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SPATIAL DYNAMICS OF A NONLOCAL DISPERSAL VECTOR DISEASE MODEL WITH SPATIO-TEMPORAL DELAY

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Abstract. This article concerns the spatial dynamics of a nonlocal dispersal vector disease model with spatio-temporal delay. We establish the existence of spreading speeds and construct some new types of solutions which are different from the traveling wave solutions. To obtain the existence of spreading speed, we follow the truncating approach to develop a comparison principle and to construct a suitable sub-solution. Our result indicates that the spreading speed coincides with the minimal wave speed of the regular traveling waves. The solutions are constructed by combining regular traveling waves and the spatially independent solutions which provide some new transmission forms of the disease.

1. Introduction

In this article, we consider the spatial dynamics, including spreading speeds and global solutions of the following nonlocal dispersal vector disease model with spatio-temporal delay

$$
\begin{gathered}
u_t(x,t) = d(J_\rho * u - u)(x,t) - au(x,t) + b[1-u(x,t)]F \star u(x,t),\\
J_\rho * u(x,t) = \int_{-\infty}^{+\infty} J_\rho(y)u(x-y,t)\mathrm{d}y,\\
F \star u(x,t) = \int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,s)u(x-y,t-s)\mathrm{d}y\mathrm{d}s,
\end{gathered}
\tag{1.1}
$$

where $u(x,t)$ represents the normalized spatial density of infectious host at location $x \in \mathbb{R}$ and at time t, $d > 0$ is the dispersal rate, $a > 0$ is the cure or recovery rate of the infected host, and $b > a$ is the host-vector contact rate. The term $d(J_\rho * u - u)$ is called the nonlocal dispersal and represents transportation due to long range dispersion mechanisms, $J_\rho(\cdot)$ is a ρ-parameterized symmetric kernel given by $\frac{1}{\rho}J(\frac{y}{\rho})$, where ρ represents the nonlocal dispersal distance if $\rho > 0$ and no dispersal if $\rho = 0$. $F(\cdot,\cdot)$ is the convolution kernel function used to describe the spatio-temporal delay.

By a global solution we mean a solution defined for $t \geq t_0$ and all $x \in \mathbb{R}$. To describe this condition some authors use the term "entire solution" which can be

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mistaken as an entire function of complex variables. Throughout this paper, we assume that the kernel functions J and F satisfy the following assumptions:

(J1) $J \in L^1(\mathbb{R})$ is a positive even function with $\int_{-\infty}^{+\infty} J(y)\mathrm{d}y = 1$. Moreover, for any $\lambda \in [0, \hat{\lambda})$,
$$\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}\mathrm{d}y < +\infty,$$
and $\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}\mathrm{d}y \to +\infty$ as $\lambda \to \hat{\lambda}^-$, where $\hat{\lambda}$ may be $+\infty$.

(F1) $F \in C(\mathbb{R} \times (0, +\infty), \mathbb{R}^+)$ with $F(y, s) = F(-y, s) \geq 0$, and satisfies $\int_0^{+\infty} \int_{-\infty}^{+\infty} F(y, s)\mathrm{d}y\mathrm{d}s = 1$. In addition, for any $c \geq 0$, there exists some $\tilde{\lambda} := \tilde{\lambda}(c) > 0$ such that
$$\int_0^{+\infty} \int_{-\infty}^{+\infty} F(y, s)e^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s < +\infty,$$
for all $\lambda \in [0, \tilde{\lambda})$.

From the assumptions (J1) and (F1), we can see that (1.1) has two constant equilibria $u \equiv 0$ and $u \equiv 1 - a/b =: K$.

When $F(y, s) = \delta(y)\delta(s - \tau)$, Equation (1.1) is reduced to the following nonlocal dispersal equation with constant delay
$$u_t(x, t) = d(J_\rho * u - u)(x, t) - au(x, t) + b[1 - u(x, t)]u(x, t - \tau). \tag{1.2}$$
In 2009, Pan et al [24] proved the existence of traveling wavefronts of (1.2) with speed $c \geq c^*$ by Schauder's fixed point theorem and upper-lower solution technique. Furthermore, if taking $a = 0$ and $\tau = 0$ (no time-delay) in (1.2), then we obtain the Fisher-KPP equation with nonlocal dispersal
$$u_t(x, t) = d(J_\rho * u - u)(x, t) + bu(x, t)[1 - u(x, t)], \tag{1.3}$$
which was considered by a number of researchers, see Carr and Chmaj [3], Coville et al [4, 5], Pan [23], Schumacher [29, 30], Yagisita [39] for traveling wave solutions, and Li et al [16] for global solutions.

We would like to point out that (1.1) is the nonlocal dispersal counterpart of the following classical host-vector model
$$u_t(x, t) = d\Delta u(x, t) - au(x, t) + b[1 - u(x, t)]F \star u(x, t), \tag{1.4}$$
which was presented by Ruan and Xiao [28] for a disease without immunity in which the current density of infectious vectors is related to the number of infectious hosts at earlier times. The reader is referred to [4, 9] for the discussion of the relationship between nonlocal dispersal operators and random dispersal operators. In fact, for J compactly supported and $0 < \rho \ll 1$, we have
$$\begin{aligned}(J_\rho * u - u)(x, t) &= \int_{-\infty}^{+\infty} \frac{1}{\rho} J\big(\frac{y}{\rho}\big)[u(x - y, t) - u(x, t)]\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} J(y)[u(x - \rho y, t) - u(x, t)]\mathrm{d}y \\ &= \frac{1}{2}\rho^2 \int_{-\infty}^{+\infty} J(y)y^2\mathrm{d}y \frac{\partial^2 u(x, t)}{\partial x^2} + o(\rho^2).\end{aligned}$$

It is well known, in epidemiology, traveling wave solution and asymptotic speed of spread (sometimes called spreading speed) are two fundamental mathematical tools that have been shown to be useful for the description of the transmission of the disease. In particular, the spreading speed can help us understand how fast the disease spreads in a spatial environment [1, 21, 27]. Thus, they are among the central problems investigated for (1.1) and (1.4) and are quite well understood for (1.4). Ruan and Xiao [28] proved the existence of traveling wave solutions of (1.4) with some special delay kernels. Combining the comparison method and the finite time-delay approximation, Zhao and Xiao [42] established the existence of the spreading speed for the solutions of (1.4) with initial functions having compact supports, and showed that the spreading speed coincides with its minimal wave speed for monotone waves. For the other related results on (1.4), we can refer the readers to Huang and Huo [14], Lv and Wang [18], Peng and Song [25], Peng et al [26] and Zhang [40]. It is very necessary to point out that when the habitat is divided into discrete regions and the population density is measured at one point (e.g., center) in each region, then (1.4) is reduced to the system

$$\begin{aligned} \frac{\mathrm{d}u_n(t)}{\mathrm{d}t} &= d[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)]-au_n(t) \\ &\quad + b[1-u_n(t)]\int_0^{+\infty}\sum_{j\in\mathbb{Z}}F_j(s)u_{n-j}(t-s)\mathrm{d}s. \end{aligned} \tag{1.5}$$

Xu and Weng [37] obtained the spreading speed and the strictly monotonic traveling waves for the system (1.5), and confirmed that the spreading speed coincides with the minimal wave speed for traveling wavefronts.

In addition to the traveling wave solutions and spreading speeds, another important issue in epidemic dynamics is the interaction between traveling wave solutions, which can provide some new transmission forms of the disease. Mathematically, this phenomenon can be described by a class of global solutions that are defined for all space and time. In recent years, there were many works devoted to the global solutions for various evolution equations, see e.g., [6, 7, 10, 11, 12, 13, 15, 17, 22, 16, 35, 36] and the references cited therein. More recently, Li et al. [19] established the global solutions of (1.5) by the combinations of traveling waves and the spatially independent solution.

In this article, we mainly focus on the spatial dynamics of nonlocal dispersal vector disease model (1.1), and investigate whether it is consistent with that of random diffusion equation (1.4). Recently, Xu and Xiao [38] have obtained the existence, nonexistence and uniqueness of the regular traveling wave solutions of (1.1). To the best of our knowledge, the issues on existence of spreading speed and global solutions for nonlocal dispersal model (1.1) have not been addressed. This is the motivation of the current study. Inspired by [34, 37], we establish the existence of spreading speed by using the truncating technique associated with the comparison method and constructing sub-solution. We can also confirm that the spreading speed coincides with the minimal wave speed of regular traveling waves of (1.1), which has been founded in many reaction-diffusion equations, lattice differential equations, and integral equations, see, e.g., [20, 31, 33, 42, 34, 37, 41] and references therein. In the second part of this paper, based on the results of regular traveling wave solutions in [38], we construct the global solutions of (1.1) via the combinations of traveling waves and the spatially independent solution. In order to establish the global solution, we shall consider the solutions $u^n(x,t)$ of a sequence of initial value problems of (1.1). However, the convergence of $\{u^n(x,t)\}$

is not ensured. Hence, we try to find a convergent subsequence of $\{u^n(x,t)\}$. Since the solutions $\{u^n(x,t)\}$ are not smooth enough with respect to x, we have to make $\{u^n(x,t)\}$ possess a property which is similar to a global Lipschitz condition with respect to x (see Lemma 4.4).

The remaining part of this paper is organized as follows. In Section 2, we obtain the well-posedness of the solution for the initial value problem of (1.1) and develop a comparison principle. In Section 3, the existence of spreading speed for model (1.1) is established. Section 4 is devoted to constructing the global solutions of (1.1) and investigating the qualitative properties of them.

2. Initial value problem of (1.1)

In this section, we establish the existence, uniqueness of solutions and the comparison principle for the initial value problem of (1.1). Obviously, the initial value problem of (1.1) can be written as

$$\begin{gathered} u_t(x,t) = -(d+b)u(x,t) + G[u](x,t),\ (x,t)\in\mathbb{R}\times[\kappa,+\infty),\\ u(x,s)=\phi(x,s),\quad (x,s)\in\mathbb{R}\times(-\infty,\kappa], \end{gathered}\tag{2.1}$$

where $\kappa\in\mathbb{R}$ is any given constant denoted the initial time and $\phi(x,s)\in C(\mathbb{R}\times(-\infty,\kappa],\mathbb{R}^+)$ is a given initial function, and $G: C(\mathbb{R}^2,[0,K])\to C(\mathbb{R}^2,\mathbb{R}^+)$ is defined by

$$G[u](x,t) = (b-a)u(x,t) + dJ_\rho * u(x,t) + b[1-u(x,t)]F\star u(x,t). \tag{2.2}$$

It is easy to see that (2.1) is equivalent to the integral equation

$$u(x,t) = e^{-(d+b)(t-\kappa)}\phi(x,\kappa) + \int_\kappa^t e^{-(d+b)(t-s)}G[u](x,s)\mathrm{d}s, \tag{2.3}$$

$(x,t)\in\mathbb{R}\times[\kappa,+\infty)$.

Lemma 2.1. *G is a nondecreasing operator on $C(\mathbb{R}^2,[0,K])$, and for any $u\in C(\mathbb{R}\times[\kappa,+\infty),[0,K])$, we have*

$$0\le G[u](x,t)\le(d+b)K.$$

Proof. Suppose that $0\le u(x,t)\le v(x,t)\le K$ for $(x,t)\in\mathbb{R}^2$. By some simple computations, we have

$$\begin{aligned} &G[v](x,t)-G[u](x,t)\\ &=(b-a)[v(x,t)-u(x,t)] + d\int_{-\infty}^{+\infty}J_\rho(y)[v(x-y,t)-u(x-y,t)]\mathrm{d}y\\ &\quad - b[v(x,t)-u(x,t)]\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)v(x-y,t-s)\mathrm{d}y\mathrm{d}s\\ &\quad + b[1-u(x,t)]\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)[v(x-y,t-s)-u(x-y,t-s)]\mathrm{d}y\mathrm{d}s\\ &\ge\Big[(b-a)-b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)v(x-y,t-s)\mathrm{d}y\mathrm{d}s\Big][v(x,t)-u(x,t)]\\ &\ge\Big[(b-a)-bK\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)\mathrm{d}y\mathrm{d}s\Big][v(x,t)-u(x,t)]=0, \end{aligned}$$

which implies that $G[v](x,t) \geq G[u](x,t)$ for all $(x,t) \in \mathbb{R}^2$. Moreover, for any $u \in C(\mathbb{R} \times [\kappa, +\infty), [0, K])$, due to the above nondecreasing property of G, we obtain

$$\begin{aligned}
0 &\leq G[u](x,t) \leq G[K](x,t) \\
&= (b-a)K + dK \int_{-\infty}^{+\infty} J_\rho(y)\mathrm{d}y + b(1-K)K \int_0^{+\infty} \int_{-\infty}^{+\infty} F(y,s)\mathrm{d}y\mathrm{d}s \\
&= (b-a)K + dK + b(1-K)K = (d+b)K.
\end{aligned}$$

The proof is complete. □

Theorem 2.2 (Existence and Uniqueness)**.** *For any given initial functions $\phi \in C(\mathbb{R} \times (-\infty, \kappa], [0, K])$, (2.1) has a unique solution $u(x,t;\phi) \in C(\mathbb{R}^2, [0, K])$.*

Proof. For $u \in C(\mathbb{R}^2, [0, K])$ and $\phi \in C(\mathbb{R} \times (-\infty, \kappa], [0, K])$, define a set

$$S = \Big\{ u \in C(\mathbb{R}^2, [0, K]) : u(x,s) = \phi(x,s) \text{ for } (x,s) \in \mathbb{R} \times (-\infty, \kappa] \Big\}$$

and an operator

$$H[u](x,t) = \begin{cases} \phi(x,\kappa)e^{-(d+b)(t-\kappa)} \\ + \int_\kappa^t e^{-(d+b)(t-s)} G[u](x,s)\mathrm{d}s & \text{for } (x,t) \in \mathbb{R} \times [\kappa, +\infty), \\ \phi(x,t) & \text{for } (x,t) \in \mathbb{R} \times (-\infty, \kappa]. \end{cases}$$

According to Lemma 2.1, for any $u \in S$, we have

$$0 \leq H[u](x,t) \leq Ke^{-(d+b)(t-\kappa)} + (d+b)K \int_\kappa^t e^{-(d+b)(t-s)}\mathrm{d}s = K.$$

Thus, $H(S) \subseteq S$.

For $\tau > 0$, define

$$\Gamma_\tau = \big\{ u \in C(\mathbb{R}^2, \mathbb{R}) : \sup_{(x,t) \in \mathbb{R} \times [\kappa, +\infty)} |u(x,t)|e^{-\tau t} < +\infty \big\}.$$

It is clear that Γ_τ is a Banach space equipped with the norm

$$\|u\|_\tau = \sup_{(x,t) \in \mathbb{R} \times [\kappa, +\infty)} |u(x,t)|e^{-\tau t},$$

and S is a closed subset of Γ_τ.

For $u, v \in S$, let $w(x,t) = u(x,t) - v(x,t)$ for $(x,t) \in \mathbb{R} \times [\kappa, +\infty)$, then one has

$$\begin{aligned}
&|H[u](x,t) - H[v](x,t)| \\
&= \Big| \int_\kappa^t e^{-(d+b)(t-s)} [G[u](x,s) - G[v](x,s)]\mathrm{d}y\mathrm{d}s \Big| \\
&= \Big| \int_\kappa^t e^{-(d+b)(t-s)} \Big[(b-a)w(x,s) + d\int_{-\infty}^{+\infty} J_\rho(y)w(x-y,s)\mathrm{d}y \\
&\quad - bw(x,s) \int_0^{+\infty} \int_{-\infty}^{+\infty} F(y,\iota)u(x-y,s-\iota)\mathrm{d}y\mathrm{d}\iota \\
&\quad + b[1-v(x,s)] \int_0^{+\infty} \int_{-\infty}^{+\infty} F(y,\iota)w(x-y,s-\iota)\mathrm{d}y\mathrm{d}\iota \Big]\mathrm{d}s \Big| \\
&\leq \int_\kappa^t e^{-(d+b)(t-s)} \Big[(b-a)|w(x,s)| + d\int_{-\infty}^{+\infty} J_\rho(y)|w(x-y,s)|\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&+ b|w(x,s)| \int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)u(x-y,s-\iota)\mathrm{d}y\mathrm{d}\iota \\
&+ b\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)|w(x-y,s-\iota)|\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&|H[u](x,t)-H[v](x,t)|e^{-\tau t} \\
&\le \int_\kappa^t e^{-(d+b+\tau)(t-s)}\Big\{e^{-\tau s}\Big[(b-a)|w(x,s)| + d\int_{-\infty}^{+\infty} J_\rho(y)|w(x-y,s)|\mathrm{d}y \\
&\quad + b|w(x,s)|\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)u(x-y,s-\iota)\mathrm{d}y\mathrm{d}\iota\Big] \\
&\quad + b\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)|w(x-y,s-\iota)|e^{-\tau(s-\iota)}e^{-\tau\iota}\mathrm{d}y\mathrm{d}\iota\Big\}\mathrm{d}s \\
&\le \int_\kappa^t e^{-(d+b+\tau)(t-s)}\mathrm{d}s\Big[(b-a) + d\int_{-\infty}^{+\infty} J_\rho(y)\mathrm{d}y + bK\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)\mathrm{d}y\mathrm{d}\iota \\
&\quad + b\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)e^{-\tau\iota}\mathrm{d}y\mathrm{d}\iota\Big]\|w\|_\tau \\
&\le \frac{1}{d+b+\tau}[1-e^{-(d+b+\tau)(t-\kappa)}](b-a+d+bK+b)\|w\|_\tau \\
&\le \frac{d+3b-2a}{d+b+\tau}\|w\|_\tau.
\end{aligned}
$$

It then follows that

$$
\|H(u)-H(v)\|_\tau \le \frac{d+3b-2a}{d+b+\tau}\|w\|_\tau.
$$

Since $\lim_{\tau\to+\infty}\frac{d+3b-2a}{d+b+\tau}=0$, we can choose $\varrho\in(0,1)$ such that

$$
\|H(u)-H(v)\|_\tau \le \varrho\|w\|_\tau \text{ for large } \tau.
$$

Thus, H is a contracting map. By Banach contracting mapping theorem, H has a unique fixed point u in Γ_τ if τ is sufficiently large, which is the unique solution of (2.1). The proof is complete. ☐

To establish the spreading speeds and global solutions, we need the comparison principle for the initial value problem (2.1).

Lemma 2.3 (Comparison Principle)**.** *Let $u(x,t;\phi_u)$ and $v(x,t;\phi_v)$ be solutions of the initial value problem* (2.1) *with initial value $\phi_u,\phi_v\in C(\mathbb{R}\times(-\infty,\kappa],[0,K])$, respectively. If $\phi_u(x,s)\ge\phi_v(x,s)$ for all $(x,s)\in\mathbb{R}\times(-\infty,\kappa]$, then $u(x,t;\phi_u)\ge v(x,t;\phi_v)$ for all $(x,t)\in\mathbb{R}^2$.*

Proof. Let $\omega(x,t)=v(x,t;\phi_v)-u(x,t;\phi_u)$ for all $(x,t)\in\mathbb{R}\times[\kappa,+\infty)$. By Theorem 2.2, $\omega(x,t)$ is continuous and bounded. Define $\bar\omega(t)=\sup_{x\in\mathbb{R}}\omega(x,t)$ for any $t\in\mathbb{R}$. Hence, $\bar\omega(t)$ is continuous on $\mathbb{R}$. We shall show that $\bar\omega(t)\le 0$ for all $t\ge\kappa$. Assume, for the sake of contradiction, that this is not true. Then there must exist $t_0>\kappa$ such that $\bar\omega(t_0)>0$ and

$$
\bar\omega(t_0)e^{-M_0t_0}=\sup_{t\ge0}\bar\omega(t)e^{-M_0t}>\bar\omega(\hat t)e^{-M_0\hat t},\quad \hat t\in[\kappa,t_0), \tag{2.4}
$$

where M_0 is a constant satisfying $M_0 > 2(b-a) > 0$. It follows that there exists a sequence of points $\{x_n\}_{n\in\mathbb{N}^+}$ such that $\omega(x_n,t_0) > 0$ and $\lim_{n\to+\infty}\omega(x_n,t_0) = \bar{\omega}(t_0)$. At the same time, select $\{t_n\}_{n\in\mathbb{N}^+}$ as a sequence in $[\kappa,t_0]$ such that

$$\omega(x_n,t_n)e^{-M_0t_n} = \max_{t\in[\kappa,t_0]}\{\omega(x_n,t)e^{-M_0t}\}. \tag{2.5}$$

Then it follows from (2.4) that $\lim_{n\to+\infty}t_n = t_0$. Since

$$\omega(x_n,t_0)e^{-M_0t_0} \le \omega(x_n,t_n)e^{-M_0t_n} \le \bar{\omega}(t_n)e^{-M_0t_n} \le \bar{\omega}(t_0)e^{-M_0t_0},$$

we have

$$\omega(x_n,t_0)e^{-M_0(t_0-t_n)} \le \omega(x_n,t_n) \le \bar{\omega}(t_0)e^{-M_0(t_0-t_n)}.$$

Letting $n\to+\infty$, we obtain $\lim_{n\to+\infty}\omega(x_n,t_n) = \bar{\omega}(t_0)$. Then (2.5) implies that for each $n\in\mathbb{N}^+$,

$$0 \le \frac{\partial}{\partial t}\{\omega(x_n,t)e^{-M_0t}\}\Big|_{t=t_n-} = e^{-M_0t_n}\Big(\frac{\partial\omega(x_n,t)}{\partial t}\Big|_{t=t_n} - M_0\omega(x_n,t_n)\Big).$$

Thus, we have

$$\begin{aligned}
M_0\omega(x_n,t_n) &\le \frac{\partial\omega(x_n,t)}{\partial t}\Big|_{t=t_n}\\
&= -(d+a)\omega(x_n,t_n) + dJ_\rho*\omega(x_n,t_n)\\
&\quad - b\omega(x_n,t_n)\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)v(x_n-y,t_n-s;\phi_v)\mathrm{d}y\mathrm{d}s\\
&\quad + b[1-u(x_n,t_n;\phi_u)]\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)\omega(x_n-y,t_n-s)\mathrm{d}y\mathrm{d}s\\
&\le -(d+a)\omega(x_n,t_n) + dJ_\rho*\omega(x_n,t_n) + (b-a)\bar{\omega}(t_n)\\
&\quad + b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)\omega(x_n-y,t_n-s)\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{2.6}$$

By (2.4), we have $\bar{\omega}(\hat{t}) \le \bar{\omega}(t_0)e^{-M_0(t_0-\hat{t})}$ for $\hat{t}\in[\kappa,t_0)$. Letting $n\to+\infty$ in (2.6), we obtain

$$M_0\bar{\omega}(t_0) \le \Big(b-2a+b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)e^{-M_0s}\mathrm{d}y\mathrm{d}s\Big)\bar{\omega}(t_0) \le 2(b-a)\bar{\omega}(t_0),$$

which together with $\bar{\omega}(t_0) > 0$ implies that $M_0 \le 2(b-a)$. That is a contradiction and indicates that $\omega(x,t)\le 0$ for $(x,t)\in\mathbb{R}\times[\kappa,+\infty)$. Note that $\omega(x,s) = \phi_v(x,s)-\phi_u(x,s)\le 0$ for $(x,s)\in\mathbb{R}\times(-\infty,\kappa]$. Therefore, $u(x,t;\phi_u)\ge v(x,t;\phi_v)$ for all $(x,t)\in\mathbb{R}^2$ and we complete the proof. □

Remark 2.4. For the initial value problem (2.1) with

$$G[u](x,t) = (b-a)u(x,t) + dJ_\rho*u(x,t) + bF\star u(x,t),$$

which is actually the corresponding linearized system, we still can obtain the results on the existence and uniqueness of solution, and the comparison principle, that is to say Theorem 2.2 and Lemma 2.3 yet hold.

3. Spreading speed

In this section, we shall establish the existence of the spreading speed for (1.1). To start with, we give the definition of spreading speed.

Definition 3.1. Assume that $u(x,t;\phi)$ is the solution of (1.1) with the initial value ϕ. We call a number $c^*>0$ the spreading speed of (1.1), if the following properties are valid:

(i) for any $c>c^*$,
$$\limsup_{t\to+\infty,|x|\ge ct} u(x,t;\phi)=0; \tag{3.1}$$
(ii) for any $c\in(0,c^*)$,
$$\liminf_{t\to+\infty,|x|\le ct} u(x,t;\phi)\ge K. \tag{3.2}$$

Next, we define
$$\begin{aligned}
&\Delta(\lambda,c)\\
&=c\lambda-d\Big(\int_{-\infty}^{+\infty}J_\rho(y)e^{-\lambda y}\mathrm{d}y-1\Big)+a-b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)e^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s\\
&=c\lambda-d\Big(\int_{-\infty}^{+\infty}J(y)e^{-\lambda\rho y}\mathrm{d}y-1\Big)+a-b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)e^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s.
\end{aligned}$$
Note that $\Delta(0,c)=a-b<0$ for all $c>0$, $\Delta(\lambda,c)\to-\infty$ as $\lambda\to\hat\lambda$ by (J1) and (F1). Moreover, by a direct calculation, we have, for all $\lambda\in(0,\hat\lambda)$ and $c>0$,
$$\begin{aligned}
\frac{\partial\Delta(0,c)}{\partial\lambda}&=c+b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)cs\mathrm{d}y\mathrm{d}s>0,\\
\frac{\partial\Delta(\lambda,c)}{\partial c}&=\lambda+b\lambda\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)se^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s>0,\\
\frac{\partial^2\Delta(\lambda,c)}{\partial\lambda^2}&=-d\rho^2\int_{-\infty}^{+\infty}J(y)y^2e^{-\lambda\rho y}\mathrm{d}y\\
&\quad-b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)(y+cs)^2e^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s<0.
\end{aligned}$$
Based on the above properties of $\Delta(\lambda,c)$, we can get the following conclusion easily.

Lemma 3.2. *There exist a positive pair (λ_*,c^*) such that*
$$\Delta(\lambda_*,c^*)=0,\quad \frac{\partial\Delta(\lambda_*,c^*)}{\partial\lambda}=0.$$
Furthermore,

(i) *if $c\in(c^*,+\infty)$, then $\Delta(\lambda,c)=0$ has two different real roots $\lambda_1(c)$, $\lambda_2(c)$ with $0<\lambda_1(c)<\lambda_*<\lambda_2(c)<\hat\lambda\le+\infty$ and*
$$\Delta(\lambda,c)\begin{cases}>0 & \text{for } \lambda\in(\lambda_1(c),\lambda_2(c)),\\ <0 & \text{for } \lambda\in[0,\lambda_1(c))\cup(\lambda_2(c),\hat\lambda),\end{cases}$$
(ii) *if $c\in(0,c^*)$, then $\Delta(\lambda,c)<0$ for all $\lambda>0$.*

In a recent paper, Xu and Xiao [38] studied the regular traveling waves of (1.1).

Lemma 3.3. *Assume that* (J1) *and* (F1) *hold. Then for $c > c^*$,* (1.1) *has a unique positive regular traveling wave $u(x,t) = U_c(x+ct)$, while it has no regular traveling wave for $c < c^*$; for $c = c^*$,* (1.1) *has a positive traveling wave $u(x,t) = U_{c^*}(x+c^*t)$, and all these traveling waves are strictly increasing, and satisfy*

$$\lim_{\xi\to-\infty} U_c(\xi) = 0, \quad \lim_{\xi\to+\infty} U_c(\xi) = K \quad \text{for } c \geq c^*.$$

Furthermore, $\lim_{\xi\to-\infty} U_c(\xi)e^{-\lambda_1(c)\xi} = 1$ *and* $\lim_{\xi\to-\infty} U_c'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$ *for* $c > c^*$.

In the following, we shall show that c^* is the spreading speed of (1.1). For convenience, we take the initial time $\kappa = 0$ in the rest part of this Section. Since the proof is rather involved, we shall split it into several steps which are formulated as lemmas.

Lemma 3.4. *Assume $c > c^*$ and $\phi \in C(\mathbb{R}\times(-\infty,0],[0,K])$. Then the following statements hold:*

(i) *if* $\limsup_{x\to-\infty,s\leq 0}\phi(x,s)e^{-\lambda x} < +\infty$ *for* $\lambda > \lambda_1(c)$*, then* $\limsup_{t\to+\infty,x\leq -ct} u(x,t;\phi) = 0$;

(ii) *if* $\limsup_{x\to+\infty,s\leq 0}\phi(x,s)e^{\lambda x} < +\infty$ *for* $\lambda > \lambda_1(c)$*, then* $\limsup_{t\to+\infty,x\geq ct} u(x,t;\phi) = 0$.

Proof. (i) Define a sequence $\{u^{(n)}(x,t)\}_{n\in\mathbb{N}}$ as

$$u^{(n)}(x,t) = H[u^{(n-1)}](x,t) \text{ for } (x,t)\in\mathbb{R}^2,$$

with

$$u^{(0)}(x,t) = \begin{cases} \phi(x,t) & \text{for } (x,t)\in\mathbb{R}\times(-\infty,0],\\ \phi(x,0) & \text{for } (x,t)\in\mathbb{R}\times(0,+\infty).\end{cases}$$

By an argument similar to that of Theorem 2.2, we can obtain that $u^{(n)}(x,t) \in C(\mathbb{R}^2,[0,K])$ and $\lim_{n\to+\infty} u^{(n)}(x,t) = u(x,t)$ for $(x,t)\in\mathbb{R}\times[0,+\infty)$ is a solution of (2.1).

For any $c > c^*$, take $c_1 \in (c^*, c)$. Since $\limsup_{x\to-\infty,s\leq 0}\phi(x,s)e^{-\lambda x} < +\infty$, combining the fact $u^{(0)}(x,t)\in[0,K]$ for all $(x,t)\in\mathbb{R}^2$, we can choose $M > 0$ such that

$$u^{(0)}(x,t)e^{-\lambda(x+c_1|t|)} \leq u^{(0)}(x,t)e^{-\lambda x} \leq M \quad \text{for } (x,t)\in\mathbb{R}^2. \tag{3.3}$$

Without loss of generality, we assume that $\lambda \in (\lambda_1(c), \lambda_*)$, then choose suitable $c_1 \in (c^*, c)$ such that $\Delta(\lambda, c_1) = 0$. For $(x,t)\in\mathbb{R}\times(0,+\infty)$, by the definition of $u^{(1)}(x,t)$ and (3.3), we obtain

$$\begin{aligned}
&u^{(1)}(x,t)e^{-\lambda(x+c_1|t|)}\\
&= e^{-\lambda(x+c_1t)}\Big\{u^{(0)}(x,t)e^{-(d+b)t} + \int_0^t e^{-(d+b)(t-s)}\Big[(b-a)u^{(0)}(x,s)\\
&\quad + d\int_{-\infty}^{+\infty} J_\rho(y)u^{(0)}(x-y,s)\mathrm{d}s\\
&\quad + b[1-u^{(0)}(x,s)]\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)u^{(0)}(x-y,s-\iota)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\Big\}\\
&\leq e^{-(d+b+\lambda c_1)t}\Big\{u^{(0)}(x,t)e^{-\lambda x} + \int_0^t e^{(d+b+\lambda c_1)s}\Big[(b-a)u^{(0)}(x,s)e^{-\lambda(x+c_1s)}
\end{aligned}$$

$$
\begin{aligned}
&+ d\int_{-\infty}^{+\infty} J_\rho(y)u^{(0)}(x-y,s)e^{-\lambda(x-y+c_1 s)}e^{-\lambda y}\mathrm{d}y + b[1-u^{(0)}(x,s)] \\
&\times \int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)u^{(0)}(x-y,s-\iota)e^{-\lambda(x-y+c_1(s-\iota))}e^{-\lambda(y+c_1\iota)}\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\Big\} \\
&\le Me^{-(d+b+\lambda c_1)t}\Big\{1+\int_0^t e^{(d+b+\lambda c_1)s}\Big[(b-a)+d\int_{-\infty}^{+\infty} J_\rho(y)e^{-\lambda y}\mathrm{d}y \\
&+ b\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)e^{-\lambda(y+c_1\iota)}\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\Big\} \\
&= Me^{-(d+b+\lambda c_1)t}\Big\{1+\frac{e^{(d+b+\lambda c_1)t}-1}{d+b+\lambda c_1}\Big[(b-a)+d\int_{-\infty}^{+\infty} J_\rho(y)e^{-\lambda y}\mathrm{d}y \\
&+ b\int_0^{+\infty}\int_{-\infty}^{+\infty} F(y,\iota)e^{-\lambda(y+c_1\iota)}\mathrm{d}y\mathrm{d}\iota\Big]\Big\} \\
&= Me^{-(d+b+\lambda c_1)t}\Big\{1+\frac{e^{(d+b+\lambda c_1)t}-1}{d+b+\lambda c_1}\Big[b+c_1\lambda+d-\Delta(\lambda,c_1)\Big]\Big\} = M.
\end{aligned}
$$

Note that for $(x,t)\in\mathbb{R}\times(-\infty,0]$,

$$
u^{(1)}(x,t)e^{-\lambda(x+c_1|t|)} = \phi(x,t)e^{-\lambda(x+c_1|t|)} \le M.
$$

An induction argument yields

$$
u^{(n)}(x,t)e^{-\lambda(x+c_1|t|)} \le M \text{ for } (x,t)\in\mathbb{R}^2.
$$

Letting $n\to+\infty$, we have $u(x,t)e^{-\lambda(x+c_1|t|)}\le M$. Hence, when $x\le -ct$, we have

$$
0\le u(x,t)\le Me^{\lambda(x+c_1|t|)}\le Me^{-\lambda(c-c_1)t}\to 0 \quad \text{as } t\to+\infty.
$$

Thus, we have $\limsup_{t\to+\infty,x\le -ct} u(x,t;\phi)=0$, if $\limsup_{x\to-\infty,s\le 0}\phi(x,s)e^{-\lambda x}<+\infty$ for $\lambda>\lambda_1(c)$.

(ii) By a similar discussion as (i), we can prove $\limsup_{t\to+\infty,x\ge ct} u(x,t;\phi)=0$, if $\limsup_{x\to+\infty,s\le 0}\phi(x,s)e^{\lambda x}<+\infty$ for $\lambda>\lambda_1(c)$. We omit the details here and the proof is then complete. □

From the statements (i) and (ii) of Lemma 3.4, we obtain that (3.1) holds. In order to prove (3.2), we shall take the truncating approach to develop a comparison principle and to construct a suitable sub-solution of (2.3). This method is first used by Aronson and Weinberger [1, 2] and Dikemann [8] for partial differential equations. Recently, Weng et al. [34], Xu and Weng [37] apply this method to delay lattice equations.

For any $T>0$ and $\varphi\in C(\mathbb{R}^2,[0,K])$, define

$$
E[\varphi](x,t,T)=\int_0^T e^{-(d+b)s}G[\varphi](x,t-s)\mathrm{d}s \text{ for } (x,t)\in\mathbb{R}\times[T,+\infty), \tag{3.4}
$$

where G is defined by (2.2).

Lemma 3.5 (Comparison Principle). *Let $\varphi\in C(\mathbb{R}^2,[0,K])$ be such that for any $\bar t>T$, $\operatorname{supp}\varphi(x,t)=\{x\in\mathbb{R}:\varphi(x,t)\ne 0$ for all $t\in[T,\bar t]\}$ is bounded and*

$$
E[\varphi](x,t,T)\ge\varphi(x,t) \quad \text{for all } (x,t)\in\mathbb{R}\times(T,+\infty). \tag{3.5}
$$

If there exists $t_0 > 0$ such that the solution $u(x,t)$ of (2.3) satisfies $u(x,t_0) > 0$ and $u(x,t_0+t) \geq \varphi(x,t)$ for all $(x,t) \in \mathbb{R} \times (-\infty, T]$, then

$$u(x,t_0+t) \geq \varphi(x,t) \quad \text{for } (x,t) \in \mathbb{R}^2. \tag{3.6}$$

Proof. Define $\hat{t} = \sup\{t \geq T : u(x,t_0+t) \geq \varphi(x,t) \text{ for all } x \in \mathbb{R}\}$. We shall show that $\hat{t} = +\infty$. Otherwise, if $\hat{t} < +\infty$, then there exists a sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}^+}$ such that (a) $x_n \in \operatorname{supp}\varphi(\cdot, t_n)$; (b) $t_n \to \hat{t}$ as $n \to +\infty$; (c) $0 \leq u(x_n, t_0 + t_n) < \varphi(x_n, t_n)$. By the boundedness of $\operatorname{supp}\varphi(x,t)$, we can obtain that $\{x_n\}_{n\in\mathbb{N}^+}$ contains a converge subsequence $\{x_{n_k}\}_{k\in\mathbb{N}^+}$ such that $\{x_{n_k}\} \to \hat{x}$ as $n \to +\infty$. By (a) and (c), $\hat{x} \in \operatorname{supp}\varphi(\cdot,\hat{t})$ and

$$u(\hat{x}, t_0+\hat{t}) \leq \varphi(\hat{x},\hat{t}). \tag{3.7}$$

On the other hand, since $\hat{t} \geq T$, $t_0 > 0$, by (2.3), (3.5) and the definition of $\hat{t}$, one has

$$\begin{aligned}
u(\hat{x}, t_0+\hat{t}) &= u(\hat{x},t_0)e^{-(d+b)(t_0+\hat{t})} + \int_{t_0}^{t_0+\hat{t}} e^{(d+b)(s-t_0-\hat{t})} G[u](\hat{x},s)\mathrm{d}s \\
&> \int_0^{\hat{t}} e^{(d+b)(s-\hat{t})} G[u](\hat{x}, s+t_0)\mathrm{d}s \\
&= \int_0^{\hat{t}} e^{-(d+b)s} G[u](\hat{x}, \hat{t}+t_0-s)\mathrm{d}s \\
&\geq \int_0^{T} e^{-(d+b)s} G[u](\hat{x}, \hat{t}+t_0-s)\mathrm{d}s \\
&\geq \int_0^{T} e^{-(d+b)s} G[\varphi](\hat{x}, \hat{t}-s)\mathrm{d}s \\
&= E[\varphi](\hat{x},\hat{t},T) \geq \varphi(\hat{x},\hat{t}),
\end{aligned}$$

which contradicts (3.7). Hence, $\hat{t} = +\infty$ and we complete the proof. $\square$

Define the multivariate function

$$\begin{aligned}
K_c(h,T,l,X,\lambda) &= \int_0^T e^{-(d+b+\lambda c)s}\Big[(b-a) + d\int_{-X}^{X} J_\rho(y)e^{-\lambda y}\mathrm{d}y \\
&\quad + h\int_0^l \int_{-X}^{X} F(y,\iota)e^{-\lambda(y+c\iota)}\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \\
&= \frac{1-e^{-(d+b+\lambda c)T}}{d+b+\lambda c}\Big[(b-a) + d\int_{-X}^{X} J_\rho(y)e^{-\lambda y}\mathrm{d}y \\
&\quad + h\int_0^l \int_{-X}^{X} F(y,\iota)e^{-\lambda(y+c\iota)}\mathrm{d}y\mathrm{d}\iota\Big].
\end{aligned}$$

Lemma 3.6. *For any $c \in (0, c^*)$, there exists $T > 0$, $h \in (0,b)$, $l > 0$ and $X > 0$ such that*

$$K_c(h,T,l,X,\lambda) > 1 \quad \text{for } \lambda \in \mathbb{R}. \tag{3.8}$$

Proof. Obviously, when $\lambda \geq 0$, for any $T > 0$, $h \in (0,b)$, $l > 0$ and $X > 0$, we obtain

$$K_c(h,T,l,X,\lambda) \geq \frac{d\int_{-X}^{X} J_\rho(y)e^{-\lambda y}\mathrm{d}y}{d+b+\lambda c}[1-e^{-(d+b+\lambda c)T}]$$

$$
\begin{aligned}
&= \frac{d\int_0^X J_\rho(y)(e^{-\lambda y}+e^{\lambda y})\mathrm{d}y}{d+b+\lambda c}[1-e^{-(d+b+\lambda c)T}] \\
&\geq \frac{d\int_0^X J_\rho(y)e^{\lambda y}\mathrm{d}y}{d+b+\lambda c}[1-e^{-(d+b+\lambda c)T}].
\end{aligned}
$$

Since

$$
\lim_{\lambda\to+\infty} \frac{d\int_0^X J_\rho(y)e^{\lambda y}\mathrm{d}y}{d+b+\lambda c}(1-e^{-(d+b+\lambda c)T}) = \lim_{\lambda\to+\infty} \frac{d\int_0^X J_\rho(y)ye^{\lambda y}\mathrm{d}y}{c} = +\infty,
$$

we obtain

$$
\lim_{\lambda\to+\infty} K_c(h,T,l,X,\lambda) = +\infty.
$$

Then by the continuity of $K_c(h,T,l,X,\lambda)$, we can choose $\lambda_0>0$, $T_0>0$, $l_0>0$, $X_0>0$ and $b_0\in(0,b)$ such that

$$
K_c(h,T,l,X,\lambda) > 1 \quad \text{for } \lambda\geq\lambda_0,\ T\geq T_0,\ l\geq l_0, X\geq X_0,\ h\in(b_0,b).
$$

If (3.8) is not true, then there exist sequences $\{h_n\}_{n\in\mathbb{N}^+}$, $\{T_n\}_{n\in\mathbb{N}^+}$, $\{l_n\}_{n\in\mathbb{N}^+}$, $\{X_n\}_{n\in\mathbb{N}^+}$, $\{\lambda_n\}_{n\in\mathbb{N}^+}$ satisfying $h_n\to b$, $T_n\to+\infty$, $l_n\to+\infty$, $X_n\to+\infty$ as $n\to+\infty$ and $\lambda_n\in[0,\lambda_0)$ such that

$$
K_c(h_n,T_n,l_n,X_n,\lambda_n) \leq 1. \tag{3.9}
$$

Since $\lambda_n\in[0,\lambda_0)$ is bounded, we can choose a subsequence $\{\lambda_{n_k}\}$ such that $\lim_{k\to+\infty}\lambda_{n_k}=\bar\lambda\in[0,\lambda_0]$. According to (ii) of Lemma 3.2, $\Delta(\bar\lambda,c)<0$ for $c\in(0,c^*)$. Then by a direct calculation, we have

$$
\begin{aligned}
&K_c(h_n,T_n,l_n,X_n,\lambda_n) \\
&= \frac{1-e^{-(d+b+\lambda_n c)T_n}}{d+b+\lambda_n c}\Big[(b-a)+d\int_{-X_n}^{X_n} J_\rho(y)e^{-\lambda_n y}\mathrm{d}y \\
&\quad + h_n\int_0^{l_n}\int_{-X_n}^{X_n} F(y,\iota)e^{-\lambda_n(y+c\iota)}\mathrm{d}y\mathrm{d}\iota\Big] \\
&\to \frac{1}{d+b+\bar\lambda c}\Big[(b-a)+d\int_{-\infty}^{\infty} J_\rho(y)e^{-\bar\lambda y}\mathrm{d}y + b\int_0^\infty\int_{-\infty}^{\infty} F(y,\iota)e^{-\bar\lambda(y+c\iota)}\mathrm{d}y\mathrm{d}\iota\Big] \\
&= \frac{d+b+\bar\lambda c-\Delta(\bar\lambda,c)}{d+b+\bar\lambda c} > 1 \quad \text{as } n\to+\infty,
\end{aligned}
$$

which contradicts to (3.9). Hence, $K_c(h,T,l,X,\lambda)>1$ for $\lambda\geq0$. On the other hand, for $\lambda<0$, by L'Hospital's rule, we have

$$
\begin{aligned}
\lim_{\lambda\to-\infty} K_c(h,T,l,X,\lambda) &= \lim_{\lambda\to-\infty} \frac{1-e^{-(d+b+\lambda c)T}}{d+b+\lambda c} d\int_0^X J_\rho(y)e^{-\lambda y}\mathrm{d}y \\
&= \lim_{\lambda\to-\infty}\Big[Te^{-(d+b+\lambda c)T} d\int_0^X J_\rho(y)e^{-\lambda y}\mathrm{d}y \\
&\quad + \frac{d\int_0^X J_\rho(y)ye^{-\lambda y}\mathrm{d}y[e^{-(d+b+\lambda c)T}-1]}{c}\Big] = +\infty.
\end{aligned}
$$

By a similar discussion with $\lambda\geq0$, we can obtain $K_c(h,T,l,X,\lambda)>1$ for $\lambda<0$. The proof is complete. □

Define a function with two parameters $w \in \mathbb{R}$ and $\beta > 0$ as

$$f(x, w, \beta) = \begin{cases} e^{-wx} \sin(\beta x), & x \in [0, \frac{\pi}{\beta}], \\ 0, & x \in \mathbb{R} \setminus [0, \frac{\pi}{\beta}]. \end{cases} \tag{3.10}$$

Lemma 3.7. *Assume that $c \in (0, c^*)$. Then there exist $T > 0$, $l > 0$, $X > 0$, $\beta_0 > 0$, $h \in (0, b)$ and a continuous function $\bar{w} = \bar{w}(\beta)$ defined on $[0, \beta_0]$ such that*

$$\begin{aligned} &\int_0^T e^{-(d+b)s}\Big[(b-a)f(x+cs) + d\int_{-X}^{X} J_\rho(y) f(x+y+cs)\mathrm{d}y \\ &+ h\int_0^l \int_{-X}^{X} F(y,\iota) f(x+y+cs+c\iota)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \ge f(x) \quad \text{for } x \in \mathbb{R}, \end{aligned} \tag{3.11}$$

where $f(x) = f(x, \bar{w}(\beta), \beta)$.

Proof. Define the function

$$\begin{aligned} L(\lambda) = \int_0^T e^{-(d+b)s}\Big[&(b-a)e^{-\lambda cs} + d\int_{-X}^{X} J_\rho(y) e^{-\lambda(y+cs)}\mathrm{d}y \\ &+ h\int_0^l \int_{-X}^{X} F(y,\iota) e^{-\lambda(y+cs+c\iota)}\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s. \end{aligned}$$

If λ is a real number, then by Lemma 3.6, we have

$$L(\lambda) = K_c(h, T, l, X, \lambda) > 1.$$

If λ is a complex number $w + i\beta$, then

$$L(w+i\beta) = \mathrm{Re}\{L(w+i\beta)\} + i\,\mathrm{Im}\{L(w+i\beta)\},$$

where

$$\begin{aligned} \mathrm{Re}\{L(w+i\beta)\} = \int_0^T e^{-(d+b)s}\Big[&(b-a)e^{-wcs}\cos(\beta cs) \\ &+ d\int_{-X}^{X} J_\rho(y) e^{-w(y+cs)}\cos[\beta(y+cs)]\mathrm{d}y \\ &+ h\int_0^l \int_{-X}^{X} F(y,\iota) e^{-w(y+cs+c\iota)}\cos[\beta(y+cs+c\iota)]\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s, \\ \mathrm{Im}\{L(w+i\beta)\} = -\int_0^T e^{-(d+b)s}\Big[&(b-a)e^{-wcs}\sin(\beta cs) \\ &+ d\int_{-X}^{X} J_\rho(y) e^{-w(y+cs)}\sin[\beta(y+cs)]\mathrm{d}y \\ &+ h\int_0^l \int_{-X}^{X} F(y,\iota) e^{-w(y+cs+c\iota)}\sin[\beta(y+cs+c\iota)]\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s. \end{aligned}$$

For $\lambda \in \mathbb{R}$, direct computation leads to

$$\begin{aligned} L''(\lambda) = \int_0^T e^{-(d+b)s}\Big[&(b-a)(cs)^2 e^{-\lambda cs} + d\int_{-X}^{X} J_\rho(y)(y+cs)^2 e^{-\lambda(y+cs)}\mathrm{d}y \\ &+ h\int_0^l \int_{-X}^{X} F(y,\iota)(y+cs+c\iota)^2 e^{-\lambda(y+cs+c\iota)}\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s > 0. \end{aligned} \tag{3.12}$$

Combining the fact that $\lim_{|\lambda|\to+\infty} L(\lambda) = +\infty$, it follows that $L(\lambda)$ can achieve its minimum, say at $\lambda = w_0$. Thus

$$L'(w_0) = 0. \tag{3.13}$$

Now we define the function

$$g(w,\beta) = \begin{cases} \mathrm{Im}\{L(w+i\beta)\}/\beta & \text{for } \beta \neq 0,\\ L'(w) & \text{for } \beta = 0. \end{cases}$$

By (3.12) and (3.13), we have $g(w_0,0) = L'(w_0) = 0$ and $\frac{\partial g(w_0,0)}{\partial w} = L''(w_0) > 0$. Hence, the implicit function theorem implies that there exists a β_1 and a continuous function $\bar{w} = \bar{w}(\beta)$ defined on $[0,\beta_1]$ with $\bar{w}(0) = w_0$ such that $g(\bar{w}(\beta),\beta) = 0$ for $\beta \in [0,\beta_1]$. Hence,

$$\mathrm{Im}\{L(\bar{w}(\beta)+i\beta)\} = 0 \quad \text{for } \beta \in [0,\beta_1]. \tag{3.14}$$

Since $L(w_0) > 1$, we can choose $\beta_2 \in (0,\beta_1)$ sufficiently small so that

$$\mathrm{Re}\{L(\bar{w}(\beta)+i\beta)\} > 1 \quad \text{for } \beta \in [0,\beta_2]. \tag{3.15}$$

Let $0 < \beta < \beta_0 := \min\{\beta_2, \frac{\pi}{X+c^*(T+l)}\}$. Then for $|y| < X$, $s \in (0,T)$, $\iota \in (0,l)$, $x \in [0,\pi/\beta]$, we have

$$-\frac{\pi}{\beta} \le -X \le x+y+cs \le x+y+cs+c\iota \le \frac{\pi}{\beta} + X + c^*(T+l) \le \frac{2\pi}{\beta}.$$

Since $\sin(\beta x) \le 0$ for $x \in [-\frac{\pi}{\beta},0] \cup [\frac{\pi}{\beta},\frac{2\pi}{\beta}]$; from (3.14) and (3.15), for $x \in [0,\frac{\pi}{\beta}]$, we have

$$\begin{aligned}
&\int_0^T e^{-(d+b)s}\Big[(b-a)f(x+cs) + d\int_{-X}^X J_\rho(y)f(x+y+cs)\mathrm{d}y\\
&+ h\int_0^l\int_{-X}^X F(y,\iota)f(x+y+cs+c\iota)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\\
&= \int_0^T e^{-(d+b)s}\Big[(b-a)e^{-\bar{w}(\beta)(x+cs)}\sin[\beta(x+cs)]\\
&\quad + d\int_{-X}^X J_\rho(y)e^{-\bar{w}(\beta)(x+y+cs)}\sin[\beta(x+y+cs)]\mathrm{d}y\\
&\quad + h\int_0^l\int_{-X}^X F(y,\iota)e^{-\bar{w}(\beta)(x+y+cs+c\iota)}\sin[\beta(x+y+cs+c\iota)]\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\\
&= e^{-\bar{w}(\beta)x}\Big\{\int_0^T e^{-(d+b)s}\Big[(b-a)e^{-\bar{w}(\beta)cs}\sin[\beta(x+cs)]\\
&\quad + d\int_{-X}^X J_\rho(y)e^{-\bar{w}(\beta)(y+cs)}\sin[\beta(x+y+cs)]\mathrm{d}y\\
&\quad + h\int_0^l\int_{-X}^X F(y,\iota)e^{-\bar{w}(\beta)(y+cs+c\iota)}\sin[\beta(x+y+cs+c\iota)]\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\Big\}\\
&= e^{-\bar{w}(\beta)x}\sin(\beta x)\,\mathrm{Re}\{L(\bar{w}(\beta)+i\beta)\} - e^{-\bar{w}(\beta)x}\cos(\beta x)\,\mathrm{Im}\{L(\bar{w}(\beta)+i\beta)\}\\
&> e^{-\bar{w}(\beta)x}\sin(\beta x) = f(x).
\end{aligned}$$

The proof is complete. $\square$

Consider the family of functions

$$
\begin{aligned}
R(x,w,\beta,\gamma) &:= \max_{\eta\ge -\gamma} f(x+\eta,w,\beta)\\
&= \begin{cases} M_f & \text{for } x\le \gamma+\mu,\\ f(x-\gamma,w,\beta) & \text{for } \gamma+\mu<x<\gamma+\frac{\pi}{\beta},\\ 0 & \text{for } x\ge \gamma+\frac{\pi}{\beta}, \end{cases}
\end{aligned} \tag{3.16}
$$

where $M_f=M_f(w,\beta)=\max\{f(x,w,\beta): x\in[0,\frac{\pi}{\beta}]\}$ and $\mu=\mu(w,\beta)$ is the maximum point of M_f. Now we give a lemma which in fact provides a sub-solution of (2.3).

Lemma 3.8. *Assume that $c\in(0,c^*)$. Then there exist $T>0$, $\beta>0$, $w\in\mathbb{R}$, $A>0$ and $\delta_0>0$ such that for any $t\ge T$ and $\delta\in(0,\delta_0)$,*

$$
E[\delta\varphi](x,t,T)\ge \delta\varphi(x,t), \tag{3.17}
$$

where E is defined by (3.4) *and $\varphi(x,t)=R(|x|,w,\beta,A+ct)$ for $(x,t)\in\mathbb{R}^2$.*

Proof. According to Lemma 3.7, we can choose $T>0$, $l>0$, $X>0$, $\beta_0>0$, $h\in(0,b)$ and a function $\bar w=\bar w(\beta)$ defined on $[0,\beta_0]$ such that (3.11) holds. Note $b(1-u)>h$ for $u\in(0,1-h/b)$. Take $A=2X+c^*l$ and choose $\delta_0\in(0,\min\{1,\frac{b-h}{bM_f}\})$. Suppose that $\delta\in(0,\delta_0)$ and $t\ge T$. Then

$$
\begin{aligned}
&E[\delta\varphi](x,t,T)\\
&=\delta\int_0^T e^{-(d+b)s}\Big[(b-a)\varphi(x,t-s)+d\int_{-\infty}^{+\infty}J_\rho(y)\varphi(x-y,t-s)\mathrm{d}y\\
&\quad+b[1-\delta\varphi(x,t-s)]\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,\iota)\varphi(x-y,t-s-\iota)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s\\
&\ge\delta\int_0^T e^{-(d+b)s}\Big[(b-a)\varphi(x,t-s)+d\int_{-X}^{X}J_\rho(y)\varphi(x-y,t-s)\mathrm{d}y\\
&\quad+h\int_0^l\int_{-X}^{X}F(y,\iota)\varphi(x-y,t-s-\iota)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s.
\end{aligned} \tag{3.18}
$$

We now consider the following four cases.

Case 1. $|x|\le A+\mu+c(t-T-l)-X$, $s\in[0,T]$, $\iota\in[0,l]$, $|y|\le X$. In this case,

$$
|x-y|\le A+\mu+c(t-T-l)\le A+\mu+c(t-s-\iota)\le A+\mu+c(t-s).
$$

It then follows from (3.18), Lemma 3.6 and the definition of $\varphi(x,t)$ that

$$
\begin{aligned}
&E[\delta\varphi](x,t,T)\\
&\ge\delta M_f\Big[(b-a)+d\int_{-X}^{X}J_\rho(y)\mathrm{d}y+h\int_0^l\int_{-X}^{X}F(y,\iota)\mathrm{d}y\mathrm{d}\iota\Big]\frac{1-e^{-(d+b)T}}{d+b}\\
&=\delta M_f K_c(h,T,l,X,0)\\
&>\delta M_f\ge\delta\varphi(x,t).
\end{aligned}
$$

Case 2. $A+\mu+c(t-T-l)-X\le x\le A+\frac{\pi}{\beta}+ct$. With the evenness of $J_\rho(\cdot)$ and $F(\cdot,\iota)$, by Lemma 3.7, we have

$$
E[\delta\varphi](x,t,T)\ge\delta\int_0^T e^{-(d+b)s}\Big[(b-a)\max_{\eta\ge -A-c(t-s)}f(|x|+\eta)
$$

$$
\begin{aligned}
&+ d\int_{-X}^{X} J_\rho(y) \max_{\eta\geq -A-c(t-s)} f(|x-y|+\eta)\mathrm{d}y \\
&+ h\int_0^l\int_{-X}^{X} F(y,\iota) \max_{\eta\geq -A-c(t-s-\iota)} f(|x-y|+\eta)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \\
&= \delta\int_0^T e^{-(d+b)s}\Big[(b-a)\max_{\eta\geq -A-ct} f(x+cs+\eta) \\
&+ d\int_{-X}^{X} J_\rho(y) \max_{\eta\geq -A-ct} f(x-y+cs+\eta)\mathrm{d}y \\
&+ h\int_0^l\int_{-X}^{X} F(y,\iota) \max_{\eta\geq -A-ct} f(x-y+cs+c\iota+\eta)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \\
&= \delta\int_0^T e^{-(d+b)s}\Big[(b-a)\max_{\eta\geq -A-ct} f(x+cs+\eta) \\
&+ d\int_{-X}^{X} J_\rho(y) \max_{\eta\geq -A-ct} f(x+y+cs+\eta)\mathrm{d}y \\
&+ h\int_0^l\int_{-X}^{X} F(y,\iota) \max_{\eta\geq -A-ct} f(x+y+cs+c\iota+\eta)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \\
&\geq \delta \max_{\eta\geq -A-ct} f(x+\eta) \\
&= \delta R(|x|,w,\beta,A+ct) = \delta\varphi(x,t).
\end{aligned}
$$

Case 3. $-(A+\frac{\pi}{\beta}+ct)\leq x\leq -(A+\mu+c(t-T-l)-X)$. In this case,

$$
\begin{aligned}
E[\delta\varphi](x,t,T) &\geq \delta\int_0^T e^{-(d+b)s}\Big[(b-a)\max_{\eta\geq -A-ct} f(|x|+cs+\eta) \\
&+ d\int_{-X}^{X} J_\rho(y) \max_{\eta\geq -A-ct} f(|x-y|+cs+\eta)\mathrm{d}y \\
&+ h\int_0^l\int_{-X}^{X} F(y,\iota) \max_{\eta\geq -A-ct} f(|x-y|+cs+c\iota+\eta)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \\
&= \delta\int_0^T e^{-(d+b)s}\Big[(b-a)\max_{\eta\geq -A-ct} f(|x|+cs+\eta) \\
&+ d\int_{-X}^{X} J_\rho(y) \max_{\eta\geq -A-ct} f(-x+y+cs+\eta)\mathrm{d}y \\
&+ h\int_0^l\int_{-X}^{X} F(y,\iota) \max_{\eta\geq -A-ct} f(-x+y+cs+c\iota+\eta)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \\
&\geq \delta \max_{\eta\geq -A-ct} f(-x+\eta) \\
&= \delta \max_{\eta\geq -A-ct} f(|x|+\eta) = \delta\varphi(x,t).
\end{aligned}
$$

Case 4. $|x|\geq A+\frac{\pi}{\beta}+ct$. By (3.16), we have $\varphi(x,t)=0$. Hence, (3.17) holds naturally.

From the above discussion, we obtain (3.17) and the proof is complete. $\square$

Lemma 3.9. *Define a recursive sequence* $\{U^{(n)}(x,t,l,X)\}_{n\in N}$ *by*

$$
\begin{aligned}
&U^{(n+1)}(x,t,l,X)\\
&=\int_0^t e^{-(d+b)s}\Big[(d+b-a)U^{(n)}(x,t-s,l,X)+b[1-U^{(n)}(x,t-s,l,X)]\\
&\quad\times\int_0^l\int_{-X}^X U^{(n)}(x-y,t-s-\iota,l,X)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s \quad \text{for } x\in\mathbb{R},\ t>0;
\end{aligned}
$$

and

$$U^{(n)}(x,t,l,X)=0 \quad \text{for } x\in\mathbb{R},\ t\le 0,$$

with

$$U^{(0)}(x,t,l,X)\in[0,K) \quad \text{for } (x,t)\in\mathbb{R}^2.$$

Then for any $\epsilon>0$*, there exist* $\bar{t}(\epsilon)>0$*,* $\bar{l}(\epsilon)>0$*,* $\bar{X}(\epsilon)>0$ *and* $\bar{N}(\epsilon)\in\mathbb{N}^+$ *such that*

$$U^{(n)}(x,t,l,X)>K-\epsilon \quad \text{for } t\ge n(\bar{t}(\epsilon)+l),\ l\ge\bar{l}(\epsilon),\ X\ge\bar{X}(\epsilon),\ n\ge\bar{N}(\epsilon).$$

Proof. Since $U^{(0)}(x,t,l,X)\in[0,K)$ and $1-e^{-(d+b)t}\in(0,1)$ for $t>0$, an induction argument implies that $U^{(n)}(x,t,l,X)\in(0,K)$ for all $(x,t)\in\mathbb{R}\times(0,+\infty)$ and $n\in\mathbb{N}^+$. Noting that $b(1-\nu)\nu>a\nu$ for $\nu\in(0,K)$, $(d+b-a)\nu+b(1-\nu)\nu>(d+b)\nu$ for $\nu\in(0,K)$. Taking any $\epsilon\in(0,K)$, we obtain

$$\inf\Big\{\frac{(d+b-a)\nu+b(1-\nu)\nu}{(d+b)\nu}:\nu\in(0,K-\epsilon]\Big\}>1.$$

Furthermore, by choosing $\xi(\epsilon)\in(0,1)$, we can obtain

$$\xi(\epsilon)[(d+b-a)\nu+b(1-\nu)\nu]>(d+b)\nu \quad \text{for } \nu\in(0,K-\epsilon]. \tag{3.19}$$

Define a sequence $\{q_n\}_{n\in N}$ as follows:

$$q_0=U^{(0)}(x,t,l,X),\quad q_{n+1}=\frac{\xi(\epsilon)}{d+b}[d+b-a+b(1-q_n)]q_n. \tag{3.20}$$

Then the following statements hold:

(a) If $0\le q_n\le K-\epsilon$, then $q_{n+1}\ge q_n$;
(b) If $q_n>K-\epsilon$, then $q_{n+1}\ge\frac{\xi(\epsilon)}{d+b}[d+b-a+b(1-K+\epsilon)](K-\epsilon)\ge K-\epsilon$, since $h(\nu)=[d+b-a+b(1-\nu)]\nu$ is increasing in $\nu\in[0,K)$.

Next, we shall show that $q_n>K-\epsilon$ for large n. In fact, if not, then we can obtain that $q_n\le K-\epsilon$ for all $n\in\mathbb{N}$. By (a), $\{q_n\}_{n\in\mathbb{N}}$ is monotone increasing and bounded, hence $\lim_{n\to+\infty}q_n<+\infty$ exists and denoted by q, then we have

$$q=\frac{\xi(\epsilon)}{d+b}[d+b-a+b(1-q)]q,$$

which contradicts to (3.19). Hence there exists $\bar{N}(\epsilon)>0$ such that $q_n>K-\epsilon$ for all $n\ge\bar{N}(\epsilon)$.

By (F1), we can choose $\bar{t}(\epsilon)>0$, $\bar{l}(\epsilon)>0$ and $\bar{X}(\epsilon)>0$ sufficiently large such that

$$\big(1-e^{-(d+b)\bar{t}(\epsilon)}\big)\int_0^{\bar{l}(\epsilon)}\int_{-\bar{X}(\epsilon)}^{\bar{X}(\epsilon)}F(y,\iota)\mathrm{d}y\mathrm{d}\iota\ge\xi(\epsilon).$$

For any $l\ge\bar{l}(\epsilon)$, $X\ge\bar{X}(\epsilon)$, if $U^{(n)}(x,t,l,X)\ge q_n$ for some n and all $t>n(\bar{t}(\epsilon)+l)$, then for all $t>(n+1)(\bar{t}(\epsilon)+l)$, we can obtain

$$U^{(n+1)}(x,t,l,X)$$

$$
\begin{aligned}
&\geq \int_0^{\bar{t}(\epsilon)} e^{-(d+b)s}\mathrm{d}s\Big[(d+b-a)+b[1-q_n]\int_0^{\bar{l}(\epsilon)}\int_{-\bar{X}(\epsilon)}^{\bar{X}(\epsilon)} F(y,\iota)\mathrm{d}y\mathrm{d}\iota\Big]q_n\\
&= \frac{1-e^{-(d+b)\bar{t}(\epsilon)}}{d+b}\Big[(d+b-a)+b(1-q_n)\int_0^{\bar{l}(\epsilon)}\int_{-\bar{X}(\epsilon)}^{\bar{X}(\epsilon)} F(y,\iota)\mathrm{d}y\mathrm{d}\iota\Big]q_n\\
&\geq \frac{\xi(\epsilon)}{d+b}[(d+b-a)+b(1-q_n)]q_n = q_{n+1}.
\end{aligned}
$$

By (3.20), $U^{(0)}(x,t,l,X) = q_0$, then induction leads to

$$
U^{(n)}(x,t,l,X) \geq q_n > K-\epsilon,
$$

for $l \geq \bar{l}(\epsilon)$, $X \geq \bar{X}(\epsilon)$, $n \geq \bar{N}(\epsilon)$ and $t \geq n(\bar{t}(\epsilon)+l)$. The proof is complete. $\square$

Theorem 3.10. *Suppose that $\phi(x,s) \in C(\mathbb{R}\times(-\infty,0],[0,K])$ and $\operatorname{supp}\phi = \{x \in \mathbb{R} : \phi(x,s) \neq 0$ for $s \leq 0\}$ is compact. Then for any $c \in (0,c^*)$, we have*

$$
\liminf_{t\to+\infty,|x|\leq ct} u(x,t;\phi) \geq K. \tag{3.21}
$$

Proof. Let $c_2 \in (0,c^*)$, by Lemma 3.8, there exist $T>0$, $\beta>0$, $w\in\mathbb{R}$, $A>0$ and $\delta_0>0$ such that for any $t\geq T$ and $\delta\in(0,\delta_0)$,

$$
E[\delta\varphi](x,t,T) \geq \delta\varphi(x,t),
$$

where $\varphi(x,t) = R(|x|,w,\beta,A+c_2t)$ for $(x,t)\in\mathbb{R}^2$.

Since $\operatorname{supp}\phi$ is compact, by (2.3), there exists $t_0>0$ such that $u(x,t;\phi)>0$ for $(x,t)\in\mathbb{R}\times[t_0,+\infty)$. In the following, we denote $u(x,t;\phi)$ by $u(x,t)$ for simplicity. Then we choose $\delta_1\in(0,\delta_0)$ sufficiently small such that

$$
\delta_1 q < K \quad \text{and} \quad u(x,t+t_0) \geq \delta_1\varphi(x,t) \quad \text{for } (x,t)\in\operatorname{supp}\varphi(x,T)\times(-\infty,T],
$$

where q is defined in Lemma 3.9 and $\operatorname{supp}\varphi(x,T)$ is bounded from Lemma 3.8. Hence, by Lemma 3.5,

$$
u(x,t_0+t) \geq \delta_1\varphi(x,t) \quad \text{for } (x,t)\in\operatorname{supp}\varphi(x,T)\times\mathbb{R}.
$$

Then combining the definition of $\varphi(x,t)$, we have

$$
u(x,t_0+t) \geq \delta_1 M_f \text{ for } |x| \leq A+c_2t+\mu,\ t\in\mathbb{R}. \tag{3.22}
$$

By (2.3), we have

$$
\begin{aligned}
&u(x,t+t_0) \geq \int_0^{t+t_0} e^{-(d+b)(t+t_0-s)}G[u](x,s)\mathrm{d}s\\
&\geq \int_0^t e^{-(d+b)s}\Big[(b-a)u(x,t+t_0-s)+d\int_{-X}^{X}J_\rho(y)u(x-y,t+t_0-s)\mathrm{d}y\\
&\quad + b[1-u(x,t+t_0-s)]\int_0^l\int_{-X}^{X}F(y,\iota)u(x-y,t+t_0-s-\iota)\mathrm{d}y\mathrm{d}\iota\Big]\mathrm{d}s.
\end{aligned} \tag{3.23}
$$

Put $U^{(0)}(x,t,l,X) = \delta_1 M_f$ and define $U^{(n)}(x,t,l,X)$ as Lemma 3.9. By induction and using (3.22) and (3.23), we obtain

$$
u(x,t+t_0) \geq U^{(n)}(x,t,l,X) \quad \text{for } |x| \leq A+c_2t+\mu-nX,\ t\geq 0. \tag{3.24}
$$

Then from Lemma 3.9 and (3.24), for any $\epsilon > 0$, there exist $\bar{t}(\epsilon) > 0$, $\bar{l}(\epsilon) > 0$, $\bar{X}(\epsilon)$ and $\bar{N}(\epsilon) \in \mathbb{N}^+$ such that $u(x,t) > K - \epsilon$ for $t \geq t_0 + n(\bar{t}(\epsilon) + l)$ and $|x| \leq A + c_2(t - t_0) + \mu - \bar{N}(\epsilon)\bar{X}(\epsilon)$. Since $c_2 > c$, we define

$$\check{t} = \max\big\{t_0 + n(\bar{t}(\epsilon) + l), \frac{1}{c_2 - c}(\bar{N}(\epsilon)\bar{X}(\epsilon) + c_2 t_0 - A - \mu)\big\}.$$

Then by (3.24), we obtain

$$u(x,t) > K - \epsilon \quad \text{for } t \geq \check{t},\ |x| \leq ct.$$

Hence, with the arbitrariness of ϵ, we have

$$\liminf_{t \to +\infty, |x| \leq ct} u(x,t;\phi) \geq K.$$

The proof is complete. $\square$

Remark 3.11. Combining Lemma 3.4, Theorem 3.10 and Definition 3.1, we obtain that c^* is the spreading speed of model (1.1). From Lemma 3.2, c^* is uniquely determined by the system

$$\Delta(\lambda, c) = 0, \quad \frac{\partial \Delta(\lambda, c)}{\partial \lambda} = 0. \tag{3.25}$$

In fact, comparing with Theorem 3.2 of Xu and Xiao [38], we know that the spreading speed c^* coincides with the minimal wave speed of monotone regular traveling waves for model (1.1). By Lemma 3.2, we know that c^* depends on the nonlocal dispersal distance ρ and dispersal rate d. Moreover, using the evenness of J, we can obtain that

$$\begin{aligned}
\frac{\mathrm{d}c^*}{\mathrm{d}\rho} &= \frac{d\lambda_* \int_0^{+\infty} yJ(y)(e^{\rho\lambda_* y} - e^{-\rho\lambda_* y})\mathrm{d}y}{\lambda_* + b\lambda_* \int_0^{+\infty}\int_{-\infty}^{+\infty} sF(y,s)e^{-\lambda_*(y+c^*s)}\mathrm{d}y\mathrm{d}s} > 0 \text{ for } \rho > 0,\\
\frac{\mathrm{d}c^*}{\mathrm{d}d} &= \frac{\int_{-\infty}^{+\infty} J(y)e^{-\lambda\rho y}\mathrm{d}y - 1}{\lambda_* + b\lambda_* \int_0^{+\infty}\int_{-\infty}^{+\infty} sF(y,s)e^{-\lambda_*(y+c^*s)}\mathrm{d}y\mathrm{d}s} > 0 \text{ for } d > 0,
\end{aligned}$$

which indicates that the stronger the diffusive ability of the infectious host is, the greater the speed at which the disease spreads.

4. Global solutions

In this section, we have another two assumptions on the kernel functions J and F:

(J2) There exists a positive constant $M_1 > 0$ such that

$$\int_{-\infty}^{+\infty} |J(y+h) - J(y)|\mathrm{d}y \leq M_1 h \qquad \text{for all } h \geq 0.$$

(F2) There exists a positive constant $M_2 > 0$ such that

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} |F(y+h,s) - F(y,s)|\mathrm{d}y\mathrm{d}s \leq M_2 h \quad \text{for all } h \geq 0.$$

Remark 4.1. The conditions (J2) and (F2) are used to prove Lemma 4.4, which imply the sequence of solutions of Cauchy problem (2.1) are equicontinuous in x. In fact, if $J' \in L^1(\mathbb{R})$, $F'_y(y,s) \in L^1(\mathbb{R} \times [0,+\infty))$, then (J2) and (F2) hold naturally (see [16, 32]). In other words, (J2) and (F2) are relatively weak.

4.1. **Existence of spatially independent solutions.** In this subsection, we consider the corresponding spatially independent equation of (1.1):

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -au(t) + b[1-u(t)]\int_0^{+\infty} f(s)u(t-s)\mathrm{d}s, \tag{4.1}$$

where $f(s) = \int_{-\infty}^{+\infty} F(y,s)\mathrm{d}y$. Define

$$\Lambda(\lambda) = \lambda + a - b\int_0^{+\infty} f(s)e^{-\lambda s}\mathrm{d}s.$$

Obviously, $\Lambda(0) = a-b < 0$, $\Lambda(+\infty) = +\infty$, and $\Lambda'(\lambda) = 1 + b\int_0^{+\infty} sf(s)e^{-\lambda s}\mathrm{d}s > 0$. Combining these facts, we can conclude that $\Lambda(\lambda) = 0$ admits one and only one real root λ^* such that $\Lambda(\lambda) < 0$ for $0 \le \lambda < \lambda^*$ and $\Lambda(\lambda) > 0$ for $\lambda > \lambda^*$.

Theorem 4.2. *Assume that* (F1) *holds. Then* (4.1) *admits a heteroclinic solution* $\Theta(t)$ *satisfying*

$$\Theta(+\infty) = K, \quad \Theta(t) \le Ke^{\lambda^* t}, \quad \lim_{t\to-\infty} \Theta(t)e^{-\lambda^* t} = K, \quad \Theta'(t) > 0 \quad \text{for } t \in \mathbb{R},$$

where λ^* *is the unique positive real root of* $\Lambda(\lambda) = 0$.

Proof. Note that (4.1) is similar to [19, (1.2)] with $f(s) = \Sigma_{j\in\mathbb{Z}} F_j(s)$. Following the same process with the proof of [19, Lemma 1.1], we can prove this theorem easily and so we omit the details here. $\square$

4.2. **Existence and qualitative properties of global solutions.** In this subsection, we shall derive the global solutions by combining the traveling wave solutions and the spatially independent solution of (1.1).

For any $n \in \mathbb{N}$, $c_1, c_2 > c^*$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, and $\chi_1, \chi_2, \chi_3 \in \{0,1\}$ satisfying $\chi_1 + \chi_2 + \chi_3 \ge 2$, denote

$$\phi^n(x,s) = \max\{\chi_1 U_{c_1}(x + c_1 s + \gamma_1), \chi_2 U_{c_2}(-x + c_2 s + \gamma_2), \chi_3\Theta(s+\gamma_3)\} \tag{4.2}$$

for $(x,s) \in \mathbb{R}\times(-\infty,-n]$ and

$$\underline{u}(x,t) = \max\{\chi_1 U_{c_1}(x + c_1 t + \gamma_1), \chi_2 U_{c_2}(-x + c_2 t + \gamma_2), \chi_3\Theta(t+\gamma_3)\} \tag{4.3}$$

for $(x,t) \in \mathbb{R}^2$. For $(x,t) \in \mathbb{R}^2$, define

$$\begin{aligned}
\Pi_1(x,t) &= \chi_1 U_{c_1}(x + c_1 t + \gamma_1) + \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2t+\gamma_2)} + \chi_3 K e^{\lambda^*(t+\gamma_3)},\\
\Pi_2(x,t) &= \chi_1 B_{c_1} e^{\lambda_1(c_1)(x+c_1t+\gamma_1)} + \chi_2 U_{c_2}(-x + c_2 t + \gamma_2) + \chi_3 K e^{\lambda^*(t+\gamma_3)},\\
\Pi_3(x,t) &= \chi_1 B_{c_1} e^{\lambda_1(c_1)(x+c_1t+\gamma_1)} + \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2t+\gamma_2)} + \chi_3\Theta(t+\gamma_3),
\end{aligned}$$

where $B_{c_i} = \inf\{B > 0 : B \ge U_{c_i}(\xi)e^{-\lambda_1(c_i)\xi}$ for any $\xi \in \mathbb{R}\}$ and $c_i > c^*$ with $i = 1,2$. Obviously, by Lemma 3.3 we have $B_{c_i} \ge \lim_{\xi\to-\infty} U_{c_i}(\xi)e^{-\lambda_1(c_i)\xi} = 1$.

Let $u^n(x,t)$ be the unique solution of the Cauchy problem (2.1) with initial value $\phi^n(x,s)$ with $(x,s) \in \mathbb{R}\times(-\infty,-n]$. Note that $u^n(x,t) \le u^{n+1}(x,t)$ for $(x,t) \in \mathbb{R}\times[-n,+\infty)$ by (2.3). Now we give the following estimates of $u^n(x,t)$ for $(x,t) \in \mathbb{R}\times[-n,+\infty)$.

Lemma 4.3. *Suppose that both* (J1) *and* (F1) *hold. Then* $u^n(x,t)$ *satisfies that*

$$\underline{u}(x,t) \le u^n(x,t) \le \min\Big\{K, \Pi_1(x,t), \Pi_2(x,t), \Pi_3(x,t)\Big\}$$

for all $(x,t) \in \mathbb{R}\times[-n,+\infty)$.

Proof. According to Theorem 2.2 and Lemma 2.3, we have $\underline{u}(x,t) \le u^n(x,t) \le K$ for all $(x,t) \in \mathbb{R} \times [-n,+\infty)$. Then we only need to illustrate

$$u^n(x,t) \le \min\{\Pi_1(x,t), \Pi_2(x,t), \Pi_3(x,t)\}, \ x \in \mathbb{R}, \ t \ge -n. \tag{4.4}$$

Here, we only prove $u^n(x,t) \le \Pi_1(x,t)$ for all $x \in \mathbb{R}$ and $t \ge -n$. The other cases can also be treated in a similar way. We first assume that $\chi_1 = 1$. Set

$$W^n(x,t) = u^n(x,t) - U_{c_1}(x + c_1 t + \gamma_1) \quad \text{for } x \in \mathbb{R}, \ t \ge -n.$$

By a direct calculation, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} W^n(x,t) \\
&= d(J_\rho * W^n - W^n)(x,t) + bF \star W^n(x,t) \\
&\quad + bU_{c_1}(x + c_1 t + \gamma_1) \int_0^{+\infty} \int_{-\infty}^{+\infty} F(y,s) U_{c_1}(x - y + c_1(t-s) + \gamma_1) \mathrm{d}y \mathrm{d}s \\
&\quad - bu^n(x,t) \int_0^{+\infty} \int_{-\infty}^{+\infty} F(y,s) u^n(x-y, t-s) \mathrm{d}y \mathrm{d}s, \quad x \in \mathbb{R}, \ t \ge -n.
\end{aligned}$$

Since $u^n(x,t) \ge \underline{u}(x,t) \ge U_{c_1}(x + c_1 t + \gamma_1)$ for any $(x,t) \in \mathbb{R} \times [-n,+\infty)$, we have $W^n(x,t) \ge 0$ for all $(x,t) \in \mathbb{R} \times [-n,+\infty)$. In addition, by the translation invariance, we obtain $u^n(x-y,t-s) \ge U_{c_1}(x-y+c_1(t-s)+\gamma_1)$ for any $x, \ y \in \mathbb{R}$, $t \ge -n$ and $s \ge 0$. Thus, for all $(x,t) \in \mathbb{R} \times [-n,+\infty)$,

$$\frac{\partial}{\partial t} W^n(x,t) \le d(J_\rho * W^n - W^n)(x,t) + bF \star W^n(x,t). \tag{4.5}$$

Let $\vartheta(x,t) = \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2 t+\gamma_2)} + \chi_3 K e^{\lambda^*(t+\gamma_3)}$. Then we can easily verify that $\vartheta(x,t)$ satisfies

$$\begin{gathered}
\frac{\partial}{\partial t} \vartheta(x,t) = d(J_\rho * \vartheta - \vartheta)(x,t) - a\vartheta(x,t) + bF \star \vartheta(x,t), \\
\vartheta(x,s) = \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2 s+\gamma_2)} + \chi_3 K e^{\lambda^*(s+\gamma_3)}, \quad x \in \mathbb{R}, \ t \le -n.
\end{gathered} \tag{4.6}$$

By the definition of $\phi^n(x,s)$, combining Lemma 3.3 and Theorem 4.2, we obtain

$$\begin{aligned}
W^n(x,s) &= \phi^n(x,s) - U_{c_1}(x + c_1 s + \gamma_1) \le \chi_2 U_{c_2}(-x + c_2 s + \gamma_2) + \chi_3 \Theta(s + \gamma_3) \\
&\le \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2 s+\gamma_2)} + \chi_3 K e^{\lambda^*(s+\gamma_3)} = \vartheta(x,s), \quad x \in \mathbb{R}, \ s \le -n.
\end{aligned}$$

Then through Remark 2.4, we have

$$W^n(x,t) \le \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2 t+\gamma_2)} + \chi_3 K e^{\lambda^*(t+\gamma_3)}, \quad x \in \mathbb{R}, \ t \ge -n.$$

Hence, we obtain

$$u^n(x,t) \le U_{c_1}(x + c_1 t + \gamma_1) + \chi_2 B_{c_2} e^{\lambda_1(c_2)(-x+c_2 t+\gamma_2)} + \chi_3 K e^{\lambda^*(t+\gamma_3)} = \Pi_{\chi_1}(x,t),$$

for all $(x,t) \in \mathbb{R} \times [-n,+\infty)$.

For $\chi_1 = 0$, using the fact that $\chi_1 + \chi_2 + \chi_3 \ge 2$, we have $\chi_2 = \chi_3 = 1$. That is $u^n(x,t) \le B_{c_2} e^{\lambda_1(c_2)(-x+c_2 t+\gamma_2)} + K e^{\lambda^*(t+\gamma_3)}$, the conclusion holds obviously and so we complete the proof. $\square$

Next, we need to provide some a priori estimates uniform in n of $u^n(x,t)$, which allow us to pass to the limit as $n \to +\infty$. From Lemma 3.3, we can get that

$|U_c'| \le \frac{2(d+b)K}{c}$. Then by (4.2), we know that $\phi^n(x,s)$ is globally Lipschitz in x and there exists a positive constant C which is independent of n such that

$$|\phi^n(x_1,s)-\phi^n(x_2,s)| \le C|x_1-x_2| \quad \text{for all } x_1,x_2\in\mathbb{R} \text{ and } s\le -n.$$

Lemma 4.4. *Suppose that* (J1)-(J2), (F1)-(F2) *hold. Let $u^n(x,t)$ be the unique solution of the Cauchy problem* (2.1) *with initial value $\phi^n \in C(\mathbb{R}\times(-\infty,-n],[0,K])$. Then there exist positive constants $C_i>0$ $(i=1,2,3,4)$ which are independent of n such that for any $x\in\mathbb{R}$ and $t>-n$,*

$$|u_t^n(x,t)|\le C_1, \quad |u_t^n(x,t+h)-u_t^n(x,t)|\le C_2h, \tag{4.7}$$

$$|u^n(x+h,t)-u^n(x,t)|\le C_3h, \quad |u_t^n(x+h,t)-u_t^n(x,t)|\le C_4h \tag{4.8}$$

for any $h\ge 0$.

Proof. From (2.1), and applying Theorem 2.2 and Lemma 2.1, we obtain $|u_t^n(x,t)|\le 2(d+b)K := C_1$ for any $(x,t)\in\mathbb{R}\times(-n,+\infty)$ easily. For any $h\ge 0$, by (1.1) , we have

$$\begin{aligned}
&|u_t^n(x,t+h)-u_t^n(x,t)| \\
&\le d\int_{-\infty}^{+\infty} J_\rho(x-y)\big|u^n(y,t+h)-u^n(y,t)\big|\mathrm{d}y + (d+a)\big|u^n(x,t+h)-u^n(x,t)\big| \\
&\quad + b\int_0^{+\infty}\int_{-\infty}^{+\infty} F(x-y,s)\big|u^n(y,t+h-s)-u^n(y,t-s)\big|\mathrm{d}y\mathrm{d}s \\
&\quad + b\Big|u^n(x,t+h)\int_0^{+\infty}\int_{-\infty}^{+\infty} F(x-y,s)u^n(y,t+h-s)\mathrm{d}y\mathrm{d}s \\
&\quad - u^n(x,t)\int_0^{+\infty}\int_{-\infty}^{+\infty} F(x-y,s)u^n(y,t-s)\mathrm{d}y\mathrm{d}s\Big| \\
&\le 2(d+b)(2d+a+b)Kh \\
&\quad + b\big|u^n(x,t+h)-u^n(x,t)\big|\int_0^{+\infty}\int_{-\infty}^{+\infty} F(x-y,s)u^n(y,t+h-s)\mathrm{d}y\mathrm{d}s \\
&\quad + bu^n(x,t)\int_0^{+\infty}\int_{-\infty}^{+\infty} F(x-y,s)\big|u^n(y,t+h-s)-u^n(y,t-s)\big|\mathrm{d}y\mathrm{d}s \\
&\le [2(d+b)(2d+a+b)K + 4b(d+b)K^2]h := C_2h,
\end{aligned}$$

for any $x\in\mathbb{R}$ and $t>-n$.

We now consider (4.8). Let $w^n(x,t)=u^n(x+h,t)-u^n(x,t)$. Then by (2.1), we have

$$\begin{aligned}
&w_t^n(x,t) \\
&= d\int_{-\infty}^{+\infty}[J_\rho(y+h)-J_\rho(y)]u^n(x-y,t)\mathrm{d}y \\
&\quad - \Big(d+a+b\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)u^n(x+h-y,t)\mathrm{d}y\mathrm{d}s\Big)w^n(x,t) \\
&\quad + b\int_0^{+\infty}\int_{-\infty}^{+\infty}[F(y+h,s)-F(y,s)]u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s \\
&\quad - bu^n(x,t)\int_0^{+\infty}\int_{-\infty}^{+\infty}[F(y+h,s)-F(y,s)]u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s.
\end{aligned} \tag{4.9}$$

By (J2) and (F2),

$$\int_{-\infty}^{+\infty}|J_\rho(y+h)-J_\rho(y)|u^n(x-y,t)\mathrm{d}y\le \frac{KM_1}{\rho}h,$$
$$\int_0^{+\infty}\int_{-\infty}^{+\infty}[F(y+h,s)-F(y,s)]u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s\le KM_2h.$$

Now, let $\upsilon(t)$ be the solution of the Cauchy problem

$$\begin{gathered}\upsilon'(t)=-(d+b)\upsilon(t)+(dKM_1/\rho+bKM_2+bK^2M_2)h,\quad t>-n,\\ \upsilon(-n)=Ch.\end{gathered}\tag{4.10}$$

Then

$$\begin{aligned}0<\upsilon(t)&=e^{-(d+b)(t+n)}Ch+\frac{(dKM_1/\rho+bKM_2+bK^2M_2)h}{d+b}\Big(1-e^{-(d+b)(t+n)}\Big)\\ &\le\Big(C+\frac{dKM_1/\rho+bKM_2+bK^2M_2}{d+b}\Big)h:=C_3h.\end{aligned}$$

From (4.9), we know $w^n(x,t)$ satisfies

$$\begin{gathered}w^n_t(x,t)\le-(d+b)w^n(x,t)+(dKM_1/\rho+bKM_2+bK^2M_2)h,\ t>-n,\\ w^n(x,-n)=u^n(x+h,-n)-u^n(x,-n)\le Ch.\end{gathered}$$

Then the comparison method of ODE implies that for any $x\in\mathbb{R}$ and $t>-n$,

$$|u^n(x+h,t)-u^n(x,t)|=|w^n(x,t)|\le\upsilon(t)\le C_3h.$$

Moreover, for $x\in\mathbb{R}$ and $t>-n$, we have

$$\begin{aligned}&|u^n_t(x+h,t)-u^n_t(x,t)|\\ &=\Big|d\int_{-\infty}^{+\infty}[J_\rho(y+h)-J_\rho(y)]u^n(x-y,t)\mathrm{d}y-(d+a)[u^n(x+h,t)-u^n(x,t)]\\ &\quad+b\int_0^{+\infty}\int_{-\infty}^{+\infty}[F(y+h,s)-F(y,s)]u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s\\ &\quad-b[u^n(x+h,t)-u^n(x,t)]\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)u^n(x+h-y,t-s)\mathrm{d}y\mathrm{d}s\\ &\quad-bu^n(x,t)\int_0^{+\infty}\int_{-\infty}^{+\infty}[F(y+h,s)-F(y,s)]u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s\Big|\\ &\le d\int_{-\infty}^{+\infty}|J_\rho(y+h)-J_\rho(y)|u^n(x-y,t)\mathrm{d}y+(d+a)|u^n(x+h,t)-u^n(x,t)|\\ &\quad+b\int_0^{+\infty}\int_{-\infty}^{+\infty}|F(y+h,s)-F(y,s)|u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s\\ &\quad+b|u^n(x+h,t)-u^n(x,t)|\int_0^{+\infty}\int_{-\infty}^{+\infty}F(y,s)u^n(x+h-y,t-s)\mathrm{d}y\mathrm{d}s\\ &\quad+bu^n(x,t)\int_0^{+\infty}\int_{-\infty}^{+\infty}|F(y+h,s)-F(y,s)|u^n(x-y,t-s)\mathrm{d}y\mathrm{d}s\\ &\le\Big(\frac{dKM_1}{\rho}+(d+a)C_3+bKM_2+bKC_3+bK^2M_2\Big)h:=C_4h.\end{aligned}$$

The proof is complete. □

Theorem 4.5. *Assume that* (J1)-(J2), (F1)-(F2) *hold. Then for any* $c_1, c_2 > c^*$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, *and* $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ *satisfying* $\chi_1 + \chi_2 + \chi_3 \geq 2$, *there exists an global solution* $u(x,t) := u_{c_1,c_2,\gamma_1,\gamma_2,\gamma_3,\chi_1,\chi_2,\chi_3}(x,t)$ *of* (1.1) *such that*

$$\begin{aligned} &\max\{\chi_1 U_{c_1}(x + c_1 t + \gamma_1),\ \chi_2 U_{c_2}(-x + c_2 t + \gamma_2),\ \chi_3 \Theta(t + \gamma_3)\} \\ &\leq u(x,t) \leq \min\{K,\ \Pi_1(x,t),\ \Pi_2(x,t),\ \Pi_3(x,t)\} \quad \text{for } (x,t) \in \mathbb{R}^2. \end{aligned} \tag{4.11}$$

Furthermore, the global solution has the following properties: (i) $u_t(x,t) > 0$ *for any* $(x,t) \in \mathbb{R}^2$ *and* $u(x,t) \to K$ *as* $t \to +\infty$ *uniformly in* x.

(ii) *If* $\chi_3 = 0$, *then*

$$\lim_{t \to -\infty} \sup_{x \geq 0} |u(x,t) - U_{c_1}(x + c_1 t + \gamma_1)| = 0, \tag{4.12}$$

$$\lim_{t \to -\infty} \sup_{x \leq 0} |u(x,t) - U_{c_2}(-x + c_2 t + \gamma_2)| = 0. \tag{4.13}$$

(iii) *If* $\chi_1 = 1$, *then* $\lim_{x \to +\infty} \sup_{t \geq \breve{t}} u(x,t) = K$ *for any* $\breve{t} \in \mathbb{R}$.

(iv) *If* $\chi_2 = 1$, *then* $\lim_{x \to -\infty} \sup_{t \geq \breve{t}} u(x,t) = K$ *for any* $\breve{t} \in \mathbb{R}$.

(v) *For any* $|x| \leq X_0$ *with any* $X_0 \in \mathbb{R}^+$, $u(x,t) = O(e^{h(c_1,c_2,\lambda^*)t})$ *as* $t \to -\infty$, *where*

$$h(c_1, c_2, \lambda^*) = \begin{cases} \min\{c_1\lambda_1(c_1), c_2\lambda_1(c_2), \lambda^*\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1,1,1); \\ \min\{c_1\lambda_1(c_1), c_2\lambda_1(c_2)\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1,1,0); \\ \min\{c_1\lambda_1(c_1), \lambda^*\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1,0,1); \\ \min\{c_2\lambda_1(c_2), \lambda^*\}, & \text{if } (\chi_1, \chi_2, \chi_3) = (0,1,1). \end{cases}$$

(vi) $u(x,t)$ *is nondecreasing with respect to* γ_i, $i = 1, 2, 3$ *for each pair of* $(x,t) \in \mathbb{R}^2$. *Moreover,* $u(x,t)$ *converges to* K *as* $\gamma_1 \to +\infty$ (*or as* $\gamma_2 \to +\infty$, *or as* $\gamma_3 \to +\infty$) *uniformly for any* $(x,t) \in \mathbb{R} \times [\breve{t}, +\infty)$ *for any* $\breve{t} \in \mathbb{R}$.

Proof. By Lemma 4.3, we have

$$\underline{u}(x,t) \leq u^n(x,t) \leq \min\{K, \Pi_1(x,t), \Pi_2(x,t), \Pi_3(x,t)\},$$

for any $x \in \mathbb{R}$ and $t \geq -n$. By the priori estimates in Lemma 4.4, the Arzela-Ascoli Theorem implies that there exists a subsequence $\{u^{n_k}(x,t)\}$ of $\{u^n(x,t)\}$ such that $u^{n_k}(x,t)$ converges to a function $u(x,t)$ uniformly in $(x,t) \in \mathbb{R} \times [-n_k, +\infty)$ as $k \to +\infty$. Following the fact that $u^n(x,t) \leq u^{n+1}(x,t)$ for $(x,t) \in \mathbb{R} \times [-n, +\infty)$, we have $\lim_{n \to +\infty} u^n(x,t) = u(x,t)$ for any $(x,t) \in \mathbb{R}^2$. Clearly, $u(x,t)$ is an global solution of (1.1) satisfying (4.11).

Next we shall illustrate the properties (i)-(vi) of $u(x,t)$. We firstly prove (i). For any $\iota > 0$,$x \in \mathbb{R}$ and $t \leq -n$, since both $U_c(\cdot)$ and $\Theta(\cdot)$ are strictly increasing, we have

$$\begin{aligned} &\phi^n(x,s) \\ &\leq \max\{\chi_1 U_{c_1}(x + c_1(s+\iota) + \gamma_1), \chi_2 U_{c_2}(-x + c_2(s+\iota) + \gamma_2), \chi_3 \Theta(s + \iota + \gamma_3)\} \\ &= \phi^n(x, s+\iota). \end{aligned}$$

Then by Lemma 2.3, we obtain

$$u^n(x,t;\phi^n(x,\cdot)) \leq u^n(x,t;\phi^n(x,\cdot+\iota)),$$

for $x \in \mathbb{R}$ and $t \geq -n$. On the other hand, for any $\iota > 0$, $x \in \mathbb{R}$ and $s \leq -n$, one has

$$\phi^n(x,s+\iota) \leq u^n(x,s+\iota;\phi^n(x,\cdot)).$$

Hence, using Lemma 2.3 again, we obtain

$$u^n(x,t;\phi^n(x,\cdot)) \le u^n(x,t;u^n(x,\cdot+\iota;\phi^n(x,\cdot))) = u^n(x,t+\iota;\phi^n(x,\cdot)),$$

which indicates that $u^n(x,t;\phi^n(x,\cdot))$ is increasing in $t\in[-n,+\infty)$ for each $x\in\mathbb{R}$ by the arbitrariness of $\iota>0$. Since $u(x,t)=\lim_{n\to+\infty}u^n(x,t)$ for any $(x,t)\in\mathbb{R}^2$, we obtain that $u_t(x,t)\ge 0$ for any $(x,t)\in\mathbb{R}^2$. In addition, for any $(x,t)\in\mathbb{R}^2$, we have

$$\begin{aligned} u_{tt}(x,t) &= d(J_\rho * u_t - u_t)(x,t) - au_t(x,t) - bu_t(x,t)F\star u(x,t)\\ &\quad + b[1-u(x,t)]F\star u_t(x,t)\\ &\ge -(d+a+bF\star u(x,t))u_t(x,t)\\ &\ge -(d+a+bK)u_t(x,t) = -(d+b)u_t(x,t). \end{aligned}$$

By using ODE theory, we obtain

$$u_t(x,t) \ge u_t(x,r)e^{-(d+b)(t-r)} \quad \text{for } t>r \text{ and } x\in\mathbb{R}, \tag{4.14}$$

where $r\in\mathbb{R}$ is fixed. Suppose that there exists a point $(x_0,t_0)\in\mathbb{R}^2$ such that $u_t(x_0,t_0)=0$, then from (4.14), we have $u_t(x_0,t)=0$ for $t\le t_0$, which indicts that $\lim_{t\to-\infty}u(x_0,t)=u(x_0,t_0)>0$. However, we know $\lim_{t\to-\infty}u(x_0,t)=0$ by (4.11), which leads to a contradiction. Hence, $u_t(x,t)>0$ for any $(x,t)\in\mathbb{R}^2$. The second part of (i) can be derived by (4.11). Thus, we complete the proof of (i).

Now we verify (ii). When $\chi_3=0$, then $\chi_1=\chi_2=1$. For $x\ge 0$ and $t\in\mathbb{R}$, by (4.11), we have

$$0\le u(x,t)-U_{c_1}(x+c_1t+\gamma_1) \le B_{c_2}e^{\lambda_1(c_1)(-x+c_2t+\gamma_2)} \le B_{c_2}e^{\lambda_1(c_1)(c_2t+\gamma_2)} \to 0,$$

as $t\to-\infty$, which leads to (4.12). We can also get (4.13) by a similar proof.

For (iii), when $\chi_1=1$, by (4.11), we have

$$U_{c_1}(x+c_1t+\gamma_1)\le u(x,t)\le K.$$

From Lemma 3.3, we have $\lim_{x\to+\infty}U_{c_1}(x+c_1t+\gamma_1)=K$ for $t\ge\check{t}$ with any $\check{t}\in\mathbb{R}$. Then $\lim_{x\to+\infty}\sup_{t\ge\check{t}}u(x,t)=K$ for any $\check{t}\in\mathbb{R}$ and we complete the proof of (iii). The argument for (iv) is similar as that for (iii).

For (v), we only illustrate the case of $(\chi_1,\chi_2,\chi_3)=(1,0,1)$ and the other cases can be illustrated similarly. When $(\chi_1,\chi_2,\chi_3)=(1,0,1)$, without loss of generality, we assume that $c_1\lambda_1(c_1)\le\lambda^*$. By (4.11), we have

$$\begin{aligned} &\max\{U_{c_1}(x+c_1t+\gamma_1)e^{-c_1\lambda_1(c_1)t},\ \Theta(t+\gamma_3)e^{-c_1\lambda_1(c_1)t}\}\\ &\le u(x,t)e^{-c_1\lambda_1(c_1)t}\\ &\le \min\{Ke^{-c_1\lambda_1(c_1)t},\ \Pi_1'(x,t),\ \Pi_2'(x,t),\ \Pi_3'(x,t)\} \end{aligned} \tag{4.15}$$

for $(x,t)\in\mathbb{R}^2$, where

$$\begin{aligned} \Pi_1'(x,t) &= [U_{c_1}(x+c_1t+\gamma_1)+Ke^{\lambda^*(t+\gamma_3)}]e^{-c_1\lambda_1(c_1)t},\\ \Pi_2'(x,t) &= [B_{c_1}e^{\lambda_1(c_1)(x+c_1t+\gamma_1)}+Ke^{\lambda^*(t+\gamma_3)}]e^{-c_1\lambda_1(c_1)t},\\ \Pi_3'(x,t) &= [B_{c_1}e^{\lambda_1(c_1)(x+c_1t+\gamma_1)}+\Theta(t+\gamma_3)]e^{-c_1\lambda_1(c_1)t}. \end{aligned}$$

For each $x\in\mathbb{R}$, and as $t\to-\infty$, by Lemma 3.3 and Theorem 4.2, one has

$$U_{c_1}(x+c_1t+\gamma_1)e^{-c_1\lambda_1(c_1)t}\to e^{\lambda_1(c_1)(x+\gamma_1)},$$

$$\Theta(t+\gamma_3)e^{-c_1\lambda_1(c_1)t} = \Theta(t+\gamma_3)e^{-\lambda^*(t+\gamma_3)}e^{(\lambda^*-c_1\lambda_1(c_1))t}e^{\lambda^*\gamma_3}\to 0,$$

$$\Pi_1'(x,t) \to e^{\lambda_1(c_1)(x+\gamma_1)},$$
$$\Pi_2'(x,t) \to B_{c_1} e^{\lambda_1(c_1)(x+\gamma_1)} \geq e^{\lambda_1(c_1)(x+\gamma_1)},$$
$$\Pi_3'(x,t) \to B_{c_1} e^{\lambda_1(c_1)(x+\gamma_1)} \geq e^{\lambda_1(c_1)(x+\gamma_1)}.$$

Furthermore, by (4.15) we obtain

$$u(x,t)e^{-c_1\lambda_1(c_1)t} \to e^{\lambda_1(c_1)(x+\gamma_1)} \quad \text{as } t \to -\infty,$$

which indicates that $u(x,t) = O(e^{h(c_1,c_2,\lambda^*)t})$ as $t \to -\infty$ for any $|x| \leq X_0$ with any $X_0 \in \mathbb{R}^+$. This completes the proof of (v).

Finally, we shall prove (vi). By Lemma 3.3 and Theorem 4.2, we know that both $U_c(\cdot)$ and $\Theta(\cdot)$ are strictly increasing and $U_{c_1}(+\infty) = U_{c_2}(+\infty) = \Theta(+\infty) = K$. Then combining the proof of (4.11) we can derive (vi). The proof is complete. $\square$

Remark 4.6. The global solutions of (1.1) derived in Theorem 4.5 reveal some new transmission forms of the disease. For example, from (i)-(iv) of Theorem 4.5, we can obtain that the global solution behaves like a traveling wave of (1.1) which travels from the right and another traveling wave of (1.1) which travels from the left. Specifically, this global solution tends to K as $t \to +\infty$, and when $t \to -\infty$, this global solution tends to $U_{c_1}(x+c_1t+\gamma_1)$ on $x \geq 0$ while it tends to $U_{c_2}(-x+c_2t+\gamma_2)$ on $x \leq 0$.

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