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BIFURCATION OF LIMIT CYCLES FOR CUBIC REVERSIBLE SYSTEMS

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Abstract. This article is concerned with the bifurcation of limit cycles of a class of cubic reversible system having a center at the origin. We prove that this system has at least four limit cycles produced by the period annulus around the center under cubic perturbations.

1. Introduction

One of the main problems in the qualitative theory of real planar differential systems is the determination of limit cycles. For polynomial differential systems, the problem of the maximum number of limit cycles arises in the context of the second part of the Hilbert's 16th problem. A classical way to obtain limit cycles is perturbing a polynomial differential system which has a center.

In this article we study the bifurcation of limit cycles of a cubic systems under small cubic perturbations. We consider system

$$\dot{x} = \frac{H_y(x,y)}{R(x,y)} + \varepsilon f(x,y,\varepsilon), \quad \dot{y} = -\frac{H_x(x,y)}{R(x,y)} + \varepsilon g(x,y,\varepsilon), \tag{1.1}$$

where $H(x,y)$ is a first integral of system (1.1) with $\varepsilon = 0$ and integrating factor $R(x,y)$, $f(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ are cubic polynomials in x,y with coefficients depending analytically on the small parameter ε.

We assume that the unperturbed system of (1.1) has at least one centre which is surrounded by a continuous set of period annuli Γ_h of real algebraic curve $H(x,y) = h, h \in (h_1,h_2)$. As well know, the maximum number of limit cycles produced by period annuli of system (1.1) with $\varepsilon = 0$ is reduced to counting the number of zeros of the displacement function

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + O(\varepsilon^3), \tag{1.2}$$

where $d(h,\varepsilon)$ is defined below, which is parameterized by the Hamiltonian value h. The number of zeros of the first non-vanish Melnikov function $M_k(h)$ in (1.2) determine the upper bound of limit cycles in (1.1) produced from periodic orbits of the unperturbed system (1.1). As usual, we call the the upper bound of limit cycles *cyclicity* and the first non-vanish Melnikov function *the generating function.*

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Most of the results concerned with the cyclicity of the period annulus is for planar quadratic systems under quadratic perturbations, in particular for quadratic systems with centers of genus one. We refer to [2, 4, 5, 6, 7], [10, 13], [16, 17, 20, 19], the survey paper [9] and references therein. But there are less results concerned with bifurcation from the periodic orbits of cubic system. The authors of [11] investigated the upper bound of limit cycles that bifurcate from the periodic orbits of cubic reversible isochronous centers having all their orbits formed by conics inside the class of all polynomial systems of degree n. The paper [3] study the maximum number of limit cycles from cubic Pleshkan's isochronous system S_1^* under a small polynomial perturbations of degree n. In [18], the authors study the number of limit cycles produced by the period annulus of a cubic reversible isochronous center under cubic perturbations.

In this article, we will study the cubic reversible system

$$\begin{aligned} \dot{x} &= -y(1-x)(1-2x), \\ \dot{y} &= x - 2x^2 + 2x^3 + y^2, \end{aligned} \tag{1.3}$$

which has a first integral of the form

$$H(x,y) = \frac{(x-1)^2(x^2+y^2)}{(2x-1)^2}, \tag{1.4}$$

with the integrating factor $R(x,y) = \frac{2(1-x)}{(2x-1)^3}$. It is easy to know that the origin is a center of system (1.3), $x = 1$ and $x = \frac{1}{2}$ are two invariant lines. Hence, there is an unbounded period annulus surrounding the center of system (1.3) and $h \in (0, +\infty)$. Chavarriga and Sabatini [1] have proved that the origin is a reversible isochronous center.

The main purpose of this article is to deal with the bifurcation of limit cycles of system (1.3) under cubic polynomial perturbations. We consider the following perturbating system:

$$\begin{aligned} \dot{x} &= -y(1-x)(1-2x) + \varepsilon f(x,y,\varepsilon), \\ \dot{y} &= x - 2x^2 + y^2 + 2x^3 + \varepsilon g(x,y,\varepsilon), \end{aligned} \tag{1.5}$$

where

$$f(x,y,\varepsilon) = \sum_{i+j=1}^{3} a_{ij}(\varepsilon) x^i y^j, \quad g(x,y,\varepsilon) = \sum_{i+j=1}^{3} b_{ij}(\varepsilon) x^i y^j$$

with $a_{ij}(\varepsilon)$ and $b_{ij}(\varepsilon)$ depending analytically on the small parameter ε. By (1.2) we know that the Abelian integrals of system (1.5) is

$$I(h) = \oint_{\Gamma_h} R(x,y) f(x,y,0) dy - R(x,y) g(x,y,0) dx, \tag{1.6}$$

where Γ_h is the compact component of $H(x,y) = h$, defined by (1.4).

The following theorem is the main result of this article.

Theorem 1.1. *For cubic perturbed systems* (1.5)*, the maximum number of zeros in $h \in (0, +\infty)$, counting multiplicities, of the Abelian integral $I(h)$ in* (1.6) *is equal to four. Moreover, for each $k = 0, 1, 2, 3, 4$, there exist perturbations such that $I(h)$ have exactly k zeros.*

To prove this theorem, we shall change the Abelian integral $I(h)$ in (1.6) to a linear combination of five integrals in (2.2) and introduce some definitions of Chebyshev system and lemmas in Section 2. In Section 3, we shall prove that the five integrals in (2.2) form an extended complete Chebychev system. Accordingly, we obtain the number of zeros of the generating function by some purely algebraic computations.

It follows from (1.4) that system (1.3) is reversible. Hence, Theorem 1.1 and (1.2) imply the following result.

Theorem 1.2. *The least upper bound for the number of limit cycles of system* (1.5) *bifurcating from the period annulus of unpertubing system* (1.3) *is equal to four. Moreover, for each* $k = 0, 1, 2, 3, 4$*, there exist perturbations such that exactly* k *limit cycles produced by the period annulus of system* (1.3)

2. The generating function and preliminary results

To study the bifurcation of limit cycles of system (1.5), we need to calculate the number of zeros of the Abelian integral $I(h)$. The author of [4] use Chebyshev property to study the number of zeros of Abelian integrals of several classes of planar quadratic systems under quadratic perturbations. This method is valid for some restricted forms of the first integrals.

We write the first integral $H(x, y)$ in (1.4) as

$$H(x, y) = A(x) + B(x)y^2, \tag{2.1}$$

where $A(x) = \frac{(x(x-1))^2}{(2x-1)^2}$ and $B(x) = \frac{(x-1)^2}{(2x-1)^2}$. There exists a period annulus by the set of ovals $\Gamma_h \in \{(x, y)|H(x, y) = h\}$ around the origin, which is parameterized by the Hamiltonian value $h \in (0, +\infty)$.

Lemma 2.1 ([4])**.** *Let* Γ_h *be an oval inside the level curve* $\{A(x) + B(x)y^{2m} = h\}$ *and we consider a function* F *such that* $\frac{F}{A'}$ *is analytic at* $x = 0$*. Then, for any* $k \in \mathbb{N}$*,*

$$\int_{\Gamma_h} F(x)y^{k-2}dx = \int_{\Gamma_h} G(x)y^k dx\,,$$

where $G(x) = \frac{2}{k}(\frac{BF}{A'})'(x) - (\frac{B'F}{A'})(x)$*.*

Using above lemma, we have the following proposition.

Proposition 2.2. *The generating function* $I(h)$ *defined by* (1.6) *can be rewritten as*

$$I(h) = \mu_0 J_0(h) + \mu_1 J_1(h) + \mu_2 J_2(h) + \mu_3 J_3(h) + \mu_4 J_4(h), \tag{2.2}$$

where

$$J_0(h) = \int_{\Gamma_h} \frac{x^2 y}{(1-2x)^4}dx, \quad J_1(h) = \int_{\Gamma_h} \frac{xy}{(1-2x)^4}dx,$$
$$J_2(h) = \int_{\Gamma_h} \frac{y}{(1-2x)^4}dx, \quad J_3(h) = \int_{\Gamma_h} \frac{y^3}{(1-2x)^4}dx,$$
$$J_4(h) = \int_{\Gamma_h} \frac{xy^3}{(1-2x)^3}dx,$$

whith μ_0, μ_1, μ_2*,* μ_3 *and* μ_4 *are arbitrary constants.*

Proof. Integrating by parts, for any $i, j \geq 0$, we obtain

$$\begin{aligned}\int_{\Gamma_h} R(x,y)x^i y^j dy &= \frac{1}{j+1}\int_{\Gamma_h} R(x,y)x^i dy^{j+1}\\ &= \frac{2}{j+1}\int_{\Gamma_h} \frac{[ix^{i-1}+(5-3i)x^i+2(i-2)x^{1+i}]y^{j+1}}{(2x-1)^4}dx.\end{aligned}$$

Since the system (1.3) is reversible,

$$\int_{\Gamma_h} \frac{x^i y^j}{(2x-1)^4}dx = 0, \quad \text{for } j = 2n,\, n \in \mathbb{N}.$$

The Abel integral $I(h)$ (also known as the first order Melnikov function) is the divergence integral. By direct computation we have

$$\begin{aligned}I(h) &= \int_{\Gamma_h} Rf(x,y,0)dy - Rg(x,y,0)dx\\ &= \int\int_{int\Gamma_h} [(Rf)_x + (Rg)_y]dxdy\\ &= \int_{\Gamma_h} \frac{(\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3 x^3+\alpha_4 x^4)y}{(2x-1)^4}dx + \int_{\Gamma_h} \frac{(\beta_0+\beta_1 x+\beta_2 x^2)y^3}{(2x-1)^4}dx,\end{aligned}$$

where α_i and β_j are arbitrary constants independent on ε.

It is easy to show that $\frac{xy^3}{(2x-1)^4}$ and $\frac{x^2y^3}{(2x-1)^4}$ can be expressed as line combination of $\frac{y^3}{(2x-1)^4}$, $\frac{y^3}{(2x-1)^3}$ and $\frac{xy^3}{(2x-1)^3}$. By using lemma 2.1 and solving the differential equation

$$\frac{1}{(2x-1)^3} = \frac{2}{3}\Big(\frac{BF}{A'}\Big)'(x) - \Big(\frac{B'F}{A'}\Big)(x),$$

we obtain

$$F(x) = \frac{x(1-2x+2x^2)(3+2C-4Cx+2Cx^2)}{2(-1+2x)^4},$$

where C ia a constant. Taking $C = 0$, we have

$$\int_{\Gamma_h} \frac{y^3}{(2x-1)^3}dx = \int_{\Gamma_h} \frac{3x(1-2x+2x^2)y}{2(-1+2x)^4}dx.$$

Similarly, we obtain

$$\int_{\Gamma_h} \frac{xy^3}{(2x-1)^3}dx = -\int_{\Gamma_h} \frac{3x(1-2x+2x^2)y}{2(-1+2x)^3}dx.$$

Obviously, there exist constants c_i and d_j such that

$$\frac{3x(1-2x+2x^2)}{2(-1+2x)^4} = \sum_{i=0}^{3} \frac{c_i x^i}{(-1+2x)^4},$$

and

$$\frac{3x(1-2x+2x^2)}{2(-1+2x)^3} = \sum_{j=0}^{4} \frac{d_j x^i}{(-1+2x)^4}.$$

Thus, it follows from the above analysis we obtain the expression (2.3) and the proof is complete. □

To prove Theorem 1.1, now we introduce some definitions and lemmas. The reader is referred to [4] and [14] for details.

Definition 2.3. Let $\varphi_0(x), \varphi_1(x), \dots, \varphi_{n-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$.

(a) $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{n-1}(x))$ is a Chebyshev system (for short, a T-system) on L if any nontrivial linear combination
$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \dots + \alpha_{n-1}\varphi_{n-1}(x)$$
has at most $n-1$ isolated zeros for $x \in L$.

(b) $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{n-1}(x))$ is a complete Chebyshev system (for short, a CT-system) on L if $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{k-1}(x))$ is a T-system for all $k = 1, 2, \dots, n$.

(c) $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{n-1}(x))$ is an extend complete Chebyshev system (for short, an ECT-system) on L if for all $k = 1, 2, \dots, n$, any nontrivial linear combination
$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \dots + \alpha_{n-1}\varphi_{k-1}(x)$$
has at most $k-1$ isolated zeros on L counted with multiplicities.

Definition 2.4. Let $\varphi_0(x), \varphi_1(x), \dots, \varphi_{k-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$. The continuous Wronskian of $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{k-1}(x))$ at $x \in L$ is
$$W[\varphi_0, \varphi_1, \dots, \varphi_{k-1}](x) = \det(\varphi_j^{(i)}(x))_{0\le i,j\le k-1} = \begin{vmatrix} \varphi_0(x) & \dots & \varphi_{k-1}(x) \\ \varphi_0'(x) & \dots & \varphi_{k-1}'(x) \\ \dots & \dots & \dots \\ \varphi_0^{(k-1)}(x) & \dots & \varphi_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

The following two lemmas are found in [14] and [8] for instance.

Lemma 2.5. $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{n-1}(x))$ *is an ECT-system on interval L, then, for each $k = 1, 2, \dots, n-1$, there exists a linear combination with exactly k simple zeros on L.*

Lemma 2.6. $(\varphi_0(x), \varphi_1(x), \dots, \varphi_{n-1}(x))$ *is an ECT-system on L if and only if, for each $k = 1, 2, \dots, n$,*
$$W[\varphi_k](x) \neq 0 \quad \textit{for all } x \in L.$$

From (2.1) we can see that the projection of the period annulus Γ_h on the x-axis is $x \in (-\infty, \frac{1}{2})$ and $xA'(x) > 0$ for any $x \in (-\infty, \frac{1}{2})\backslash 0$. Hence there exists an analytic involution $\sigma(x)$ ($\sigma \circ \sigma = Id$ and $\sigma \neq Id$) such that
$$A(x) = A(\sigma(x)) \quad \text{for all } x \in (-\infty, 1/2).$$
Let $t = \sigma(x)$, then
$$A(x) - A(t) = \frac{(x-t)(-1+x+t)p(x)q(x)}{(-1+2x)^2(-1+2t)^2} = 0,$$
where $p(x) = -x - t + 2xt$ and $q(x) = 1 - x - t + 2xt$. Since $\sigma(x)$ is an involution, $\sigma(0) = 0$. It is easy to know that
$$t = \sigma(x) = \frac{x}{2x-1}. \tag{2.3}$$

Using [4, Theorem B] directly, we have the following result.

Proposition 2.7. *Let the Ablian integrals*

$$I_i(h)=\int_{\Gamma_h}\varphi_i(x)y^{2m-1}dx,\quad i=0,1,2,3,4.$$

where f_i be analytic functions in $(-\infty,\frac{1}{2})$, $m\in\mathbb{Z}$ and Γ_h be the oval surrounding the origin inside the level curve $\{A(x)+B(x)y^2=h\}$, $h\in(0,+\infty)$. Suppose that

$$L_i(x)=\Big(\frac{\varphi_i}{A'B^{\frac{2m-1}{2}}}\Big)(x)-\Big(\frac{\varphi_i}{A'B^{\frac{2m-1}{2}}}\Big)(\sigma(x)).$$

Then (J_0,J_1,J_2,J_3,J_4) is an ECT-system on the interval $(0,+\infty)$ if $m>3$ and (L_0,L_1,L_2,L_3,L_4) is a CT-system on $(0,1/2)$.

3. Proof of Theorem 1.1

In this section we apply proposition 2.7 to prove that (J_0,J_1,J_2,J_3,J_4) in (2.2) is an ECT-system on $(0,+\infty)$. However, we find that proposition 2.7 can not directly be applied for (J_0,J_1,J_2,J_3,J_4). To solve this problem, by lemma 2.1 and (2.1), we firstly change J_0,J_1 and J_2 to

$$\begin{aligned}
J_0(h)&=\int_{\Gamma_h}\frac{x^2y}{(1-2x)^4}dx=\int_{\Gamma_h}-\frac{(1+x-6x^2+6x^3)y^3}{3(-1+2x)^3(1-2x+2x^2)^2}dx,\\
J_1(h)&=\int_{\Gamma_h}\frac{xy}{(1-2x)^4}dx=\int_{\Gamma_h}-\frac{2(2-5x+4x^2)y^3}{3(-1+2x)^3(1-2x+2x^2)^2}dx,\\
J_2(h)&=\int_{\Gamma_h}\frac{y}{(1-2x)^4}dx=\int_{\Gamma_h}-\frac{(-1+7x-14x^2+10x^3)y^3}{3x^2(-1+2x)^3(1-2x+2x^2)^2}dx.
\end{aligned}$$

Then, by applying twice Lemma 2.1 to $J_0(h)$ and taking $m=4$, we obtain

$$\begin{aligned}
J_0(h)&=\frac{1}{h^2}\bar{J}_0(h)=\frac{1}{h^2}\int_{\Gamma_h}\frac{(1+x-6x^2+6x^3)(A(x)+B(x)y^2)^2y^3}{3(1-2x+2x^2)^2(1-2x)^7}dx\\
&=\frac{1}{h^2}\int_{\Gamma_h}\frac{(1+x-6x^2+6x^3)[(A(x))^2y^3+2A(x)B(x)y^5+(B(x))^2y^7]dx}{3(1-2x+2x^2)^2(1-2x)^7}.\\
&=\frac{1}{h^2}\int_{\Gamma_h}\varphi_0(x)y^7dx,
\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}
\varphi_0(x)=\frac{-2(-1+x)^4}{35(-1+2x)^7(1-2x+2x^2)^6}\Big(&8-41x-20x^2+740x^3-2856x^4\\
&+5966x^5-8092x^6+7668x^7-5240x^8+2544x^9-800x^{10}+128x^{11}\Big).
\end{aligned}$$

In the same way, we obtain

$$J_i(h)=\frac{1}{h^2}\bar{J}_i(h)=\frac{1}{h^2}\int_{\Gamma_h}\varphi_i(x)y^7dx,\quad i=1,2,3,4,\tag{3.2}$$

where

$$\begin{aligned}
\varphi_1(x)=\frac{-2(-1+x)^4}{35(-1+2x)^7(1-2x+2x^2)^6}\Big(&32-311x+1364x^2-3504x^3\\
&+5816x^4-6576x^5+5296x^6-3168x^7+1408x^8-416x^9+64x^{10}\Big),
\end{aligned}$$

$$\begin{aligned}\varphi_2(x) &= \frac{2(-1+x)^4}{35x^2(-1+2x)^7(1-2x+2x^2)^6}\Big(4-60x+368x^2-1249x^3\\ &\quad +2588x^4-3360x^5+2680x^6-1248x^7+344x^8-88x^9+16x^{10}\Big),\\ \varphi_3(x) &= \frac{(-1+x)^4}{35(-1+2x)^8(1-2x+2x^2)^4}\Big(48-339x+1118x^2-2196x^3\\ &\quad +2848x^4-2552x^5+1592x^6-632x^7+128x^8\Big),\\ \varphi_4(x) &= \frac{-3x(-1+x)^4}{35(-1+2x)^7(1-2x+2x^2)^4}\Big(21-168x+624x^2-1392x^3\\ &\quad +2056x^4-2080x^5+1424x^6-608x^7+128x^8\Big).\end{aligned}$$

Clearly, $(J_0, J_1, J_2, J_3, J_4)$ is an ECT-system if and only if $(\bar{J}_0, \bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{J}_4)$ is an ECT-system on $(0, +\infty)$. It follows from proposition 2.7 that

$$\begin{aligned}L_i(x) &= \Big(\frac{\varphi_i}{A'B^{\frac{7}{2}}}\Big)(x) - \Big(\frac{\varphi_i}{A'B^{\frac{7}{2}}}\Big)(\sigma(x))\\ &= \frac{(-1+2x)^7(\varphi_i(x)-\varphi_i(\sigma(x))}{2x(-1+x)^8(1-2x+2x^2)}, \quad i=0,1,2,3,4.\end{aligned} \tag{3.3}$$

Substituting (2.2) into (3.3), by direct computation we have

$$L_0(x) = \frac{8(-1+2x)^3p_0(x)}{35x(-1+x)^3(1-2x+2x^2)^7}, \tag{3.4}$$

where

$$\begin{aligned}p_0(x) &= 2-24x+160x^2-720x^3+2286x^4-5232x^5+8805x^6-11070x^7\\ &\quad +10460x^8-7336x^9+3664x^{10}-1184x^{11}+192x^{12}.\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\begin{aligned}L_1(x) &= \frac{32(-1+2x)^3}{35x(-1+x)^3(1-2x+2x^2)^5}\Big(2-16x+60x^2-136x^3+206x^4\\ &\quad -216x^5+155x^6-70x^7+16x^8\Big),\\ L_2(x) &= \frac{8(-1+2x)^3}{35x^3(-1+x)^3(1-2x+2x^2)^7}\Big(-1+14x-78x^2+208x^3-124x^4\\ &\quad -1048x^5+4372x^6-9824x^7+15434x^8-18172x^9+16264x^{10}\\ &\quad -10896x^{11}+5232x^{12}-1632x^{13}+256x^{14}),\\ L_3(x) &= \frac{-8(-1+2x)^2}{35x(-1+x)^3(1-2x+2x^2)^5}\Big(6-60x+316x^2-1088x^3+2634x^4\\ &\quad -4604x^5+5861x^6-5380x^7+3436x^8-1392x^9+280x^{10}\Big),\\ L_4(x) &= \frac{3744x^3(-1+2x)^9}{35(-1+x)^3(1-2x+2x^2)^5}.\end{aligned}$$

By Proposition 2.7, we need to check that $(L_0, L_1, L_2, L_3, L_4)$ is a CT-system on $(0, 1/2)$. From definition 2.3, it is easy to show that if $(L_0, L_1, L_2, L_3, L_4)$ is an ECT-system on $(0, 1/2)$, then $(L_0, L_1, L_2, L_3, L_4)$ is a CT-system on $(0, 1/2)$. Moreover, it follows from definitions of $L_i(x)$ and the involution $\sigma(x)$ that $L_i(\sigma(x)) = -L_i(x)$. Therefore, we have

Lemma 3.1. *$(L_0, L_1, L_2, L_3, L_4)$ is a CT-system on the interval $(0, 1/2)$ if and only if $(L_0, L_1, L_2, L_3, L_4)$ is a CT-system on $(-\infty, 0)$.*

Lemma 3.2. *$(L_0(x), L_1(x), L_2(x), L_3(x), L_4(x))$ is an ECT-system on $(-\infty, 0)$.*

Proof. By lemmas 2.6 and 3.1, we need only to prove that

$$W[L_i](x) \neq 0, \quad \text{for } x \in (-\infty, 0) \text{ and for each } i = 0, 1, 2, 3, 4. \tag{3.6}$$

From (3.5), we can see that all coefficients of odd degree in $p(x)$ are negative and coefficients of even degree are all positive numbers. Hence,

$$W[L_0](x) = L_0(x) \neq 0, \quad \text{for any } x \in (-\infty, 0).$$

Direct calculations show that

$$W[L_0, L_1](x) = \frac{2048(-1+2x)^6 q_1(x)}{1225x(-1+x)^5(1-2x+2x^2)^{13}},$$

where

$$\begin{aligned} q_1(x) = {}& 10 - 180x + 1540x^2 - 8320x^3 + 31825x^4 - 91630x^5 + 206144x^6 \\ & - 371368x^7 + 544647x^8 - 657510x^9 + 657886x^{10} - 547296x^{11} + 378133x^{12} \\ & - 215454x^{13} + 99596x^{14} - 36200x^{15} + 9776x^{16} - 1760x^{17} + 160x^{18}. \end{aligned}$$

By a similar process, we obtain

$$W[L_0, L_1, L_2](x) = \frac{262144(-1+2x)^{10} q_2(x)}{8575x^6(-1+x)^6(1-2x+2x^2)^{15}},$$

where

$$\begin{aligned} q_2(x) = {}& 1 - 16x + 118x^2 - 532x^3 + 1642x^4 - 3688x^5 + 6264x^6 - 8256x^7 \\ & + 8583x^8 - 7096x^9 + 4684x^{10} - 2488x^{11} + 1082x^{12} - 392x^{13} + 116x^{14} \\ & - 24x^{15} + 3x^{16}, \end{aligned}$$

$$W[L_0, L_1, L_2, L_3](x) = -\frac{3221225472(-1+2x)^9 q_3(x)}{60025x^6(-1+x)^6(1-2x+2x^2)^{21}},$$

where

$$\begin{aligned} q_3(x) = {}& 1 - 24x + 276x^2 - 2024x^3 + 10627x^4 - 42524x^5 + 134786x^6 - 347244x^7 \\ & + 740317x^8 - 1323024x^9 + 2000136x^{10} - 2574224x^{11} + 2831954x^{12} \\ & - 2668568x^{13} + 2154548x^{14} - 1488664x^{15} + 878272x^{16} - 441408x^{17} \\ & + 188784x^{18} - 68768x^{19} + 21322x^{20} - 5544x^{21} + 1148x^{22} - 168x^{23} + 14x^{24} \end{aligned}$$

and

$$W[L_0, L_1, L_2, L_3, L_4](x) = -\frac{289446436012032(-1+2x)^{14} q_4(x)}{2100875x^6(-1+x)^{12}(1-2x+2x^2)^{29}},$$

where

$$\begin{aligned} q_4(x) = {}& 35 - 1452x + 28844x^2 - 366456x^3 + 3352785x^4 - 23570688x^5 \\ & + 132619414x^6 - 614002012x^7 + 2386389535x^8 - 7903353204x^9 \\ & + 22561391864x^{10} - 56016194208x^{11} + 121833124010x^{12} \end{aligned}$$

$$
\begin{aligned}
&- 233466766984x^{13} + 396026781404x^{14} - 596927677776x^{15}\\
&+ 801995647256x^{16} - 962896901056x^{17} + 1035219064208x^{18}\\
&- 998199356992x^{19} + 864222115830x^{20} - 672259532592x^{21}\\
&+ 469892104468x^{22} - 294979366008x^{23} + 166117090858x^{24}\\
&- 83767522232x^{25} + 37733705440x^{26} - 15140306400x^{27}\\
&+ 5393826536x^{28} - 1699627568x^{29} + 471031680x^{30} - 113633072x^{31}\\
&+ 23394448x^{32} - 3964800x^{33} + 519680x^{34} - 47040x^{35} + 2240x^{36}.
\end{aligned}
$$

Fortunately, for polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$ and $p_4(x)$, we find that coefficients of odd degree in x are all positive numbers and all coefficients of even degree are all positive numbers, this show that the Wronskian $W[L_i](x)$ of (J_1, J_2, J_3, J_4) are all no-vanish on $(-\infty, 0)$ for $i = 1, 2, 3, 4$. Thus we have proved this lemma. □

Proof of Theorem 1.1. By lemma 3.2, propositions 2.2 and 2.7, it is easy to see that the Abelian integral $I(h)$ has at most four zeros. Moreover, from lemma 2.5 for each $k = 0, 1, 2, 3, 4$, there exist perturbations such that k, the number of zeros, is sharp. □

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