EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SUBLINEAR ORDINARY DIFFERENTIAL EQUATIONS AT RESONANCE

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Abstract. Using a $\mathbb{Z}_2$ type index theorem, we show the existence and multiplicity of solutions for the sublinear ordinary differential equation

$$L u(t) = \mu u(t) + W_u(t, u(t)), \quad 0 \leq t \leq L$$

with suitable periodic or boundary conditions. Here $L$ is a linear positive selfadjoint operator, $\mu$ is a parameter between two eigenvalues of this operator, and $W_u$ is the gradient of a potential function.

1. Introduction

In the study of physical, chemical and biological systems, many ordinary differential equation models can be set in the form

$$L u(t) = \mu u(t) + W_u(t, u(t)), \quad 0 \leq t \leq L, \quad (1.1)$$

(cf. [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14] and their references) where $L$ is a linear positive selfadjoint operator on $L^2([0, L], \mathbb{R}^n)$, $\mu$ is a real parameter, the potential $W(t, u) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a $C^1$-function, and $W_u(t, u) = \partial W/\partial u$ denotes the gradient of $W(t, u)$ with respect to the variable $u$. We say that (1.1) is sublinear if $\lim_{|u| \to \infty} W(t, u)/|u|^2 = 0$.

Throughout this article, $\| \cdot \|_{L^q}$ denotes the norm of the usual space $L^q([0, L], \mathbb{R}^n)$ with $1 \leq q \leq \infty$, and we always assume that, for an appropriate Hilbert space $(X, \| \cdot \|) \subset L^2$ with the corresponding inner product $\langle \cdot, \cdot \rangle$, solutions of (1.1) are exactly the critical points of the corresponding functional

$$\Phi(u) = I(u) - J(u), \quad u \in X \quad (1.2)$$

where

$$I(u) = \frac{1}{2}(\|u\|^2 - \mu \|u\|_{L^2}^2), \quad J(u) = \int_0^L W(t, u(t))dt, \quad (1.3)$$

the problem

$$L u(t) = \lambda u(t), \quad (1.4)$$
has eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$, the corresponding eigenspaces $N_j = \{v_j\}$ ($j \geq 1$) have finite dimensions. For simplicity, we first consider the case of $\dim N_j = 1$ for all $j \geq 1$, and more general case shall be discussed later. Further, we assume that there exists some $k \in N$ such that $\mu \in [\lambda_k, \lambda_{k+1})$. One says (1.1) is at resonance if $\mu = \lambda_k$, and (1.1) is sublinear.

Now we can state our main result as follows.

**Theorem 1.1.** Suppose that $(X, \| \cdot \|) \subset L^2$ is a Hilbert space, continuously embedded into $L^q$ for all $q \in [1, \infty]$, and $\{v_j(t)\}$ is an orthogonal basis in $X$ and $L^2$ such that

$$
\|v_j(t)\|^2 = 1 = \lambda_j \|v_j(t)\|_{L^2}, \quad \forall j \geq 1.
$$

Furthermore, assume that the functional $J(u) \in C^1(X, R)$ satisfies $J(0) = 0$, $J'(u)$ is a compact operator, and

1. $J(u) = J(-u)$ for all $u \in X$,
2. there exists $K > 0$ such that $|J'(u)w| \leq K\|w\|_{L^1}$ for all $u, w \in X$,
3. there exist $p \in N, M > 0, \rho > 0$ such that $M > \lambda_{k+p} - \lambda_k$ and
4. $J(u) \geq \frac{1}{2}M\|u\|_{L^2}^2$ for $\|u\|_{L^\infty} \leq \rho$.

Then, there exist at least $p$ distinct pairs $(u, -u)$ of critical points of $\Phi(u)$. If $\mu \in (\lambda_k, \lambda_{k+1})$, then (J4) can be omitted.

The above theorem will be proved using the following $Z_2$ type index theorem.

**Theorem 1.2.** Let $Y$ be a Banach space, and $f \in C^1(Y, R)$ be even satisfying the Palais-Smale condition. Suppose that: (i) there exist a subspace $V$ of $Y$ with $\dim V = r$ and $\delta > 0$ such that $\sup_{u \in V, \|u\| = \delta} J(u) = f(0)$; (ii) there exists a closed subspace $W$ of $Y$ with $\text{Codim} W = s < r$ such that $\inf_{w \in W} f(w) > -\infty$. Then $f$ possesses at least $r - s$ distinct pairs $(u, -u)$ of critical points.

For the convenience of the reader, let us recall that the functional $f$ is said to satisfy the Palais-Smale condition: if any sequence $\{u_j\}$ in $Y$ be such that $f(u_j)$ is bounded and $f'(u_j) \to 0$, possesses a convergent subsequence.

This article is organized as follows. In Section 2, we prove some lemmas for the functional $\Phi(u)$ defined by (1.2). In section 3, the proof of Theorem 1.1 and its some extensions shall be given. Section 4 is devoted to apply Theorem 1.1 to sublinear Hamiltonian systems as well as Extended Fisher-Kolmogorov type equations, and the existence and multiplicity results of their solutions shall be obtained.

2. Preliminaries

In this section, we shall study the properties of the functionals $\Phi(u), I(u), J(u)$ defined in (1.2) and (1.3).

With the hypotheses of Theorem 1.1, for all $u \in X$, we can write $u = \sum_{j=1}^{\infty} \alpha_j v_j$, thus $\|u\|^2 = \sum_{j=1}^{\infty} \alpha_j^2$, and

$$
I(u) = \frac{1}{2} \sum_{j=1}^{\infty} \alpha_j^2 [1 - \mu \int_0^L |v_j|^2 dt] = \frac{1}{2} \sum_{j=1}^{\infty} (1 - \frac{\mu}{\lambda_j}) \alpha_j^2.
$$

(2.1)
Case (i). If $\mu = \lambda_k$, then we set

$$u^+ = \sum_{j=k+1}^{\infty} \alpha_j v_j, \quad u^0 = \alpha_k v_k, \quad u^- = \sum_{j=1}^{k-1} \alpha_j v_j,$$

(2.2)

$$X^+ = \text{span}\{v_j : j \geq k + 1\}, \quad X^- = \text{span}\{v_j : 1 \leq j \leq k - 1\},$$

$$X^0 = N_k = \text{span}\{v_k\}.$$  (2.3)

Thus, we have $u = u^+ + u^0 + u^-$, $X^- = X^0 \oplus X^-.$

Case (ii). If $\lambda_k < \mu < \lambda_{k+1}$, then we let

$$u^+ = \sum_{j=k+1}^{\infty} \alpha_j v_j, \quad u^- = \sum_{j=1}^{k} \alpha_j v_j,$$

(2.4)

$$X^+ = \text{span}\{v_j : j \geq k + 1\}, \quad X^- = \text{span}\{v_j : 1 \leq j \leq k\},$$

(2.5)

so we have $u = u^+ + u^-$, $X = X^+ \oplus X^-.$

Lemma 2.1. Under the assumptions of Theorem 1.1, there exists a norm $\|\cdot\|$ of $X$, equivalent with $\|\cdot\|$, such that

$$I(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2).$$

Proof. Without loss of generality, we only consider the case $\mu = \lambda_k$ in the following. Thus

$$\left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|u^+\|^2 = \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right) \sum_{j=k+1}^{\infty} \alpha_j^2$$

$$\leq \sum_{j=k+1}^{\infty} \left(1 - \frac{\lambda_k}{\lambda_j}\right) \alpha_j^2$$

$$\leq \sum_{j=k+1}^{\infty} \alpha_j^2 = \|u^+\|^2,$$  (2.6)

$$\left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right)\|u^-\|^2 = \left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right) \sum_{j=1}^{k-1} \alpha_j^2$$

$$\leq \sum_{j=1}^{k-1} \left(\frac{\lambda_k}{\lambda_j} - 1\right) \alpha_j^2$$

$$\leq \frac{\lambda_k}{\lambda_1} \sum_{j=1}^{k-1} \alpha_j^2 = \frac{\lambda_k}{\lambda_1} \|u^-\|^2.$$  (2.7)

Let

$$\|u\|_*^2 = \sum_{j=1}^{k-1} \left(\frac{\lambda_k}{\lambda_j} - 1\right) \alpha_j^2 + \sum_{j=k+1}^{\infty} \left(1 - \frac{\lambda_k}{\lambda_j}\right) \alpha_j^2 + \alpha_k^2.$$  (2.8)

Clearly, $\|\cdot\|$ is a norm on $X$, and is equivalent with the norm $\|\cdot\|$. The corresponding inner product is

$$\langle u, w \rangle_* = \sum_{j=1}^{k-1} \left(\frac{\lambda_k}{\lambda_j} - 1\right) \alpha_j \beta_j + \sum_{j=k+1}^{\infty} \left(1 - \frac{\lambda_k}{\lambda_j}\right) \alpha_j \beta_j + \alpha_k \beta_k,$$  (2.9)
where \( u = \sum_{j=1}^{\infty} \alpha_j v_j, \) \( w = \sum_{j=1}^{\infty} \beta_j v_j \in X. \) Consequently, according to (2.1) and (2.8), one obtains

\[
I(u) = \frac{1}{2}(\|u^+\|^2_\star - \|u^-\|^2_\star).
\]

Then

\[
\Phi'(u)w = \langle u^+, w \rangle_\star - \langle u^-, w \rangle_\star - \int_0^L W_u(t, u) w \, dt, \quad \forall u, w \in X.
\]

Finally, we point out that, in the nonresonant case of \( \lambda_k < \mu < \lambda_{k+1}, \) (2.8) and (2.9) should be replaced by

\[
\|u\|^2_\star = \sum_{j=1}^{K} \left( \frac{\mu}{\lambda_j} - 1 \right) \alpha_j^2 + \sum_{j=k+1}^{\infty} \left( 1 - \frac{\mu}{\lambda_j} \right) \alpha_j^2;
\]

\[
\langle u, w \rangle_\star = \sum_{j=1}^{K} \left( \frac{\mu}{\lambda_j} - 1 \right) \alpha_j \beta_j + \sum_{j=k+1}^{\infty} \left( 1 - \frac{\mu}{\lambda_j} \right) \alpha_j \beta_j,
\]

respectively. The proof is complete.

**Lemma 2.2.** Under the assumptions of Theorem 1.1, the functional \( \Phi(u) \) satisfies the Palais-Smale condition on \( X. \)

**Proof.** We shall use the idea given by Rabinowitz [11, Theorem 4.12] and Costa [2, Proposition 3.2] for a PDE existence problem. Let \( \{u_j\} \subset X \) be such that \( \Phi(u_j) \) is bounded, and \( \Phi'(u_j) \to 0. \) We shall prove \( \{u_j\} \) has a convergent subsequence.

Setting \( u_j = u_j^+ + u_j^0 + u_j^- \) with \( u_j^+ \in X^+, \) \( u_j^0 \in X^0, \) \( u_j^- \in X^- \) for all \( j \geq 1. \) For \( j \) sufficiently large, we have

\[
\|u_j^\pm\|_\star \geq \Phi'(u_j)u_j^\pm = \langle u_j^+, u_j^\pm \rangle_\star - \langle u_j^-, u_j^\pm \rangle_\star - J'(u_j)u_j^\pm.
\]

From (J2), it follows that

\[
|J'(u_j)u_j^\pm| \leq K\|u_j^\pm\|_{L^1} \leq K_1\|u_j^\pm\|_\star,
\]

with \( K_1 > 0 \) coming from the continuous embedding \( L^1 \to (X, \|\cdot\|) \to (X, \|\cdot\|_\star). \) Combining (2.14) with \( + \) in the exponents, and (2.15) with \( + \) in the exponents, we obtain

\[
\|u_j^\pm\|_\star \geq \|u_j^\pm\|^2_\star - K_1\|u_j^\pm\|_\star,
\]

thus, \( \{u_j^\pm\} \) is bounded on \( X. \) Similarly, we also deduce that \( \{u_j^-\} \) is bounded. Therefore, there exists \( d > 0 \) such that

\[
\|u_j - u_j^0\|_\star = \|u_j^+ + u_j^-\|_\star \leq d.
\]

\[
\left| J(u_j) - J(u_j^0) \right| = \left| \int_0^1 \frac{d}{dt} J((1-t)u_j^0 + tu_j)dt \right|
\]

\[
= \left| \int_0^1 J'((1-t)u_j^0 + tu_j)(u_j - u_j^0)dt \right|
\]

\[
\leq K\|u_j - u_j^0\|_{L^1} \leq K_1\|u_j - u_j^0\|_\star
\]

\[
\leq K_1 d,
\]

which together with

\[
J(u_j^0) = \frac{1}{2}(\|u_j^+\|^2_\star - \|u_j^-\|^2_\star) - \Phi(u_j) - [J(u_j) - J(u_0)]
\]
yields $J(u_j^0)$ is bounded. By (J4), we get $\{u_j^0\}$ is bounded. Thus $\{u_j\}$ is bounded on $X$.

It should be noted that the gradient of $\Phi(u)$, $\nabla \Phi(u) : X \to X$ satisfies
\[
\nabla \Phi(u) = u - G(u)
\] (2.19)
with $G(u) : X \to X$ being a compact operator defined by
\[
\langle G(u), z \rangle = \mu \int_0^L u(t)z(t)dt + J'(u)z, \quad u, z \in X.
\]
From the boundedness of $\{u_j\}$ and (2.19), we infer that $\{u_j\}$ has at least one convergent subsequence on $X$. So the Palais-Smale condition holds. \qed

**Lemma 2.3.** Under the hypotheses of Theorem 1.1, the functional $\Phi(u)$ is bounded from below on $X^+$. \hfill \Box

**Proof.** From (J2), we have the estimate
\[
J(u) = \int_0^1 \frac{d}{dt}J(tu)dt = \int_0^1 J'(tu)u dt \leq K\|u\|_{L^1} \leq K_1\|u\|_*, \quad \forall u \in X. \tag{2.20}
\]
Then for $u \in X^+$, we infer that
\[
\Phi(u) = \frac{1}{2}\|u\|^2 - J(u) \geq \frac{1}{2}\|u\|^2 - K_1\|u\|_* \to \infty \quad (\|u\|_* \to \infty).
\] (2.21)
Namely, $\Phi(u)$ is coercive and bounded from below on $X^+$. \qed

**Lemma 2.4.** Under the assumptions of Theorem 1.1, there exists a subspace $V$ of $X$ with $\dim V = k + p$ and $\tilde{\rho} > 0$ such that $\sup_{u \in V, \|u\| = \tilde{\rho}} \Phi(u) < 0$.

**Proof.** Put
\[
V = \left\{ u = \sum_{j=1}^{k+p} \alpha_j v_j : \alpha_j \in \mathbb{R} \ (1 \leq j \leq k+p) \right\}, \tag{2.22}
\]
\[
Z = \left\{ u \in V : \sum_{j=1}^{k+p} \alpha_j^2 = \tilde{\rho}^2 \right\}, \tag{2.23}
\]
where $\tilde{\rho} = \rho/(c_\infty \sqrt{k+p})$, $c_\infty$ satisfies $\|z\|_{L^\infty} \leq c_\infty \|z\|$ for all $z \in X$.

For each $u(t) = \sum_{j=1}^{k+p} \alpha_j v_j(t) \in Z$, we have by Cauchy-Schwarz inequality
\[
|u(t)|^2 \leq \left( \sum_{j=1}^{k+p} |v_j(t)|^2 \right) \left( \sum_{j=1}^{k+p} \alpha_j^2 \right) \leq (k+p)c_\infty^2 \tilde{\rho}^2 = \rho^2, \tag{2.24}
\]
using $\|v_j\|_{L^\infty} \leq c_\infty \|v_j\| = c_\infty$ for all $j \geq 1$. Hence
\[
\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\|u\|_{L^2}^2 - J(u)
\leq \frac{1}{2}\|u\|^2 - \frac{\mu + M}{2}\|u\|_{L^2}^2
\geq \frac{1}{2} \sum_{j=1}^{k+p} \alpha_j^2 - \frac{\mu + M}{2} \sum_{j=1}^{k+p} \frac{1}{\lambda_j} \alpha_j^2
\]
\[ \frac{1}{2} \sum_{j=1}^{k+p} \frac{\lambda_j - \mu - M}{\lambda_j} \alpha_j^2 \leq \frac{1}{2} (\lambda_{k+p} - \lambda_k - M) \sum_{j=1}^{k+p} \frac{\alpha_j^2}{\lambda_j} < 0, \]

which implies \( \sup \{ \Phi(u) : u \in Z \} < 0. \)

\[ \square \]

3. PROOF AND EXTENSION OF THEOREM 1.1

Proof of Theorem 1.1. With the aid of Lemmas 2.2-2.4, by Theorem 1.2, we conclude that \( \Phi(u) \) (1.2) possesses at least \( p \) distinct pairs \((u_i, -u_i)\) of critical points.

Corollary 3.1. Under the assumptions of Theorem 1.1, if condition (J3) is replaced by

\((J3') \lim_{|u| \to 0} \frac{W(t,u)}{|u|^2} = \infty \) uniformly in \( t \in [0, L] \),

then the functional \( \Phi(u) \) defined in (1.2) has infinitely many distinct pairs \((u, -u)\) of critical points.

Proof. For any fixed \( p \in \mathbb{N} \), we may take \( M \) large enough such that \( M > \lambda_{k+p} - \lambda_k \).

By (J3'), there exists \( \rho \) sufficiently small satisfying

\[ W(t,w) \geq \frac{1}{2} M |w|^2, \quad \forall w \in \mathbb{R}^n, |w| \leq \rho \]

(3.1)

uniformly in \( t \in [0, L] \). Thus, if \( u = u(t) \in X \) with \( \|u\|_{L^\infty} \leq \rho \), then

\[ W(t,u(t)) \geq \frac{1}{2} M |u(t)|^2 \]

(3.2)

uniformly in \( t \in [0, L] \), and one obtains

\[ J(u) \geq \frac{1}{2} M \|u\|_{L^2}^2. \]

(3.3)

Therefore, in view of Theorem 1.1, the functional \( \Phi(u) \) has at least \( p \) distinct pairs \((u_i, -u_i)\) of critical points \((1 \leq i \leq p)\). Since \( p \) is arbitrary, there exist infinitely many distinct pairs \((u_i, -u_i)\) of critical points of \( \Phi(u) \) \((i = 1, 2, 3, \ldots)\). \( \square \)

Remark 3.2. For all \( \beta \in (0, 1/2), \gamma \in (0, 1) \), we can take a function \( H(s) \in C^1([0, \infty), R) \) such that

\[ s^{1+2\beta} \leq H(s) \leq s^{1+\beta}, \quad \forall s \in [0, 1], \]

(3.4)

\[ -\frac{1}{8} s^{7-1} \leq H'(s) \leq \frac{1}{8} s^{7-1} \quad \text{quad} s \in [2, \infty), \]

(3.5)

\[ H(s) \to \pm \infty \quad \text{as} \ s \to \infty. \]

(3.6)

Define \( W(t,u) = H(|u|)(\sin t)^{2m} + 2, m \geq 1 \). A straightforward computation shows that (3.4) and (3.5) imply (J1)-(J3). In addition, (J4) can be easily deduced by (3.6), see [11, Lemma 4.21].

From a carefully analyzing the constructions of \( V \) and \( Z \) in (2.22)-(2.23), we have the following result which is more general than Theorem 1.1.
Theorem 4.1. Assume that $(X, \| \cdot \|) \subset L^2$ is a Hilbert space, continuously embedded in $L^q, \forall q \in [1, \infty]$. Let $n_j = \dim N_j$ and $\{v_{j1}, v_{j2}, \ldots, v_{j_{nj}}\}$ be an orthogonal basis of $N_j (\forall j \geq 1)$ such that $\{v_{j1}(t): j \geq 1, 1 \leq i \leq n_j\}$ is an orthogonal basis in $X$ and $L^2$ with
\[
\|v_{ji}(x)\|^2 = 1 = \lambda_j \|v_{ji}(x)\|_{L^2}^2, \quad \forall j \geq 1, 1 \leq i \leq n_j.
\]
Furthermore, assume that the functional $J(u) \in C^1(X, R)$ satisfies $J(0) = 0$, $J'(u)$ is a compact operator, and (J1)-(J4) hold. Then, there exist at least $\sum_{j=k+1}^{K} n_j$ distinct pairs $(u, -u)$ of critical points of $\Phi(u)$ (if $\mu < \lambda_k, \lambda_{k+1}$, then (J4) can be omitted).

To prove this theorem, we need only changes in Lemmas 2.1 and 2.4. Especially, $V, Z$ in (2.22)-(2.23) shall be replaced by
\[
\tilde{V} = \{ u = \sum_{j=1}^{k+p} \sum_{i=1}^{n_j} \alpha_{ji} v_{ji} : \alpha_{ji} \in \mathbb{R} (1 \leq j \leq k + p, 1 \leq i \leq n_j) \}, \quad (3.7)
\]
\[
\tilde{Z} = \{ u \in \tilde{V} : \sum_{j=1}^{k+p} \sum_{i=1}^{n_j} \alpha_{ji}^2 = \rho^2 \}, \quad (3.8)
\]
respectively.

4. Applications

Application i. Given $T > 0$, we discuss the existence of $T$-periodic solutions to the second-order Hamiltonian system

\[
\ddot{u}(t) + \mu u(t) + W_u(t, u(t)) = 0, \quad t \in \mathbb{R}, \quad (4.1)
\]
where $W(t, u) \in C^1(R \times \mathbb{R}^n, R)$ is a $T$-periodic function in the variable $t$ and $W(t, 0) = 0$.

Since 1973, many authors studied periodic solutions for Hamiltonian systems via critical point theory. Clarke and Ekeland [3] studied a family of convex sublinear Hamiltonian systems where $W(t, u) = W(u)$ satisfies $\lim_{|u| \rightarrow 0} \frac{W(t, u)}{|u|^a} = \infty$, and they used the dual variational method to obtain the first variational result on periodic solutions having a prescribed minimal period. Later, Mawhin and Willem [8] made a good improvement. Rabinowitz [9, 10], Tang [13] and others proved the existence under the sublinear condition $uW(t, u) \leq aW(t, u)(0 < a < 2)$, which plays an important role. Schechter [12] assumed that $W(t, u)$ is sublinear, and $2W(t, u) - uW_u(t, u) \rightarrow -\infty (|u| \rightarrow \infty)$ or $2W(t, u) - uW_u(t, u) \leq W_0(t)$, then he proved that (4.1) has one non-constant periodic solution. Long [7] also studied this problem for bi-even sublinear potentials, and got the existence of one odd periodic solution. Li-Wang-Xiao [6] considered the existence and multiplicity of odd periodic solution for bi-even sublinear (4.1) in the case of $\mu < \lambda_1$.

Motivated by the above papers, using Theorem 3.3, we shall give a multiplicity result for (4.1) with sublinear potentials in the case of $\lambda_k \leq \mu < \lambda_{k+1}$.

Theorem 4.1. Assume that $L = T/2$, and there exists some $k \in N$ such that $(\frac{k}{L})^2 \leq \mu < (\frac{k+1}{L})^2$. Let $W(t, u) \in C^1(R \times \mathbb{R}^n, R)$ be $T$-periodic in $t$, and bi-even, namely
\[
W_u(t, u) = -W_u(-t, -u), \quad \forall t \in \mathbb{R}, \quad u \in \mathbb{R}^n.
\]
Suppose that

(W11) \( W(t, u) = W(t, -u) \) for all \( t \in \mathbb{R}, u \in \mathbb{R}^n \);
(W12) there exists \( K > 0 \) such that \( |W_u(t, u)| \leq K \) for all \( t \in \mathbb{R}, u \in \mathbb{R}^n \);
(W13) there exist \( p \in \mathbb{N}, M > 0, \rho > 0 \) such that if \( M > \frac{p(p+2k)}{L^2} \pi^2 \) then

\[
W(t, u) \geq \frac{1}{2} M |u|^2 \quad \forall t \in \mathbb{R}, |u| \leq \rho;
\]
(W14) for \( u = c \sin \frac{i \pi t}{L} \) with \( \theta_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n \) (the \( i \)-th element is 1, \( 1 \leq i \leq n \)), for all \( j \geq 1 \), \( \int_0^L W(t, u(t)) dt \to \pm \infty \) as \( |c| \to \infty \).

Then, (4.1) has \( np \)-distinct pairs \( (u(t), -u(t)) \) of odd \( T \)-periodic solutions. If \( (\frac{k \pi}{L})^2 < \mu < (\frac{(k+1) \pi}{L})^2 \), then (W14) can be omitted.

Remark 4.2. If \( W(t, u) \) satisfies

\[
W(t, u) = W(t, -u) = W(-t, -u),
\]
then \( W(t, u) \) is bi-even, and (W11) holds. For this, a typical example is, \( W(t, u) = b(t)\tilde{W}(u) \), where \( b(t) \) and \( \tilde{W}(u) \) are even in the variable \( t \), respectively.

Proof of Theorem 4.1. Firstly, consider the boundary value problem

\[
\begin{align*}
-\ddot{u}(t) &= \mu u(t) + W_u(t, u(t)), \quad 0 < t < L, \\
u(0) &= u(L) = 0.
\end{align*}
\]

(4.2)

If \( u = u(t) \) is a solution of (4.2), then we define

\[
\overline{u} = \overline{u}(t) = \begin{cases} u(t), & 0 \leq t \leq L, \\
u(-t), & -L \leq t \leq 0. \end{cases}
\]

(4.3)

By the bi-even condition, \( \overline{u} = \overline{u}(t) \) is a solution of (4.1) restricted on \([-L, L]\), so its odd extension in \((-\infty, \infty)\) is an odd \( T \)-periodic solution of (4.1).

Secondly, let \( X = H_0^1([0, L], \mathbb{R}^n) \) be the usual Hilbert space with the inner product \( \langle x, y \rangle = \int_0^L x(t) \cdot y(t) dt \) and the norm \( ||x|| = (\int_0^L |\dot{x}(t)|^2 dt)^{1/2} \). Set

\[
\Phi(u) = \frac{1}{2} \int_0^L ||\dot{u}(t)||^2 - \mu |u(t)|^2 dt - \int_0^L W(t, u(t)) dt,
\]

(4.4)

then \( \Phi(u) \in C^1(X, \mathbb{R}) \), and its critical points are the classical solutions of (4.2).

By direct computations, we know that the problem

\[
-\ddot{u}(t) = \lambda u(t), \quad u(0) = u(L) = 0
\]
possesses eigenvalues \( \lambda_j = (\frac{j \pi}{L})^2, j \geq 1 \), and the corresponding eigenfunctions are \( u_{ji} = c \theta_i \sin \frac{j \pi t}{L}, 1 \leq i \leq n, c \in \mathbb{R} \). Furthermore,

\[
\{ \theta_i \sin \frac{\pi t}{L}, \theta_i \sin \frac{2 \pi t}{L}, \theta_i \sin \frac{3 \pi t}{L}, \ldots, 1 \leq i \leq n \}
\]

(4.5)
is an orthogonal basis on both \( X \) and \( L^2 \). Since

\[
\int_0^L |\dot{u}_{ji}(t)|^2 dt = \lambda_j \frac{L}{2} = \lambda_j \int_0^L |u_{ji}(t)|^2 dt,
\]

(4.6)
writing \( v_{ji} = \sqrt{\frac{2}{L \lambda_j}} u_{ji} \), we have \( ||v_{ji}||^2 = \int_0^L |\dot{v}_{ji}|^2 dt = 1 = \lambda_j \int_0^L |v_{ji}|^2 dt \).

Noticing that

\[
\frac{p(p+2k)}{L^2} \pi^2 = \lambda_{k+p} - \lambda_k,
\]
the functional (4.4) satisfies all hypotheses of Theorem 3.3, hence it has at least $np$ distinct pairs $(u_i, -u_i)$ of critical points $(1 \leq i \leq np)$. Consequently, in the way of (4.3), the extensions of $\pm u_i(t)(1 \leq i \leq np)$ are $np$ distinct pairs of odd $T$-periodic solutions of (4.1). \hfill \Box

Application ii. We are concerned with a class of Extended Fisher-Kolmogorov type equations (see [4,5,14] and their references)

$$u^{(4)}(t) = \mu u(t) + W_u(t, u(t)) \quad 0 \leq t \leq L$$

with the boundary condition

$$u(0) = u(L) = u''(0) = u''(L) = 0,$$

which appears in the formation of spatial patterns in bistable systems.

**Theorem 4.3.** Assume that there exists some $k \in N$ such that $(\frac{k\pi}{L})^4 < \mu < (\frac{(k+1)\pi}{L})^4$. Let $W(t, u) \in C^1([0, L] \times R, R)$ satisfy the following conditions:

(W21) $W(t, u) = W(t, -u)$ for all $t \in [0, L], u \in R$;

(W22) there exists $K > 0$ such that $|W_u(t, u)| \leq K$ for all $t \in [0, L], u \in R$;

(W23) there exist $p \in N, M > 0, \rho > 0$ such that if $M > \frac{[p + k]^4 - [k]^4}{4\rho}$ then

$$W(t, u) \geq \frac{1}{2}M|u|^2 \quad \forall t \in [0, L], |u| \leq \rho;$$

(W24) for $u = c\sin \frac{j\pi t}{L}$, for all $j \geq 1$, $c \in R$, $\int_0^L W(t, u(t))dt \to \pm \infty$ as $|c| \to \infty$.

Then, (4.7) has $p$ distinct pairs $(u(t), -u(t))$ of classical solutions. If $(\frac{k\pi}{L})^4 < \mu < (\frac{(k+1)\pi}{L})^4$, then (W24) can be omitted.

**Proof.** Similarly to the proof of Theorem 4.1, we sketch it. Set

$$X = H^2(0, L) \cap H^1_0(0, L),$$

by [5] Lemma 2.1, $\|u\| = \left(\int_0^L |\ddot{u}(t)|^2 dt\right)^{1/2}$ is a norm of $X$, and

$$v_j(t) = \sin \frac{j\pi t}{L} \left(\sqrt{\frac{L}{2}} \left(\frac{j\pi}{L}\right)^2\right)^{-1}$$

is an orthogonal basis on $X$ and $L^2$ such that

$$\|v_j(t)\|^2 = 1 = \left(\frac{j\pi}{L}\right)^4 \|v_j(t)\|_{L^2}^2, \quad j \geq 1.$$  \hfill (4.10)

In addition, the problem

$$u^{(4)}(t) = \lambda u(t)$$

has eigenvalues $\lambda_j = (\frac{j\pi}{L})^4, \quad j \geq 1,$ and the corresponding eigenfunctions are exactly $v_j(t)$ in (4.9). Define the functional

$$\Phi(u) = \frac{1}{2} \int_0^L |\ddot{u}(t)|^2 dt - \frac{1}{2}\mu \int_0^L |u(t)|^2 dt - \int_0^L W(t, u(t)) dt, \quad u \in X;$$

then the critical points of $\Phi(u)$ in (4.11) are the classical solutions of the problem (4.7). Therefore, by Theorem 1.1, we have the statement in Theorem 4.3 \hfill \Box

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