundary value problems

$$-u''=\lambda[au-bu^2], \quad x\in(0,1) \tag{1.8}$$
$$u(0)=0 \tag{1.9}$$
$$u(1)=0, \tag{1.10}$$

$$-u''=\lambda[au-bu^2], \quad x\in(0,1) \tag{1.11}$$
$$u(0)=0 \tag{1.12}$$
$$u'(1)=-\lambda, \tag{1.13}$$

$$-u''=\lambda[au-bu^2], \quad x\in(0,1) \tag{1.14}$$
$$u'(0)=\lambda \tag{1.15}$$
$$u(1)=0, \tag{1.16}$$

and

$$-u''=\lambda[au-bu^2], \quad x\in(0,1) \tag{1.17}$$
$$u'(0)=\lambda \tag{1.18}$$
$$u'(1)=-\lambda. \tag{1.19}$$

We note that if $u(x)$ is a solution of (1.11) - (1.13) then $v(x):=u(1-x)$ is also solution of (1.14) - (1.16). It suffices to only consider (1.8) - (1.10), (1.11) - (1.13), and (1.17) - (1.19). The structure of positive solutions for (1.8) - (1.10) has been established in the literature (even for the higher dimensional case), see [4, 26]. For completeness, we detail the structure of positive solutions of (1.8) - (1.10) in Section 2.1 via the quadrature method introduced by Laetsch in [19]. In Sections 2.2 and 2.3, we extend the quadrature method to study (1.11) - (1.13) and (1.17) - (1.19), respectively. Section 3 is concerned with employing the mathematics software package Mathematica to generate the bifurcation curve of positive solutions of (1.5) - (1.7) as the parameter $\lambda>0$ is varied.

Throughout this paper we will consider the case when $a>b$. Computational results indicate that for certain ranges of a and b, the bifurcation curve consists of three different connected components. One component forms a closed-loop connecting two distinct points on the $\|u\|_\infty$-axis, while another component forms an ellipse or "halo". Moreover, for certain ranges of the parameter $\lambda>0$, (1.5) - (1.7) has exactly 7 positive solutions, see Figure 2. Note that (1.8) - (1.10) is portrayed in red, (1.11) - (1.13) in green, and (1.17) - (1.19) in blue.

2. Results via the quadrature method

2.1. Positive solutions of (1.8) - (1.10). Here we recall some of the one-dimensional results of [4] for positive solutions of (1.8) - (1.10) via the quadrature method. Evidently, a positive solution, $u(x)$, of (1.8) - (1.10):

$$-u''=\lambda[au-bu^2]=:\lambda f(u), \quad x\in(0,1)$$
$$u(0)=0$$
$$u(1)=0,$$

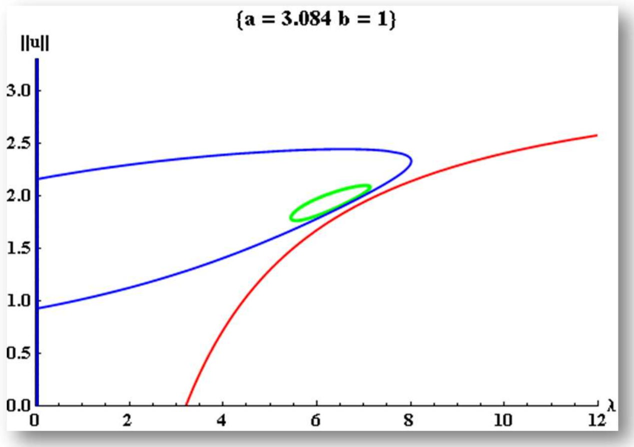

FIGURE 2. Halo-shaped bifurcation curve of positive solutions for (1.5) - (1.7)

must resemble Figure 3, where $\rho := \|u\|_\infty$.

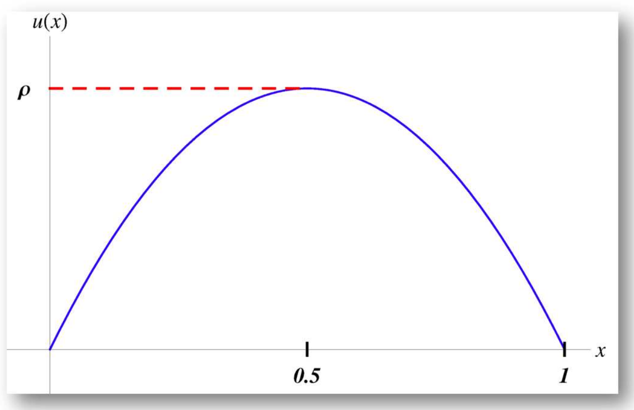

FIGURE 3. Typical solution of (1.8) - (1.10)

We now present the main result for positive solutions of (1.8) - (1.10) in Theorem 2.1.

Theorem 2.1 ([1, 19]). *Problem* (1.8) - (1.10) *has a positive solution,* $u(x)$*, with* $\|u\|_\infty = \rho$ *if and only if* $G_1(\rho) := \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = \sqrt{\lambda}$ *for some* $\lambda > 0$*, where* $F(s) := \int_0^s f(s)ds$*.*

Remark (see [1]) $G_1(\rho)$ is well defined and the included improper integral is convergent on $S := (0, \frac{a}{b})$, since $f(\rho) > 0$, $\rho \in S$ and $F(s)$ is strictly increasing on S. Moreover, $G_1(\rho)$ is a continuous and differentiable function on S.

Proof of Theorem 2.1. ($\Rightarrow$:) Assume that $u(x)$ is a positive solution to (1.8) - (1.10) with $\rho := \|u\|_\infty$. Since (1.8) is an autonomous differential equation, if there exists an $x_0 \in (0,1)$ such that $u'(x_0) = 0$ then $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ will both satisfy the initial value problem,

$$\begin{gathered} -z'' = \lambda f(z) \\ z(0) = u(x_0) \\ z'(0) = 0 \end{gathered} \tag{2.1}$$

for all $x \in [0, d)$ with $d = \min\{x_0, 1 - x_0\}$. Picard's Existence and Uniqueness Theorem asserts that $u(x_0 + x) \equiv u(x_0 - x)$. Hence, $u(x)$ must be symmetric about $x_0 = \frac{1}{2}$, $u'(x) \ge 0; x \in [0, x_0]$, and $u'(x) \le 0; x \in [x_0, 1]$. Multiplying (1.8) by $u'(x)$, gives

$$-\big[\frac{[u'(x)]^2}{2}\big]' = \lambda [F(u(x))]'. \tag{2.2}$$

Integration of (2.2) from x to $1/2$ yields,

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}, \quad x \in [0, \frac{1}{2}). \tag{2.3}$$

Integrating (2.3) from 0 to x, we have

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda} x, \quad x \in [0, \frac{1}{2}]. \tag{2.4}$$

Substitution of $x = 1/2$ into (2.4) and use of the fact that $u(1/2) = \rho$, yields,

$$G_1(\rho) := \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{\lambda}. \tag{2.5}$$

($\Leftarrow$:) Now, suppose that there exists a $\lambda > 0, \rho \in S$ such that $G_1(\rho) = \sqrt{\lambda}$. Define $u : [0, \frac{1}{2}] \to \mathbb{R}$ by

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda} x. \tag{2.6}$$

It remains to be seen that $u(x)$ is well defined and a positive solution of (1.8). It follows that the left-hand side of (2.6) is a differentiable function of u, strictly increasing from 0 to $\frac{1}{2}$ as u increases from 0 to ρ. Hence, for each $x \in [0, \frac{1}{2})$ there exists a unique $u(x)$ that satisfies (2.6). Now, use of the Implicit Function Theorem establishes that $u(x)$ is differentiable as a function of x. Differentiating (2.6), we have

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}, \quad x \in (0, \frac{1}{2}). \tag{2.7}$$

Rearranging (2.7), it yields

$$-\frac{[u'(x)]^2}{2} = \lambda[F(u(x)) - F(\rho)], \quad x \in (0, \frac{1}{2}). \tag{2.8}$$

Differentiating (2.8), we have

$$-u''(x) = f(u(x)), \quad x \in (0, 1).$$

Hence, $u(x)$ satisfies the differential equation in (1.8). It is also easy to see that $u(0) = 0$. Finally, defining $u(x)$ as a symmetric function on (0, 1) yields a positive solution to (1.8) - (1.10) with $\|u\|_\infty = \rho$ and $u(0) = 0 = u(1)$. □

To close this subsection, we recall a result about the global behavior of the bifurcation curve of positive solutions for (1.8) - (1.10). Figure 4 exemplifies the behavior of the bifurcation curve of positive solutions.

Theorem 2.2 ([4]). *Problem* (1.8) - (1.10) *has no positive solution for* $\lambda \leq \frac{\pi^2}{a}$. *Furthermore,* (1.8) - (1.10) *has a unique positive solution for* $\lambda > \frac{\pi^2}{a}$ *and this branch of positive solutions approaches infinity in the* λ*-direction as* $\rho = \|u\|_\infty \to \frac{a}{b}$.

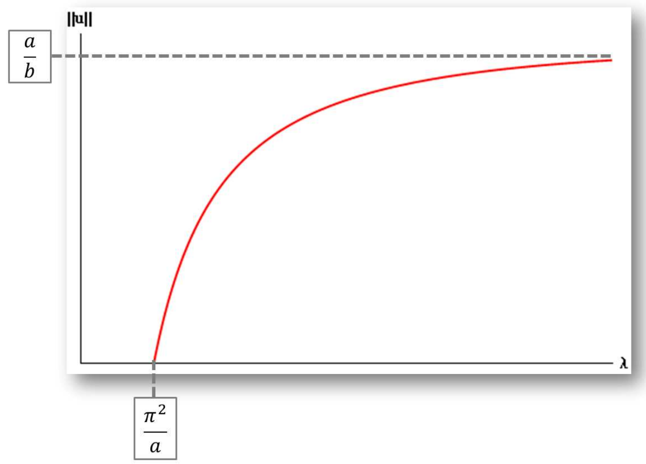

FIGURE 4. Bifurcation curve of positive solutions for (1.8) - (1.10)

2.2. **Positive solutions of** (1.11) **-** (1.13)**.** In this subsection, we extend the quadrature method to study the structure of positive solutions of (1.11) - (1.13), namely

$$\begin{gathered} -u'' = \lambda[au - bu^2] =: \lambda f(u), \quad x \in (0,1) \\ u(0) = 0 \\ u'(1) = -\lambda. \end{gathered}$$

It is clear that a positive solution, $u(x)$, of (1.11) - (1.13) will resemble Figure 5 with $\rho := \|u\|_\infty$, $u'(x_0) = 0$ for some $x_0 \in (\frac{1}{2}, 1)$, and $q := u(1)$.

We now state and prove the main result for positive solutions of (1.11) - (1.13) in Theorem 2.3.

Theorem 2.3. *Problem* (1.11) - (1.13) *has a positive solution,* $u(x)$*, with* $\rho = \|u\|_\infty$ *and* $q = u(1)$ *if and only if*

$$G_2(\rho, q) := 2[F(\rho) - F(q)] = \lambda$$

for some $\lambda > 0$*, where* $q = q(\rho) \in [0, \rho)$ *satisfies*

$$\widetilde{G_2}(\rho, q) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - 2\sqrt{F(\rho) - F(q)} = 0$$

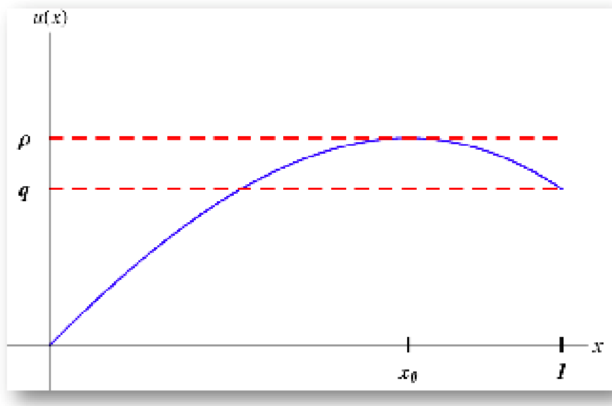

FIGURE 5. Typical solution of (1.11) - (1.13)

and $F(s) := \int_0^s f(s)ds$.

As in the previous subsection, the improper integral in $\widetilde{G_2}(\rho, q)$ is well-defined and convergent for $\rho \in S$ and $q = q(\rho) \in [0, \rho)$. Now we prove Theorem 2.3.

Proof of Theorem 2.3. ($\Rightarrow$:) Assume that $u(x)$ is a positive solution to (1.11) - (1.13) with $\rho := \|u\|_\infty$ and $q := u(1)$. Through a similar argument to the one used in the proof of Theorem 2.1, it is easy to show that if there exists an $x_0 \in (0, 1)$ such that $u'(x_0) = 0$ then $u(x)$ will be symmetric about x_0 with $u'(x) > 0$; $[0, x_0)$ and $u'(x) < 0$; $(x_0, 1]$. Now, multiplying (1.11) by u' and integrating with respect to x yields,

$$-\frac{[u'(x)]^2}{2} = \lambda F(u(x)) + C, \quad x \in [0, 1]. \tag{2.9}$$

Substituting $x = x_0$ and $x = 1$ into (2.9) gives

$$C = -\lambda F(\rho) \tag{2.10}$$

$$C = -\lambda F(q) - \frac{\lambda^2}{2}. \tag{2.11}$$

Combining (2.10) with (2.11) we have,

$$F(\rho) = F(q) + \frac{\lambda}{2}. \tag{2.12}$$

Now, substitution of (2.10) into (2.9) yields

$$\frac{[u'(x)]^2}{2} = \lambda[F(\rho) - F(u(x))], \quad x \in [0, 1]. \tag{2.13}$$

Solving for $u'(x)$ in (2.13) and using the fact that $u'(x) > 0$; $[0, x_0)$ and $u'(x) < 0$; $(x_0, 1]$ we have

$$u'(x) = \sqrt{2\lambda}\sqrt{F(\rho) - F(u(x))}, \quad x \in [0, x_0] \tag{2.14}$$

$$u'(x) = -\sqrt{2\lambda}\sqrt{F(\rho) - F(u(x))}, \quad x \in [x_0, 1]. \tag{2.15}$$

Integration of (2.14) from 0 to x and (2.15) from x_0 to x yields

$$\int_0^x \frac{u'(x)dx}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}x, \quad x \in [0, x_0] \tag{2.16}$$

$$\int_{x_0}^x \frac{u'(x)dx}{\sqrt{F(\rho) - F(u(x))}} = -\sqrt{2\lambda}(x - x_0), \quad x \in [x_0, 1]. \tag{2.17}$$

Through a change of variables and using the fact that $u(0) = 0$ and $u(x_0) = \rho$ we have

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x, \quad x \in [0, x_0] \tag{2.18}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2\lambda}(x - x_0), \quad x \in [x_0, 1]. \tag{2.19}$$

Substituting $x = x_0$ into (2.18) and $x = 1$ into (2.18) gives

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x_0 \tag{2.20}$$

$$\int_\rho^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2\lambda}(1 - x_0). \tag{2.21}$$

Now, subtraction of (2.21) from (2.20) yields,

$$2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - \sqrt{2\lambda} = 0. \tag{2.22}$$

Solving for $\sqrt{2\lambda}$ in (2.12), we have

$$\sqrt{2\lambda} = 2\sqrt{F(\rho) - F(q)}. \tag{2.23}$$

Finally, combining (2.22) and (2.23), gives

$$\widetilde{G_2}(\rho, q) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - 2\sqrt{F(\rho) - F(q)} = 0.$$

It is now readily apparent from (2.23) that

$$G_2(\rho, q) := 2[F(\rho) - F(q)] = \lambda.$$

($\Leftarrow$:) Suppose $G_2(\rho, q) = \lambda$ for some $\rho \in S$ and $\lambda > 0$ where $q = q(\rho) \in [0, \rho)$ is a solution of $\widetilde{G_2}(\rho, q) = 0$. Now, define $u(x) : [0, 1] \to \mathbb{R}$ by

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x, \quad x \in [0, x_0] \tag{2.24}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2\lambda}(x - x_0), \quad x \in [x_0, 1]. \tag{2.25}$$

We will show that $u(x)$ is a positive solution to (1.11) - (1.13). It is easy to see that the turning point given by $x_0 = \frac{1}{\sqrt{2\lambda}} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}$ is unique for fixed λ- and ρ-values. The function,

$$\frac{1}{\sqrt{2\lambda}} \int_0^u \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

is a differentiable function of u which is strictly increasing from 0 to x_0 as u increases from 0 to ρ. Thus, for each $x \in [0, x_0]$, there is a unique $u(x)$ such that

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{2.26}$$

Moreover, by the Implicit Function theorem, $u(x)$ is differentiable with respect to x. Differentiating (2.26), gives

$$u'(x) = \sqrt{2[F(\rho) - F(u(x))]}, \quad x \in (0, x_0). \tag{2.27}$$

Through a similar argument, $u(x)$ is a differentiable, decreasing function of x for $x \in (x_0, 1)$ with

$$u'(x) = -\sqrt{2[F(\rho) - F(u(x))]}, \quad x \in (x_0, 1). \tag{2.28}$$

This implies that we have,

$$\frac{-[u'(x)]^2}{2} = F(\rho) - F(u(x)), \quad x \in (0, 1).$$

Differentiating again, we have

$$-u''(x) = f(u(x)), \quad x \in (0, 1).$$

Thus, $u(x)$ satisfies (1.11). It only remains to be seen that $u(x)$ satisfies (1.12) and (1.13). However, from (2.24) it is clear that $u(0) = 0$. Since $G_2(\rho, q) = \lambda$, we have

$$2[F(\rho) - F(q(\rho))] = \lambda. \tag{2.29}$$

Substituting $x = 1$ into (2.28), gives

$$u'(1) = -\sqrt{2\lambda}\sqrt{F(\rho) - F(q)}. \tag{2.30}$$

Combining (2.29) and (2.30), we have

$$u'(1) = -\lambda.$$

Hence, $u(x)$ satisfies both (1.12) and (1.13). $\square$

With Theorem 2.3, it is imperative that we study the existence and possible multiplicity of q-values for a given $\rho \in S$. We see that the sign of

$$[\widetilde{G_2}(\rho, q)]_q = \frac{f(q) - 1}{\sqrt{F(\rho) - F(q)}}$$

is completely determined by $f(q)-1 = aq-bq^2-1$. Let $h_1(q) := f(q)-1$ and denote its roots by $q_1(a, b) := \frac{a-\sqrt{a^2-4b}}{2b}$ and $q_2(a, b) := \frac{a+\sqrt{a^2-4b}}{2b}$. Clearly, when $q_1(a, b)$ and $q_2(a, b)$ are real we must have that $0 < q_1(a, b) \leq q_2(a, b) < a/b$. Lemma 2.4 below gives a detailed description of the sign of $[\widetilde{G_2}(\rho, q)]_q$ for all possible parameter values. Its proof is just elementary algebra and is omitted.

Lemma 2.4. (1) *Let $b < 4$ and $a \leq 2\sqrt{b}$.*
- (a) *If $a < 2\sqrt{b}$ then $[\widetilde{G_2}(\rho, q)]_q < 0$ for all $\rho \in S$ and $q \in [0, \rho)$.*
- (b) *If $a = 2\sqrt{b}$ then*
  - (i) *for $\rho \in (0, q_1(a, b)]$, $[\widetilde{G_2}(\rho, q)]_q < 0$ when $q \in [0, \rho)$*
  - (ii) *for $\rho \in (q_1(a, b), \frac{a}{b})$, $[\widetilde{G_2}(\rho, q)]_q < 0$ when $q \in [0, q_1(a, b)) \cup (q_1(a, b), \rho)$ and $[\widetilde{G_2}(\rho, q_1(a, b))]_q = 0$.*

(2) *Let $a > 2\sqrt{b}$. Then*

(a) *for* $\rho \in (0, q_1(a,b)]$, $[\widetilde{G_2}(\rho,q)]_q < 0$ *when* $q \in [0,\rho)$
(b) *for* $\rho \in (q_1(a,b), q_2(a,b)]$
  (i) $[\widetilde{G_2}(\rho,q)]_q < 0$ *when* $q \in [0, q_1(a,b))$
  (ii) $[\widetilde{G_2}(\rho,q_1(a,b))]_q = 0$
  (iii) $[\widetilde{G_2}(\rho,q)]_q > 0$ *when* $q \in (q_1(a,b),\rho)$
(c) *for* $\rho \in (q_2(a,b), \frac{a}{b})$
  (i) $[\widetilde{G_2}(\rho,q)]_q < 0$ *when* $q \in [0, q_1(a,b))$
  (ii) $[\widetilde{G_2}(\rho,q_1(a,b))]_q = 0$
  (iii) $[\widetilde{G_2}(\rho,q)]_q > 0$ *when* $q \in (q_1(a,b), q_2(a,b))$
  (iv) $[\widetilde{G_2}(\rho,q_2(a,b))]_q = 0$
  (v) $[\widetilde{G_2}(\rho,q)]_q < 0$ *when* $q \in (q_2(a,b),\rho)$.

The above lemma gives sufficient conditions for nonexistence of positive solutions of (1.11) - (1.13), which are outlined in the following theorem.

Theorem 2.5. *If* $b < 4$ *and* $a \le 2\sqrt{b}$ *then* (1.11) - (1.13) *has no positive solution for any* $\lambda > 0$. *Moreover, if* $a > 2\sqrt{b}$ *then* (1.11) - (1.13) *has no positive solution,* $u(x)$*, whenever* $\|u\|_\infty \le q_1(a,b)$ *for any* $\lambda > 0$.

Proof. Let $\rho \in S$. Clearly, $\widetilde{G_2}(\rho,\rho) > 0$. But, if $a \le 2\sqrt{b}$ then Lemma 2.4 gives that $[\widetilde{G_2}(\rho,q)]_q \le 0$ for all $q \in [0,\rho)$. Hence, $\widetilde{G_2}(\rho,q) \ne 0$ for all $q \in [0,\rho)$ and Theorem 2.3 guarantees that (1.11) - (1.13) will not have a positive solution for any $\lambda > 0$. Now, if $a > 2\sqrt{b}$ and $\rho = \|u\|_\infty \le q_1(a,b)$ then Lemma 2.4 gives that $[\widetilde{G_2}(\rho,q)]_q < 0$ for all $q \in [0,\rho)$ and it follows as in the previous case that (1.11) - (1.13) will not have a positive solution for any $\lambda > 0$. □

Another consequence of Lemma 2.4 is that given a $\rho \in S$, the number of positive solutions of (1.11) - (1.13) having $\|u\|_\infty = \rho$ can be easily ascertained by computing the values of both $\widetilde{G_2}(\rho,0)$ and $\widetilde{G_2}(\rho,q_1(a,b))$, as exemplified in the following Lemma.

Lemma 2.6. *Suppose that* $a > 2\sqrt{b}$ *and* $\rho \in (q_1(a,b), \frac{a}{b})$.

(1) *Let* $\widetilde{G_2}(\rho,q_1(a,b)) > 0$. *Then* (1.11) - (1.13) *has no positive solution with* $\|u\|_\infty = \rho$ *for any* $\lambda > 0$.
(2) *Let* $\widetilde{G_2}(\rho,q_1(a,b)) = 0$. *Then* (1.11) - (1.13) *has a unique positive solution with* $\|u\|_\infty = \rho$ *and* $u(1) = q = q_1(a,b)$ *for some* $\lambda > 0$.
(3) *Let* $\widetilde{G_2}(\rho,q_1(a,b)) < 0$.
  (i) *If* $\widetilde{G_2}(\rho,0) > 0$ *then* (1.11) - (1.13) *has two positive solutions both having* $\|u\|_\infty = \rho$ *and the first with* $u(1) = q \in (0, q_1(a,b))$ *and the second has* $u(1) = q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$ *corresponding to two different* λ*-values.*
  (ii) *If* $\widetilde{G_2}(\rho,0) = 0$ *then* (1.11) - (1.13) *has two positive solutions both having* $\|u\|_\infty = \rho$ *and the first with* $u(1) = 0$ *and the second with* $u(1) = q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$ *corresponding to two different* λ*-values.*
  (iii) *If* $\widetilde{G_2}(\rho,0) < 0$ *then* (1.11) - (1.13) *has a unique positive solution with* $\|u\|_\infty = \rho$ *and* $u(1) = q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$ *for some* $\lambda > 0$.

Proof. Let $a > 2\sqrt{b}$. Notice that $\widetilde{G_2}(\rho, q)$ has a unique local minimum at $q = q_1(a,b)$ and clearly $\widetilde{G_2}(\rho,\rho) > 0$. Whenever $\rho \in (q_2(a,b), \frac{a}{b})$, $\widetilde{G_2}(\rho, q)$ strictly decreasing (from Lemma 2.4) combined with $\widetilde{G_2}(\rho,\rho) > 0$ implies that $\widetilde{G_2}(\rho, q) \neq 0$ for any $q \in [q_2(a,b), \rho)$.

(1) Since $\widetilde{G_2}(\rho, q_1(a,b)) > 0$ and $\widetilde{G_2}(\rho,\rho) > 0$ we have that $\widetilde{G_2}(\rho, q) \neq 0$ for all $q \in [0,\rho)$. Theorem 2.3 immediately gives that (1.11) - (1.13) has no positive solution with $\|u\|_\infty = \rho \in (q_1(a,b), \frac{a}{b})$ for any $\lambda > 0$.

(2) Theorem 2.3 and the fact that $\widetilde{G_2}(\rho, q_1(a,b)) = 0$ implies that (1.11) - (1.13) has a positive solution with $\|u\|_\infty = \rho$ and $u(1) = q = q_1(a,b)$. Since $q = q_1(a,b)$ is the unique local minimum for $\widetilde{G_2}(\rho, q)$ and $\widetilde{G_2}(\rho,\rho) > 0$ we have that this q is the only solution of $\widetilde{G_2}(\rho, q) = 0$. Hence, the aforementioned positive solution must be unique.

(3) For (i), we note that since $\widetilde{G_2}(\rho, 0) > 0$, $\widetilde{G_2}(\rho, q_1(a,b)) < 0$, and $\widetilde{G_2}(\rho, q)$ is strictly decreasing on $q \in [0, q_1(a,b))$ we have that $\widetilde{G_2}(\rho, q) = 0$ for a unique $q \in (0, q_1(a,b))$. Also, $\widetilde{G_2}(\rho,\rho) > 0$ and $\widetilde{G_2}(\rho, q)$ is strictly increasing on $q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$. Thus, $\widetilde{G_2}(\rho, q) = 0$ for a unique $q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$. Theorem 2.3 then guarantees that (1.11) - (1.13) has two positive solutions both with $\|u\|_\infty = \rho$ where the first solution is such that $u(1) = q \in (0, q_1(a,b))$ and the second is such that $u(1) = q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$ for two different λ-values. A similar argument proves (ii). For (iii), $\widetilde{G_2}(\rho, 0) < 0$ and $\widetilde{G_2}(\rho, q_1(a,b)) < 0$ combined with the fact that $\widetilde{G_2}(\rho, q)$ is strictly increasing on $q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$ and $\widetilde{G_2}(\rho,\rho) > 0$ implies that there can be only one solution of $\widetilde{G_2}(\rho, q) = 0$, namely, some unique $q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$. Theorem 2.3 again gives that (1.11) - (1.13) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q \in (q_1(a,b), \min\{\rho, q_2(a,b)\})$ for some $\lambda > 0$. □

Next, we compute $\widetilde{G_2}(\rho, 0)$ and $\widetilde{G_2}(\rho, q_1(a,b))$ values using Mathematica in order to conclude the shape of the bifurcation curve of positive solutions for (1.11) - (1.13). In particular, we are interested in the case when $b = 1$ and $a > 2$ is varied (note that from Theorem 2.5 we must have $a > 2$ to have the possibility of a positive solution). Our computational results indicate the following cases:

Case 1. For $b = 1$, if $a \in (2, a_1)$ (some $a_1 > 0$) then $\widetilde{G_2}(\rho, 0)$ and $\widetilde{G_2}(\rho, q_1(a,b))$ have the structure displayed in Figure 6. Computations indicate that $a_1 \approx 3.072$.

Note that Lemma 2.6 gives that (1.11) - (1.13) has no positive solution when $a \in (2, a_1)$ for any $\lambda > 0$.

Case 2. For $b = 1$, if $a = a_1$ then $\widetilde{G_2}(\rho, 0)$ and $\widetilde{G_2}(\rho, q_1(a,b))$ have the structure displayed in Figure 7.

Denote $M_1 > 0$ as the ρ-value for which $\widetilde{G_2}(M_1, q_1(a,b)) = 0$. In this case, Lemma 2.6 gives that (1.11) - (1.13) has only one positive solution with $\|u\|_\infty = M_1$, $u(1) = q_1(a,b)$, and a corresponding unique $\lambda > 0$. In this case, the bifurcation curve of positive solutions would consist of a single point.

Case 3. For $b = 1$, if $a \in (a_1, a_2)$ (for some $a_2 > a_1$), then $\widetilde{G_2}(\rho, 0)$ and $\widetilde{G_2}(\rho, q_1(a,b))$ have the structure displayed in Figure 8. Computations suggest that $a_2 \approx 3.1123$.

Denote $M_i > 0$ as the ρ-values for which $\widetilde{G_2}(M_i, q_1(a,b)) = 0$ where $i = 1, 2$. Using Lemma 2.6, we can describe the structure of positive solutions for (1.11) -

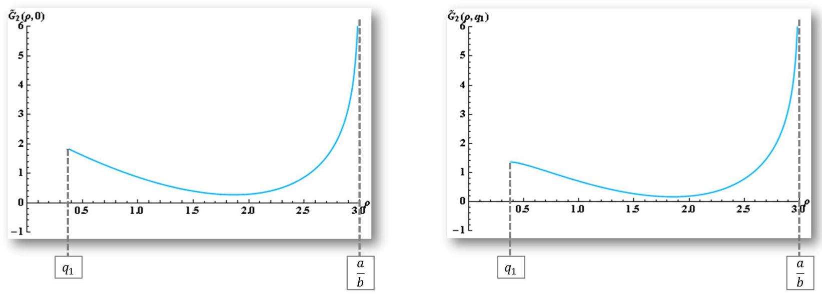

FIGURE 6. (left) ρ vs $\widetilde{G_2}(\rho, 0)$ for $a \in (2, a_1)$. (right) ρ vs $\widetilde{G_2}(\rho, q_1(a, b))$ for $a \in (2, a_1)$

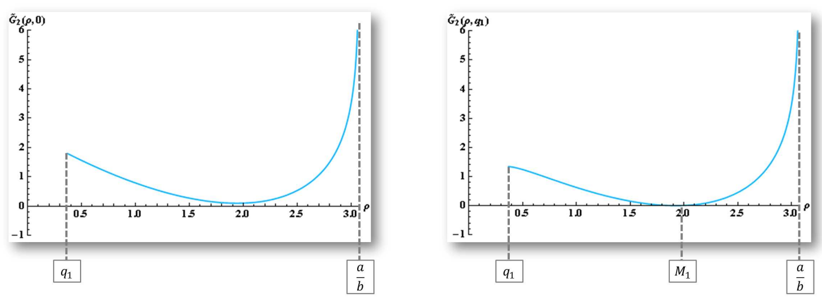

FIGURE 7. (left) ρ vs $\widetilde{G_2}(\rho, 0)$ for $a = a_1$. (right) ρ vs $\widetilde{G_2}(\rho, q_1(a, b))$ for $a = a_1$

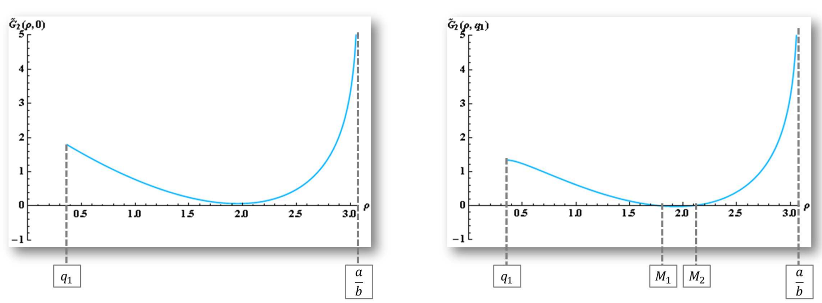

FIGURE 8. (left) ρ vs $\widetilde{G_2}(\rho, 0)$ for $a \in (a_1, a_2)$. (right) ρ vs $\widetilde{G_2}(\rho, q_1(a, b))$ for $a \in (a_1, a_2)$

(1.13) as ρ varies from M_1 to M_2. Note that Lemma 2.6 implies that (1.11) - (1.13) has no positive solution with $\|u\|_\infty = \rho$ for $\rho \in [q_1(a, b), M_1)$ and $\rho \in (M_2, \frac{a}{b})$. When $\rho = M_1$ then (1.11) - (1.13) has only one positive solution with $\|u\|_\infty = M_1$, $u(1) = q_1(a, b)$, and a corresponding unique $\lambda > 0$. For $\rho \in (M_1, M_2)$, (1.11) - (1.13) has exactly two positive solutions both with $\|u\|_\infty = M_1$ but the first having $u(1) \in (0, q_1(a, b))$ and the second having $u(1) \in (q_1(a, b), \min\{q_2(a, b), \rho\})$

for corresponding λ-values. Finally, when $\rho = M_2$ then (1.11) - (1.13) has only one positive solution with $\|u\|_\infty = M_2$, $u(1) = q_1(a,b)$, and a corresponding unique $\lambda > 0$. Thus, we have a closed loop or halo-shaped bifurcation curve of positive solutions for (1.11) - (1.13).

Case 4. For $b = 1$, if $a = a_2$ then $\widetilde{G_2}(\rho, 0)$ and $\widetilde{G_2}(\rho, q_1(a,b))$ have the structure displayed in Figure 9.

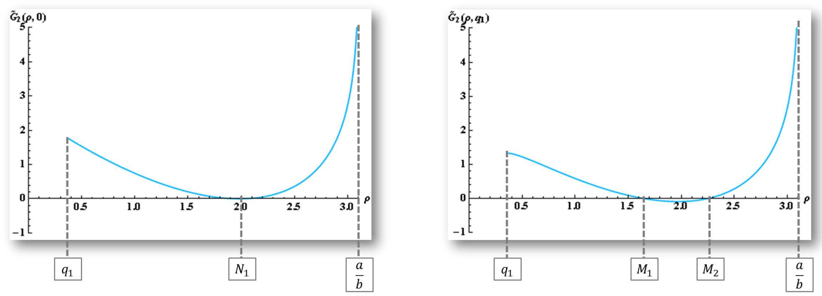

FIGURE 9. (left) ρ vs $\widetilde{G_2}(\rho, 0)$ for $a = a_2$. (right) ρ vs $\widetilde{G_2}(\rho, q_1(a,b))$ for $a = a_2$

Denote $M_i > 0$ as the ρ-values for which $\widetilde{G_2}(M_i, q_1(a,b)) = 0$ where $i = 1, 2$ and $N_1 > 0$ as the ρ-value for which $\widetilde{G_2}(N_1, 0) = 0$. Clearly, $q_1(a,b) < M_1 < N_1 < M_2 < \rho$. Using Lemma 2.6, we can easily determine that the bifurcation curve of positive solutions for (1.11) - (1.13) will have the same closed loop or halo-shape as in Case 4 with one exception. In this case, when $\rho = N_1$, (1.11) - (1.13) has exactly two positive solutions both with $\|u\|_\infty = N_1$ but the first having $u(1) = 0$ and the second having $u(1) \in (q_1(a,b), \min\{q_2(a,b), \rho\})$ for corresponding λ-values. Notice that for $\rho = N_1$ and $u(1) = 0$ this positive solution is also a solution for the Dirichlet boundary condition case, namely (1.8) - (1.10). This implies that the halo-shaped curve will connect to the Dirichlet boundary case bifurcation curve at one point, $(N_1, \lambda^*(N_1))$ for some $\lambda^*(N_1) > 0$.

Case 5. For $b = 1$, if $a \in (a_2, \infty)$ then $\widetilde{G_2}(\rho, 0)$ and $\widetilde{G_2}(\rho, q_1(a,b))$ have the structure displayed in Figure 10.

Denote $M_i > 0$ as the ρ-values for which $\widetilde{G_2}(M_i, q_1(a,b)) = 0$ and $N_i > 0$ as the ρ-values for which $\widetilde{G_2}(N_i, 0) = 0$ where where $i = 1, 2$. Clearly, $q_1(a,b) < M_1 < N_1 < N_2 < M_2 < \rho$. Again Lemma 2.6 can be used to to determine that the bifurcation curve of positive solutions for (1.11) - (1.13) will have loop structure. However, when $\rho \in (N_1, N_2)$ Lemma 2.6 gives that (1.11) - (1.13) will have only one positive solution with $\|u\|_\infty = \rho$ and $u(1) \in (q_1(a,b), \min\{q_2(a,b), \rho\})$ for a corresponding λ-value. Furthermore, when $\rho = N_i$ (1.11) - (1.13) has exactly two positive solutions both with $\|u\|_\infty = N_i$ but the first having $u(1) = 0$ and the second having $u(1) \in (q_1(a,b), \min\{q_2(a,b), \rho\})$ for corresponding λ-values with $i = 1, 2$. As in the previous case, when $\rho = N_i$ and $u(1) = 0$ this positive solution is also a solution for the Dirichlet boundary condition case for $i = 1, 2$. The bifurcation curve of positive solutions for (1.11) - (1.13) will form a loop connecting to the Dirichlet boundary condition bifurcation curve at two different points, $(N_1, \lambda^*(N_1))$

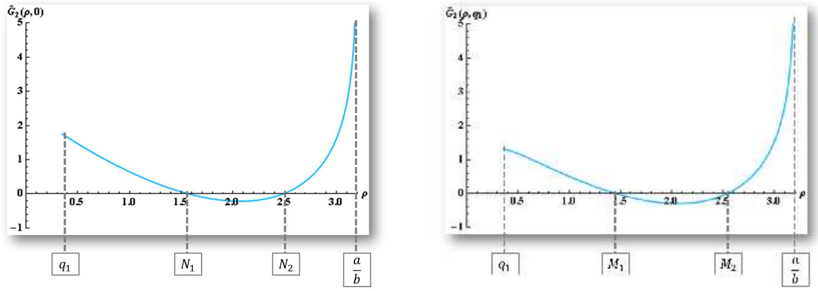

FIGURE 10. (left) ρ vs $\widetilde{G_2}(\rho, 0)$ for $a \in (a_2, \infty)$. (right) ρ vs $\widetilde{G_2}(\rho, q_1(a, b))$ for $a \in (a_2, \infty)$

and $(N_2, \lambda^{**}(N_2))$ for some $\lambda^*(N_1), \lambda^{**}(N_2) > 0$. See Section 3 for the complete evolution of the bifurcation curve of positive solutions of (1.5) - (1.7).

2.3. **Positive solutions of** (1.17) **-** (1.19)**.** We further extend the quadrature method in this section to study the structure of positive steady states of (1.17) - (1.19), namely

$$
\begin{gathered}
-u'' = \lambda[au - bu^2] =: \lambda f(u), \quad x \in (0,1) \\
u'(0) = \lambda \\
u'(1) = -\lambda.
\end{gathered}
$$

It is clear that positive solutions of (1.17) - (1.19) will resemble Figure 11 with $\rho := \|u\|_\infty$, $u'(\frac{1}{2}) = 0$, and $q := u(0) = u(1)$.

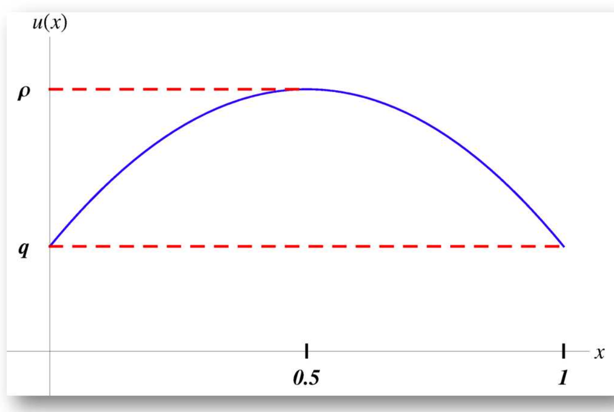

FIGURE 11. Typical solution of (1.17) - (1.19)

We now state the main result for positive solutions of (1.17) - (1.19) in Theorem 2.7.

Theorem 2.7. *Problem* (1.17) - (1.19) *has a positive solution,* $u(x)$*, with* $\rho = \|u\|_\infty$ *and* $q = u(0) = u(1)$ *if and only if*

$$G_3(\rho, q) := 2[F(\rho) - F(q)] = \lambda$$

for some $\lambda > 0$*, where* $q = q(\rho) \in [0, \rho)$ *satisfies*

$$\widetilde{G_3}(\rho, q) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - 2\int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - 2\sqrt{F(\rho) - F(q)} = 0$$

and $F(s) := \int_0^s f(s)ds$*.*

As previously noted in the remark from Section 2.1, the improper integral in $\widetilde{G_3}(\rho, q)$ is well-defined and convergent for $\rho \in S$ and $q = q(\rho) \in [0, \rho)$. The proof of Theorem 2.7 is almost identical to that of Theorem 2.3 and is omitted.

To understand the structure of positive solutions of (1.17) - (1.19) it is imperative that we study the existence and possible multiplicity of q-values for any given $\rho \in S$ as delineated in Theorem 2.7.

We see that the sign of $[\widetilde{G_3}(\rho, q)]_q = \frac{f(q)-2}{\sqrt{F(\rho)-F(q)}}$ can be completely determined by analyzing $f(q) - 2 = aq - bq^2 - 2$. Let $h_2(q) := f(q) - 2$ and denote its roots by $\bar{q}_1(a, b) := \frac{a-\sqrt{a^2-8b}}{2b}$ and $\bar{q}_2(a, b) := \frac{a+\sqrt{a^2-8b}}{2b}$. Clearly, when $\bar{q}_1(a, b)$ and $\bar{q}_2(a, b)$ are real we must have that $0 < \bar{q}_1(a, b) \le \bar{q}_2(a, b) < \frac{a}{b}$. A detailed description of the sign of $[\widetilde{G_3}(\rho, q)]_q$ for all possible parameter values is presented in Lemma 2.8. Its proof is just elementary algebra and is omitted.

Lemma 2.8. (1) *Let* $b < 8$ *and* $a \le 2\sqrt{2}\sqrt{b}$*.*

- (a) *If* $a < 2\sqrt{2}\sqrt{b}$ *then* $[\widetilde{G_3}(\rho, q)]_q < 0$ *for all* $\rho \in S$ *and* $q \in [0, \rho)$*.*
- (b) *If* $a = 2\sqrt{2}\sqrt{b}$ *then*
  - (i) *for* $\rho \in (0, \bar{q}_1(a, b)]$*,* $[\widetilde{G_3}(\rho, q)]_q < 0$ *when* $q \in [0, \rho)$
  - (ii) *for* $\rho \in (\bar{q}_1(a, b), \frac{a}{b})$*,* $[\widetilde{G_3}(\rho, q)]_q < 0$ *when* $q \in [0, \bar{q}_1(a, b)) \cup (\bar{q}_1(a, b), \rho)$ *and* $[\widetilde{G_3}(\rho, \bar{q}_1(a, b))]_q = 0$*.*

(2) *Let* $a > 2\sqrt{2}\sqrt{b}$*. Then*

- (a) *for* $\rho \in (0, \bar{q}_1(a, b)]$*,* $[\widetilde{G_3}(\rho, q)]_q < 0$ *when* $q \in [0, \rho)$
- (b) *for* $\rho \in (\bar{q}_1(a, b), \bar{q}_2(a, b)]$
  - (i) $[\widetilde{G_3}(\rho, q)]_q < 0$ *when* $q \in [0, \bar{q}_1(a, b))$
  - (ii) $[\widetilde{G_3}(\rho, \bar{q}_1(a, b))]_q = 0$
  - (iii) $[\widetilde{G_3}(\rho, q)]_q > 0$ *when* $q \in (\bar{q}_1(a, b), \rho)$
- (c) *for* $\rho \in (\bar{q}_2(a, b), \frac{a}{b})$
  - (i) $[\widetilde{G_3}(\rho, q)]_q < 0$ *when* $q \in [0, \bar{q}_1(a, b))$
  - (ii) $[\widetilde{G_3}(\rho, \bar{q}_1(a, b))]_q = 0$
  - (iii) $[\widetilde{G_3}(\rho, q)]_q > 0$ *when* $q \in (\bar{q}_1(a, b), \bar{q}_2(a, b))$
  - (iv) $[\widetilde{G_3}(\rho, \bar{q}_2(a, b))]_q = 0$
  - (v) $[\widetilde{G_3}(\rho, q)]_q < 0$ *when* $q \in (\bar{q}_2(a, b), \rho)$*.*

The above lemma determines sufficient conditions for nonexistence of positive solutions of (1.17) - (1.19), which are outlined in the following theorem.

Theorem 2.9. *If* $b < 8$ *and* $a \le 2\sqrt{2}\sqrt{b}$ *then* (1.17) - (1.19) *has no positive solution for any* $\lambda > 0$*. Moreover, if* $a > 2\sqrt{2}\sqrt{b}$ *then* (1.17) - (1.19) *has no positive solution whenever* $\|u\|_\infty \le \bar{q}_1(a, b)$ *for any* $\lambda > 0$*.*

Proof. Let $\rho \in S$. Clearly, $\widetilde{G_3}(\rho,\rho) = 0$. But, if $a \le 2\sqrt{2}\sqrt{b}$ then Lemma 2.8 gives that $[\widetilde{G_3}(\rho,q)]_q \le 0$ for all $q \in [0,\rho)$. Hence, $\widetilde{G_3}(\rho,q) \neq 0$ for all $q \in [0,\rho)$ and Theorem 2.7 guarantees that (1.17) - (1.19) will not have a positive solution. Now, if $a > 2\sqrt{2}\sqrt{b}$ and $\rho = \|u\|_\infty \le \bar{q}_1(a,b)$ then Lemma 2.8 gives that $[\widetilde{G_3}(\rho,q)]_q < 0$ for all $q \in [0,\rho)$ and it follows as in the previous case that (1.17) - (1.19) will not have a positive solution. ☐

Note that Lemma 2.8 also allows for the number of positive solutions of (1.17) - (1.19) having $\|u\|_\infty = \rho$ to be easily ascertained by computing the values of both $\widetilde{G_3}(\rho,0)$ and $\widetilde{G_3}(\rho,\bar{q}_1(a,b))$, as exemplified in the next lemma.

Lemma 2.10. *Suppose that $a > 2\sqrt{2}\sqrt{b}$ and $\rho \in (\bar{q}_1(a,b), \frac{a}{b})$.*

1. *Let $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) > 0$. Then (1.17) - (1.19) has no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$.*
2. *Let $\rho \in (\bar{q}_1(a,b), \bar{q}_2(a,b)]$.*
   - (i) *If $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) = 0$ then (1.17) - (1.19) has no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$.*
   - (ii) *Let $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) < 0$.*
     - (a) *If $\widetilde{G_3}(\rho,0) > 0$ then (1.17) - (1.19) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q \in (0,\bar{q}_1(a,b))$ for some $\lambda > 0$.*
     - (b) *If $\widetilde{G_3}(\rho,0) = 0$ then (1.17) - (1.19) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = 0$ for some $\lambda > 0$.*
     - (c) *If $\widetilde{G_3}(\rho,0) < 0$ then (1.17) - (1.19) has no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$.*
3. *Let $\rho \in (\bar{q}_2(a,b), \frac{a}{b})$.*
   - (i) *If $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) = 0$ then (1.17) - (1.19) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q = \bar{q}_1(a,b)$ for some $\lambda > 0$.*
   - (ii) *Let $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) < 0$.*
     - (a) *If $\widetilde{G_3}(\rho,0) > 0$ then (1.17) - (1.19) has two positive solutions both having $\|u\|_\infty = \rho$, the first with $u(1) = q \in (0,\bar{q}_1(a,b))$ and the second has $u(1) = q \in (\bar{q}_1(a,b),\bar{q}_2(a,b))$ corresponding to two different λ-values.*
     - (b) *If $\widetilde{G_3}(\rho,0) = 0$ then (1.17) - (1.19) has two positive solutions both having $\|u\|_\infty = \rho$ and the first with $u(1) = 0$ and the second has $u(1) = q \in (\bar{q}_1(a,b),\bar{q}_2(a,b))$ corresponding to two different λ-values.*
     - (c) *If $\widetilde{G_3}(\rho,0) < 0$ then (1.17) - (1.19) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q \in (\bar{q}_1(a,b),\bar{q}_2(a,b))$ for some $\lambda > 0$.*

Proof. Let $a > 2\sqrt{2}\sqrt{b}$. Notice that $\widetilde{G_3}(\rho,q)$ has a unique local minimum at $q = q_1(a,b)$ for $\rho \in (\bar{q_1}(a,b), \frac{a}{b})$ and a unique local maximum at $q = \bar{q}_2(a,b)$ for $\rho \in (\bar{q}_2(a,b), \frac{a}{b})$. Also, it is easy to see that $\widetilde{G_3}(\rho,\rho) = 0$.

(1) Notice that for $\rho \in (\bar{q}_1(a,b),\bar{q}_2(a,b)]$, $\widetilde{G_3}(\rho,q)$ attains its absolute minimum value at $q = \bar{q}_1(a,b)$. But, $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) > 0$ so $\widetilde{G_3}(\rho,q) \neq 0$ for any $q \in [0,\rho)$. In the case when $\rho \in (\bar{q}_2(a,b), \frac{a}{b})$, $\widetilde{G_3}(\rho,\bar{q}_1(a,b)) > 0$ implies that $\widetilde{G_3}(\rho,q) = 0$ only for $q \in (\bar{q}_2(a,b),\rho)$. But, $\widetilde{G_3}(\rho,\rho) = 0$ and $[\widetilde{G_3}(\rho,q)]_q < 0$ for $q \in (\bar{q}_2(a,b), \frac{a}{b})$.

Thus, $\widetilde{G_3}(\rho, q) \neq 0$ for any $q \in [0, \rho)$. Theorem 2.7 gives that (1.17) - (1.19) has no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$ in either case.

(2) Fix $\rho \in (\bar{q}_1(a,b), \bar{q}_2(a,b)]$. For (i), since $\widetilde{G_3}(\rho, \bar{q}_1(a,b)) = 0$ we have that $\widetilde{G_3}(\rho, \rho) \neq 0$ and thus Theorem 2.7 gives that (1.17) - (1.19) has no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$. In (ii) we assume that $\widetilde{G_3}(\rho, \bar{q}_1(a,b)) < 0$. For (a) we have that $\widetilde{G_3}(\rho, 0) > 0$. Thus, there is a unique $q \in (0, \bar{q}_1(a,b))$ such that $\widetilde{G_3}(\rho, q) = 0$. Theorem 2.7 guarantees that (1.17) - (1.19) has a positive solution with $\|u\|_\infty = \rho$ and $u(1) = q \in (0, \bar{q}_1(a,b))$. Since $[\widetilde{G_3}(\rho, q)]_q > 0$ for $q \in (\bar{q}_1(a,b), \rho)$ and $\widetilde{G_3}(\rho, \rho) = 0$ this positive solution must be unique. In case (b) we have that $\widetilde{G_3}(\rho, 0) = 0$. Theorem 2.7 again guarantees that (1.17) - (1.19) has a positive solution with $\|u\|_\infty = \rho$ and $u(1) = 0$. But, $[\widetilde{G_3}(\rho, q)]_q < 0$ for $q \in [0, \bar{q}_1(a,b))$ and $[\widetilde{G_3}(\rho, q)]_q > 0$ for $q \in (\bar{q}_1(a,b), \rho)$ combined with $\widetilde{G_3}(\rho, \rho) = 0$ gives that this positive solution is again unique. For (c) we have that $\widetilde{G_3}(\rho, 0) < 0$. Again, $[\widetilde{G_3}(\rho, q)]_q < 0$ for $q \in [0, \bar{q}_1(a,b))$ and $[\widetilde{G_3}(\rho, q)]_q > 0$ for $q \in (\bar{q}_1(a,b), \rho)$ combined with $\widetilde{G_3}(\rho, \rho) = 0$ implies that $\widetilde{G_3}(\rho, q) \neq 0$ for any $q \in [0, \rho)$. Theorem 2.7 gives that (1.17) - (1.19) has no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$.

(3) Similar arguments give the result. $\square$

Next, we compute $\widetilde{G_3}(\rho, 0)$ and $\widetilde{G_3}(\rho, \bar{q}_1(a,b))$ values using Mathematica in order to conclude the shape of the bifurcation curve of positive solutions for (1.17) - (1.19). Again, we are interested in the case when $b = 1$ and $a > 2\sqrt{2}$ is varied (note that from Theorem 2.9 we must have $a > 2\sqrt{2}$ to have the possibility of a positive solution). Our computational results indicate the following cases:

Case 1. For $b = 1$, if $a \in (2\sqrt{2}, a_2)$ (a_2 is the same value from Section 2.2) then $\widetilde{G_3}(\rho, 0)$ and $\widetilde{G_3}(\rho, \bar{q}_1(a,b))$ have the structure displayed in Figure 12.

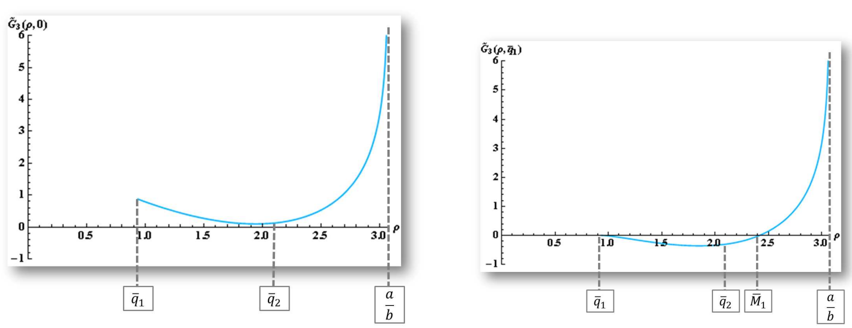

FIGURE 12. (left) ρ vs $\widetilde{G_3}(\rho, 0)$ for $a \in (2\sqrt{2}, a_2)$. (right) ρ vs $\widetilde{G_3}(\rho, \bar{q}_1(a,b))$ for $a \in (2\sqrt{2}, a_2)$

Denote $\bar{M}_1 > 0$ as the ρ-value for which $\widetilde{G_3}(\bar{M}_1, \bar{q}_1(a,b)) = 0$. From these computational results, we have that $\bar{q}_1(a,b) < \bar{q}_2(a,b) < \bar{M}_1 < \frac{a}{b}$. In this case, Lemma 2.10 gives that for $\rho \in (\bar{q}_1(a,b), \bar{q}_2(a,b)]$, (1.17) - (1.19) has only one positive solution with $\|u\|_\infty = \rho$, $u(1) = q \in (0, \bar{q}_1(a,b))$, and a corresponding unique $\lambda > 0$. Also, for $\rho \in (\bar{q}_2(a,b), \bar{M}_1)$ Lemma 2.10 gives that (1.17) - (1.19) has two positive solutions both having $\|u\|_\infty = \rho$ and the first with $u(1) = q \in (0, \bar{q}_1(a,b))$ (which connects to the branch in the case when $\rho \in (\bar{q}_1(a,b), \bar{q}_2(a,b)])$ and the second has $u(1) = q \in (\bar{q}_1(a,b), \bar{q}_2(a,b))$ corresponding to two different λ-values. Finally, when $\rho = \bar{M}_1$ Lemma2.10 implies that (1.17) - (1.19) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q = \bar{q}_1(a,b)$ for some $\lambda > 0$. Computationally, we see that the bifurcation curve of positive solutions for (1.17) - (1.19) forms a closed loop connecting two different values on the $\|u\|_\infty$-axis. An interesting feature of this case is that the closed loop structure seems to exist for every $a > 2\sqrt{2}$. This fact indicates that the nonexistence result for $a \leq 2\sqrt{2}$ from Theorem 2.9 is the best possible.

Case 2. For $b = 1$, if $a = a_2$ then $\widetilde{G_3}(\rho, 0)$ and $\widetilde{G_3}(\rho, \bar{q}_1(a,b))$ have the structure displayed in Figure 13.

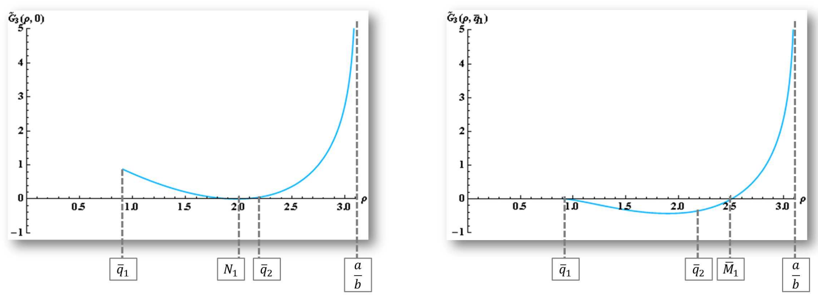

FIGURE 13. (left) ρ vs $\widetilde{G_3}(\rho, 0)$ for $a = a_2$. (right) ρ vs $\widetilde{G_3}(\rho, \bar{q}_1(a,b))$ for $a = a_2$

Denote $\bar{M}_1 > 0$ as the ρ-value for which $\widetilde{G_3}(\bar{M}_1, \bar{q}_1(a,b)) = 0$ and $N_1 > 0$ as the ρ-value for which $\widetilde{G_3}(N_1, 0) = 0$ (recall that $\widetilde{G_3}(\rho, 0) = \widetilde{G_2}(\rho, 0))$). Computationally, we have that $\bar{q}_1(a,b) < N_1 < \bar{q}_2(a,b) < \bar{M}_1 < \frac{a}{b}$. Again using Lemma 2.6, we can describe the structure of positive solutions for (1.17) - (1.19) as ρ varies from $\bar{q}_1(a,b)$ to $\bar{M}_1$. The bifurcation curve of positive solutions for (1.17) - (1.19) will be identical to that of Case 1 with one exception. When $\rho = N_1$ one of the two positive solutions of (1.17) - (1.19) will satisfy $u(1) = 0$ and thus the Dirichlet boundary condition will also be satisfied at this point for some $\lambda > 0$. In essence, the closed loop will connect to the Dirichlet boundary case at the point $(N_1, \lambda^*(N_1))$ for the same $\lambda^*(N_1) > 0$ as in Section 2.2.

Case 3. For $b = 1$, if $a \in (a_2, a_3]$ (for some $a_3 > a_2$) then $\widetilde{G_3}(\rho, 0)$ and $\widetilde{G_3}(\rho, \bar{q}_1(a,b))$ have the structure displayed in Figure 14.

Denote $\bar{M}_1 > 0$ as the ρ-value for which $\widetilde{G_3}(\bar{M}_1, \bar{q}_1(a,b)) = 0$ and $N_i > 0$ as the ρ-values for which $\widetilde{G_3}(N_i, 0) = 0$ for $i = 1, 2$. Computationally, we have that $\bar{q}_1(a,b) < N_1 < N_2 < \bar{q}_2(a,b) < \bar{M}_1 < \frac{a}{b}$ with $N_2 = \bar{q}_2(a,b)$ whenever $a = a_3$. For $\rho \in (\bar{q}_1(a,b), N_1]$ Lemma 2.10 implies that the bifurcation curve of positive solutions of (1.17) - (1.19) will be a single branch connecting a point on the $\|u\|_\infty$-axis to the Dirichlet boundary condition branch (at the point $(N_1, \lambda^*(N_1))$ from Section 2.2), since when $\rho = N_1$ the corresponding unique q-value is zero. When $\rho \in (N_1, N_2)$, (1.17) - (1.19) will have no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$. For $\rho = N_2$, (1.17) - (1.19) will have a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = 0$ (this is the point on the Dirichlet boundary case, $(N_2, \lambda^{**}(N_2))$ from

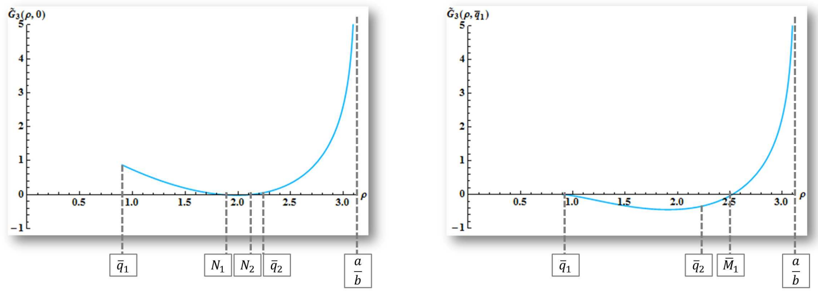

FIGURE 14. (left) ρ vs $\widetilde{G_3}(\rho, 0)$ for $a \in (a_2, a_3]$. (right) ρ vs $\widetilde{G_3}(\rho, \bar{q}_1(a, b))$ for $a \in (a_2, a_3]$

Section 2.2). Furthermore, when $\rho \in (N_2, \bar{q}_2(a, b)]$, (1.17) - (1.19) will have a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q \in (0, \bar{q}_1(a, b))$ for some $\lambda > 0$. When $\rho \in (\bar{q}_2(a, b), \bar{M}_1)$, (1.17) - (1.19) will have two positive solutions both with $\|u\|_\infty = \rho$ and with one having $u(1) = q \in (0, \bar{q}_1(a, b))$ and the other with $u(1) = q \in (\bar{q}_1(a, b), \bar{q}_2(a, b))$ corresponding to two different $\lambda-$values. Computations indicate that the bifurcation curve of positive solutions of (1.17) - (1.19) forms a loop that connects the single branch mentioned in the case when $\rho \in (N_2, \bar{q}_2(a, b)]$ to a point on the $\|u\|_\infty$-axis.

Case 4. For $b = 1$, if $a \in (a_3, \infty)$ then $\widetilde{G_3}(\rho, 0)$ and $\widetilde{G_3}(\rho, \bar{q}_1(a, b))$ have the structure displayed in Figure 15.

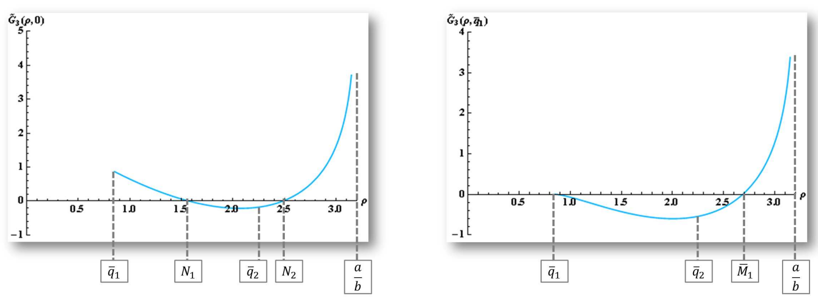

FIGURE 15. (left) ρ vs $\widetilde{G_3}(\rho, 0)$ for $a \in (a_3, \infty)$. (right) ρ vs $\widetilde{G_3}(\rho, \bar{q}_1(a, b))$ for $a \in (a_3, \infty)$

Denote $\bar{M}_1 > 0$ as the ρ-value for which $\widetilde{G_3}(\bar{M}_1, \bar{q}_1(a, b)) = 0$ and $N_i > 0$ as the ρ-values for which $\widetilde{G_3}(N_i, 0) = 0$ for $i = 1, 2$. Computationally, we have that $\bar{q}_1(a, b) < N_1 < \bar{q}_2(a, b) < N_2 < \bar{M}_1 < \frac{a}{b}$. For $\rho \in (\bar{q}_1(a, b), N_1]$ Lemma 2.10 implies that the bifurcation curve of positive solutions of (1.17) - (1.19) will be a single branch connecting a point on the $\|u\|_\infty$-axis to the Dirichlet boundary condition branch at the point $(N_1, \lambda^*(N_1))$, since when $\rho = N_1$ the corresponding unique q-value is zero. When $\rho \in (N_1, \bar{q}_2(a, b)]$, (1.17) - (1.19) will have no positive solution with $\|u\|_\infty = \rho$ for any $\lambda > 0$. For $\rho \in (\bar{q}_2(a, b), N_2)$, (1.17) - (1.19) has a unique positive solution with $\|u\|_\infty = \rho$ and $u(1) = q \in (\bar{q}_1(a, b), \bar{q}_2(a, b))$ for some $\lambda > 0$.

When $\rho \in [N_2, \bar{M}_1)$ the bifurcation curve of positive solutions of (1.17) - (1.19) has a loop that connects the point $(N_2, \lambda^{**}(N_2))$ on the Dirichlet boundary condition curve to the single branch mentioned in the case when $\rho \in (\bar{q}_2(a,b), N_2)$. Based on our computations, the bifurcation curve of positive solutions for (1.17) - (1.19) is identical in shape for both Cases 3 and 4.

3. Computational Results

In this section, we present the complete evolution of the bifurcation curve of positive solutions of (1.5) - (1.7) as the parameter $a > 0$ is varied. In this paper, we are particularly interested in the case when $b = 1$. Recalling, the Lemmas and Theorems from Section 2, we employed the mathematics software package Mathematica to computationally generate the bifurcation curve. Due to the complex nature of the formulas, these calculations were extremely computationally expensive. In what follows, the bifurcation curve of positive solutions of (1.8) - (1.10) is portrayed in red, (1.11) - (1.13) in green, and (1.17) - (1.19) in blue. Also note that the green curve represents solutions to (1.11) - (1.13) and (1.14) - (1.16) and thus counts twice. Throughout this section we will denote $\lambda_0 = \frac{\pi^2}{a}$, the critical λ-value for (1.8) - (1.10) from Theorem 2.2.

Case 1. For $b = 1$, if $a \in (0, 2\sqrt{2}]$ then there exists a $\lambda_0 > 0$ such that if

(1) $\lambda \in (\lambda_0, \infty)$ then (1.5) - (1.7) has a unique positive solution.
(2) $\lambda \in (0, \lambda_0]$ then (1.5) - (1.7) has no positive solution.

Figure 16 shows an example of Case 1.

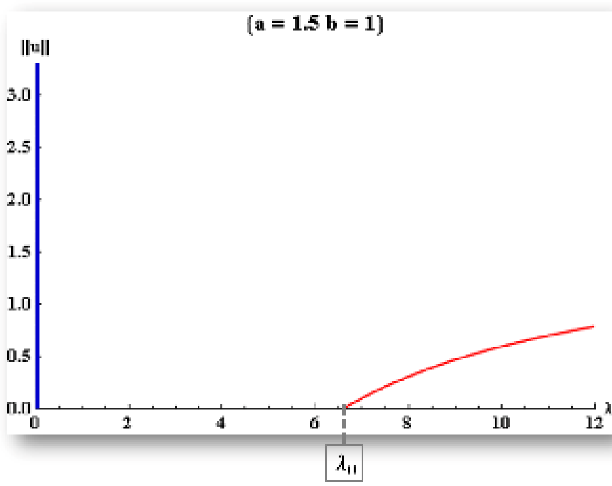

Figure 16. Bifurcation curve of positive solutions for Case 1 with $a = 1.5, b = 1$

Case 2. For $b = 1$, if $a \in (2\sqrt{2}, a_0)$ (some $a_0 \in (0, a_1)$) then there exist $\lambda_0, \lambda_1 > 0$ such that if

(1) $\lambda \in (0, \lambda_1)$ then (1.5) - (1.7) has two positive solutions.
(2) $\lambda = \lambda_1$ or $\lambda \in (\lambda_0, \infty)$ then (1.5) - (1.7) has a unique positive solution.
(3) $\lambda \in (\lambda_1, \lambda_0]$ then (1.5) - (1.7) has no positive solution.

Case 2 is illustrated in Figure 17.

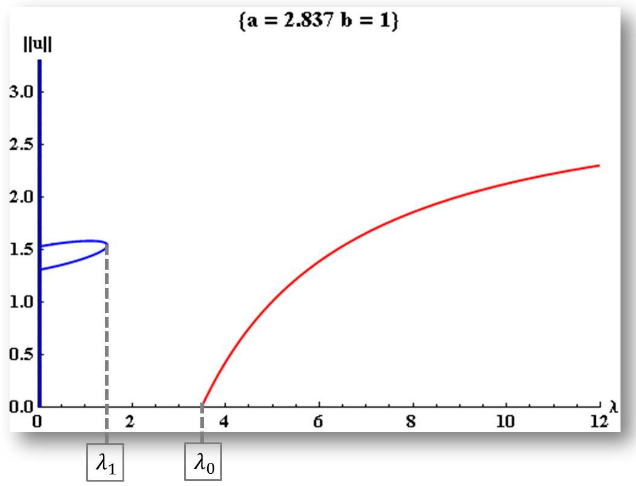

FIGURE 17. Bifurcation curve of positive solutions for Case 2 with $a = 2.837, b = 1$

Case 3. For $b = 1$, if $a = a_0$ then there exists a $\lambda_0 > 0$ such that if

(1) $\lambda \in (0, \lambda_0)$ then (1.5) - (1.7) has two positive solutions.
(2) $\lambda \in [\lambda_0, \infty)$ then (1.5) - (1.7) has a unique positive solution.

Figure 18 portrays Case 3.

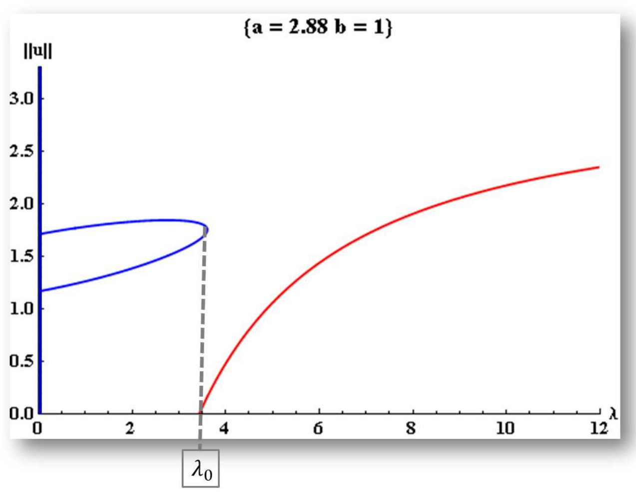

FIGURE 18. Bifurcation curve of positive solutions for Case 3 with $a = 2.88, b = 1$

Case 4. For $b = 1$, if $a \in (a_0, a_1)$ then there exist $\lambda_0, \lambda_1 > 0$ such that if

(1) $\lambda \in (\lambda_0, \lambda_1)$ then (1.5) - (1.7) has three positive solutions.

(2) $\lambda = \lambda_1$ or $\lambda \in (0, \lambda_0]$ then (1.5) - (1.7) has two positive solutions.
(3) $\lambda \in (\lambda_1, \infty)$ then (1.5) - (1.7) has a unique positive solution.

Case 4 is illustrated in Figure 19.

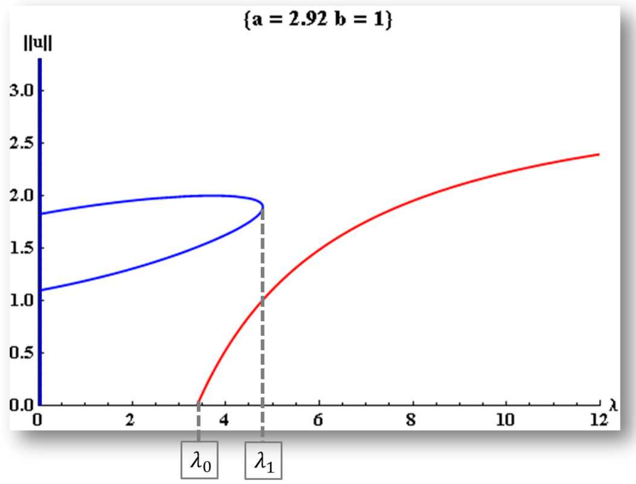

FIGURE 19. Bifurcation curve of positive solutions for Case 4 with $a = 2.92, b = 1$

Case 5. For $b = 1$, if $a = a_1$ then there exist $\lambda_0, \lambda_1, \lambda_2 > 0$ such that if

(1) $\lambda = \lambda_1$ then (1.5) - (1.7) has five positive solutions.
(2) $\lambda \in (\lambda_0, \lambda_1)$ or $\lambda \in (\lambda_1, \lambda_2)$ then (1.5) - (1.7) has three positive solutions.
(3) $\lambda = \lambda_2$ or $\lambda \in (0, \lambda_0]$ then (1.5) - (1.7) has two positive solutions.
(4) $\lambda \in (\lambda_2, \infty)$ then (1.5) - (1.7) has a unique positive solution.

Figure 20 exemplifies Case 5.

Case 6. For $b = 1$, if $a \in (a_1, a_2]$ then there exist $\lambda_i > 0$ for $i = 0, 1, 2, 3$ such that if

(1) $\lambda \in (\lambda_1, \lambda_2)$ then (1.5) - (1.7) has seven positive solutions.
(2) $\lambda = \lambda_1, \lambda_2$ then (1.5) - (1.7) has five positive solutions.
(3) $\lambda \in (\lambda_0, \lambda_1)$ or $\lambda \in (\lambda_2, \lambda_3)$ then (1.5) - (1.7) has three positive solutions.
(4) $\lambda = \lambda_3$ or $\lambda \in (0, \lambda_0]$ then (1.5) - (1.7) has two positive solutions.
(3) $\lambda \in (\lambda_3, \infty)$ then (1.5) - (1.7) has a unique positive solution.

An example of Case 6 is shown in Figures 21 and 22. Notice in Figure 22 that for $\lambda = \lambda^*(N_1)$ and the corresponding $\rho = N_1$, all four cases of the boundary conditions are satisfied. This point is where the Dirichlet boundary condition branch bifurcates into the other cases.

Case 7. For $b = 1$, if $a > a_2$ then there exist $\lambda_i > 0$ for $i = 0, 1, 2, 3, 4, 5$ such that if

(1) $\lambda \in (\lambda_1, \lambda_2]$ or $\lambda \in [\lambda_3, \lambda_4)$ then (1.5) - (1.7) has seven positive solutions.
(2) $\lambda = \lambda_1, \lambda_4$ then (1.5) - (1.7) has five positive solutions.
(3) $\lambda \in (\lambda_2, \lambda_3)$ then (1.5) - (1.7) has four positive solutions.
(4) $\lambda \in (\lambda_0, \lambda_1)$ or $\lambda \in (\lambda_4, \lambda_5)$ then (1.5) - (1.7) has three positive solutions.
(5) $\lambda = \lambda_5$ or $\lambda \in (0, \lambda_0]$ then (1.5) - (1.7) has two positive solutions.

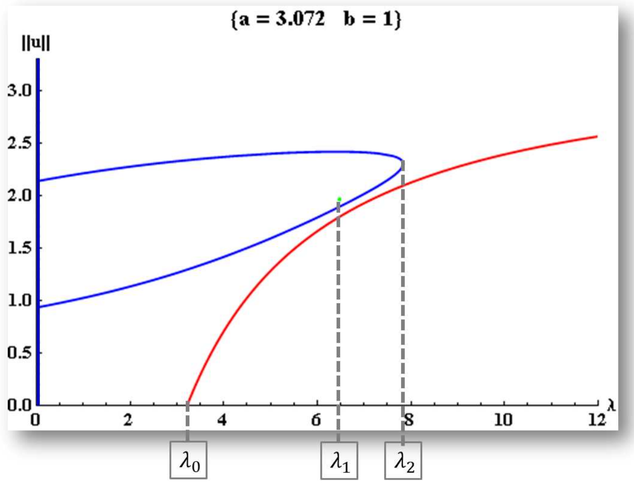

FIGURE 20. Bifurcation curve of positive solutions for Case 5 with $a = 3.072, b = 1$

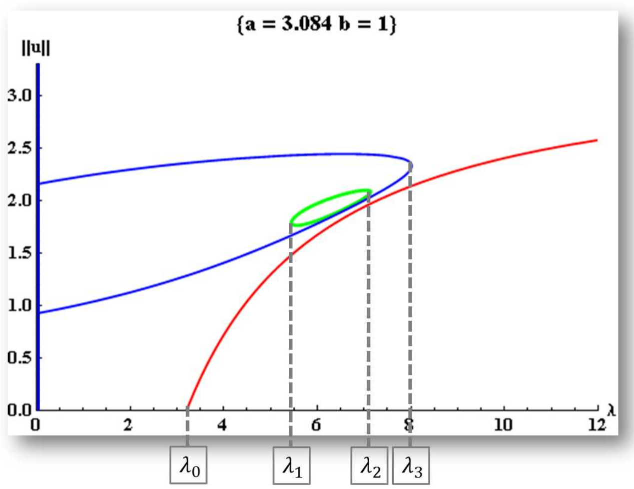

FIGURE 21. Bifurcation curve of positive solutions for Case 6 with $a = 3.084, b = 1$

(6) $\lambda \in (\lambda_5, \infty)$ then (1.5) - (1.7) has a unique positive solution.

Case 7 is illustrated in Figure 23.

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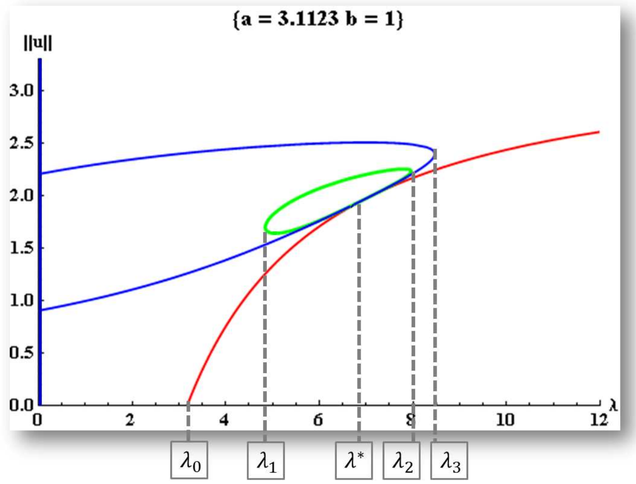

FIGURE 22. Bifurcation curve of positive solutions for Case 6 with $a = 3.1123, b = 1$

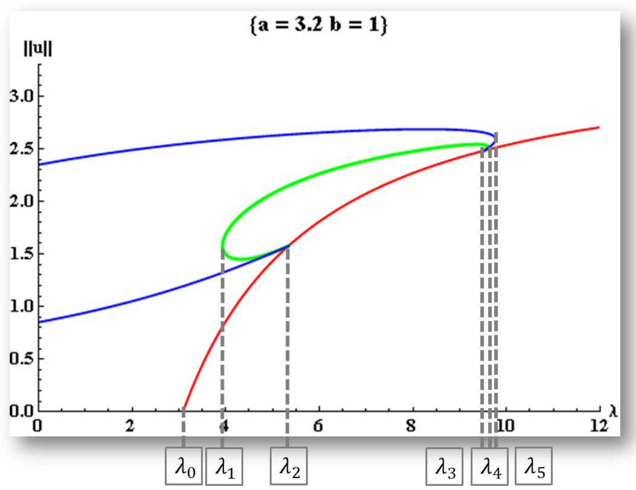

FIGURE 23. Bifurcation curve of positive solutions for Case 7 with $a = 3.2, b = 1$

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