ENTIRE FUNCTIONS SHARING SMALL FUNCTIONS WITH THEIR DIFFERENCE OPERATORS

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Abstract. We study the uniqueness for entire functions that share small functions of finite order with difference operators applied to the entire functions. In particular, we generalize of a result in [2].

1. Introduction and Main Results

In this article, we assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna’s value distribution theory [7, 9, 12]. In addition, we will use \( \rho(f) \) to denote the order of growth of \( f \) and \( \tau(f) \) to denote the type of growth of \( f \), we say that a meromorphic function \( a(z) \) is a small function of \( f(z) \) if \( T(r, a) = S(r, f) \), where \( S(r, f) = o(T(r, f)) \), as \( r \to \infty \) outside of a possible exceptional set of finite logarithmic measure, we use \( S(f) \) to denote the family of all small functions with respect to \( f(z) \). For a meromorphic function \( f(z) \), we define its shift by \( f_c(z) = f(z + c) \) (Resp. \( f_0(z) = f(z) \)) and its difference operators by

\[
\Delta_c f(z) = f(z + c) - f(z), \quad \Delta^n f(z) = \Delta^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N}, \ n \geq 2.
\]

In particular, \( \Delta_c f(z) = \Delta^n f(z) \) for the case \( c = 1 \).

Let \( f(z) \) and \( g(z) \) be two meromorphic functions, and let \( a(z) \) be a small function with respect to \( f(z) \) and \( g(z) \). We say that \( f(z) \) and \( g(z) \) share \( a(z) \) counting multiplicity (for short CM), provided that \( f(z) - a(z) \) and \( g(z) - a(z) \) have the same zeros including multiplicities.

The problem of meromorphic functions sharing small functions with their differences is an important topic of uniqueness theory of meromorphic functions (see [1, 4, 5, 6]). In 1986, Jank, Mues and Volkmann [8] proved the following result.

**Theorem 1.1.** Let \( f \) be a nonconstant meromorphic function, and let \( a \neq 0 \) be a finite constant. If \( f, f' \) and \( f'' \) share the value \( a \) CM, then \( f \equiv f' \).

Li and Yang [11] gave the following generalization of Theorem 1.1.

**Theorem 1.2.** Let \( f \) be a nonconstant entire function, let \( a \) be a finite nonzero constant, and let \( n \) be a positive integer. If \( f, f^n \) and \( f^{(n+1)} \) share the value \( a \) CM, then \( f \equiv f' \).

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Chen et al. proved a difference analogue of result of Theorem 1.1 and obtained the following results.

**Theorem 1.3.** Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a(z) \neq 0 \in S(f) \) be a periodic entire function with period \( c \). If \( f(z), \Delta_c f \) and \( \Delta_c^2 f \) share \( a(z) \) CM, then \( \Delta_c f \equiv \Delta_c^2 f \).

**Theorem 1.4.** Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a(z), b(z) \neq 0 \in S(f) \) be periodic entire functions with period \( c \). If \( f(z) - a(z), \Delta_c f(z) - b(z) \) and \( \Delta_c^2 f(z) - b(z) \) share \( 0 \) CM, then \( \Delta_c f \equiv \Delta_c^2 f \).

Recently Chen and Li generalized Theorem 1.3 and proved the following results.

**Theorem 1.5.** Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a(z) \neq 0 \in S(f) \) be a periodic entire function with period \( c \). If \( f(z), \Delta_c f \) and \( \Delta_c^n f \) \((n \geq 2)\) share \( a(z) \) CM, then \( \Delta_c f \equiv \Delta_c^n f \).

**Theorem 1.6.** Let \( f(z) \) be a nonconstant entire function of finite order. If \( f(z), \Delta_c f(z) \) and \( \Delta_c^n f(z) \) share \( 0 \) CM, then \( \Delta_c^n f(z) = C \Delta_c f(z) \), where \( C \) is a nonzero constant.

It is interesting to see what happens when \( f(z), \Delta_c^n f(z) \) and \( \Delta_c^{n+1} f(z) \) \((n \geq 1)\) share \( a(z) \) CM. The aim of this article is to give a difference analogue of result of Theorem 1.2. In fact, we prove that the conclusion of Theorems 1.5 and 1.6 remain valid when we replace \( \Delta_c f(z) \) by \( \Delta_c^{n+1} f(z) \). We obtain the following results.

**Theorem 1.7.** Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a(z) \neq 0 \in S(f) \) be a periodic entire function with period \( c \). If \( f(z), \Delta_c^n f(z) \) and \( \Delta_c^{n+1} f(z) \) \((n \geq 1)\) share \( a(z) \) CM, then \( \Delta_c^n f(z) \equiv \Delta_c^{n+1} f(z) \).

**Example 1.8.** Let \( f(z) = e^z \ln^2 z \) and \( c = 1 \). Then, for any \( a \in \mathbb{C} \), we notice that \( f(z), \Delta_c^n f(z) \) and \( \Delta_c^{n+1} f(z) \) share a CM for all \( n \in \mathbb{N} \) and we can easily see that \( \Delta_c^{n+1} f(z) \equiv \Delta_c^n f(z) \). This example satisfies Theorem 1.7.

**Remark 1.9.** In Example 1.8, we have \( \Delta_c^n f(z) \equiv \Delta_c^{n+1} f(z) \) for any integer \( m > n + 1 \). However, it remains open when \( f(z), \Delta_c^n f(z) \) and \( \Delta_c^{n+1} f(z) \) \((m > n + 1)\) share \( a(z) \) CM, the claim \( \Delta_c f(z) \equiv \Delta_c^{n+1} f(z) \) in Theorem 1.7 can be replaced by \( \Delta_c^n f(z) \equiv \Delta_c^{n+1} f(z) \) in general.

**Theorem 1.10.** Let \( f(z) \) be a nonconstant entire function of finite order, and let \( a(z), b(z) \neq 0 \in S(f) \) be periodic entire functions with period \( c \). If \( f(z) - a(z), \Delta_c^n f(z) - b(z) \) and \( \Delta_c^{n+1} f(z) - b(z) \) share \( 0 \) CM, then \( \Delta_c^n f(z) \equiv \Delta_c^{n+1} f(z) \).

**Theorem 1.11.** Let \( f(z) \) be a nonconstant entire function of finite order. If \( f(z), \Delta_c^n f(z) \) and \( \Delta_c^{n+1} f(z) \) share \( 0 \) CM, then \( \Delta_c^{n+1} f(z) \equiv C \Delta_c^n f(z) \), where \( C \) is a nonzero constant.

**Example 1.12.** Let \( f(z) = e^{a z} \) and \( c = 1 \) where \( a \neq 2k \pi i \) \((k \in \mathbb{Z})\), it is clear that \( \Delta_c^n f(z) = (e^a - 1)^n e^{a z} \) for any integer \( n \geq 1 \). So, \( f(z), \Delta_c^n f(z) \) and \( \Delta_c^{n+1} f(z) \) share \( 0 \) CM for all \( n \in \mathbb{N} \) and we can easily see that \( \Delta_c^{n+1} f(z) \equiv C \Delta_c^n f(z) \) where \( C = e^a - 1 \). This example satisfies Theorem 1.11.
2. Some lemmas

Lemma 2.1 ([10]). Let $f$ and $g$ be meromorphic functions such that $0 < \rho(f)$, $\rho(g) < \infty$ and $0 < \tau(f), \tau(g) < \infty$. Then we have

(i) If $\rho(f) > \rho(g)$, then we obtain
$$\tau(f + g) = \tau(fg) = \tau(f).$$

(ii) If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then
$$\rho(f + g) = \rho(fg) = \rho(f) = \rho(g).$$

Lemma 2.2 ([12]). Suppose $f_j(z)$ $(j = 1, 2, \ldots, n + 1)$ and $g_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 1)$ are entire functions satisfying the following two conditions:

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} = f_{n+1}(z)$;

(ii) The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1 \leq j \leq n + 1, 1 \leq k \leq n$. Furthermore, the order of $f_j(z)$ is less than the order of $e^{g_k(z)} - g_k(z)$ for $n \geq 2$ and $1 \leq j \leq n + 1, 1 \leq h < k \leq n$.

Then $f_j(z) \equiv 0$, $(j = 1, 2, \ldots, n + 1)$.

Lemma 2.3 ([5]). Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period $c$, with respect to $f(z)$,
$$m(r, \frac{\Delta_n f}{f - a}) = S(r, f),$$
where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

3. Proof of the Theorems

Proof of the Theorem 1.7. Suppose on the contrary to the assertion that $\Delta_c f(z) \neq \Delta_c^{n+1} f(z)$. Note that $f(z)$ is a nonconstant entire function of finite order. By Lemma 2.3 for $n \geq 1$, we have
$$T(r, \Delta_c^{n} f) = m(r, \Delta_c^{n} f) \leq m(r, \frac{\Delta_c^{n} f}{f}) + m(r, f) \leq T(r, f) + S(r, f).$$

Since $f(z)$, $\Delta_c^{n} f(z)$ and $\Delta_c^{n+1} f(z)$ $(n \geq 1)$ share $a(z)$ CM, then
$$\frac{\Delta_c^{n} f(z) - a(z)}{f(z) - a(z)} = e^{P(z)}, \quad (3.1)$$
$$\frac{\Delta_c^{n+1} f(z) - a(z)}{f(z) - a(z)} = e^{Q(z)}, \quad (3.2)$$
where $P$ and $Q$ are polynomials. Set
$$\varphi(z) = \frac{\Delta_c^{n+1} f(z) - \Delta_c^{n} f(z)}{f(z) - a(z)}. \quad (3.3)$$

From (3.1) and (3.2), we obtain $\varphi(z) = e^{Q(z)} - e^{P(z)}$. Then, by supposition and (3.3), we see that $\varphi(z) \neq 0$. By Lemma 2.3 we deduce that
$$T(r, \varphi) = m(r, \varphi) \leq m(r, \frac{\Delta_c^{n+1} f}{f - a}) + m(r, \frac{\Delta_c^{n} f}{f - a}) + O(1) = S(r, f). \quad (3.4)$$
Thus, by (3.4) and (3.5), we have
\[ T(r, e^Q) \leq N(r, e^Q) + N(r, e^{\varphi}) + N(r, \frac{1}{e^Q - 1}) + S(r, e^Q) \]
\[ = N(r, e^Q) + N(r, \varphi) + N(r, \varphi) + S(r, \varphi) \quad (3.5) \]
\[ = S(r, f) + S(r, e^Q). \]

Thus, by (3.4) and (3.5), we have \( T(r, e^Q) = S(r, f) \). Similarly, \( T(r, e^P) = S(r, f) \). Setting now \( g(z) = f(z) - a(z) \), from (3.1) and (3.2) we have
\[ \Delta_c^n g(z) = g(z)e^{P(z)} + a(z), \quad (3.6) \]
\[ \Delta_c^{n+1} g(z) = g(z)e^{Q(z)} + a(z). \quad (3.7) \]

By (3.6) and (3.7), we have
\[ g(z)e^{Q(z)} + a(z) = \Delta_c(\Delta_c^n g(z)) = \Delta_c(g(z)e^{P(z)} + a(z)). \]

Thus
\[ g(z)e^{Q(z)} + a(z) = g_c(z)e^{P_c(z)} - g(z)e^{P(z)}, \]
which implies
\[ g_c(z) = M(z)g(z) + N(z), \quad (3.8) \]
where \( M(z) = e^{-P_c(z)}(e^{P(z)} + e^{Q(z)}) \) and \( N(z) = a(z)e^{-P_c(z)} \). From (3.8), we have
\[ g_{2c}(z) = M_c(z)g_c(z) + N_c(z) = M_c(z)(M(z)g(z) + N(z)) + N_c(z), \]

hence
\[ g_{2c}(z) = M_c(z)M_0(z)g(z) + N^1(z), \]
where \( N^1(z) = M_c(z)N_0(z) + N_c(z) \). By the same method, we can deduce that
\[ g_{ic}(z) = \left( \prod_{k=0}^{i-1} M_{kc}(z) \right)g(z) + N^{i-1}(z) \quad (i \geq 1), \quad (3.9) \]
where \( N^{i-1}(z) (i \geq 1) \) is an entire function depending on \( a(z) \), \( e^{P(z)} \), \( e^{Q(z)} \) and their differences. Now, we can rewrite (3.6) as
\[ \sum_{i=1}^{n} C_n^i (1)^{n-i} g_{ic}(z) = (e^{P(z)} - (-1)^n)g(z) + a(z). \quad (3.10) \]

By (3.9) and (3.10), we have
\[ \sum_{i=1}^{n} C_n^i (1)^{n-i} \left( \left( \prod_{k=0}^{i-1} M_{kc}(z) \right)g(z) + N^{i-1}(z) \right) - (e^{P(z)} - (-1)^n)g(z) = a(z) \]
which implies
\[ A(z)g(z) + B(z) = 0, \quad (3.11) \]
where
\[ A(z) = \sum_{i=1}^{n} C_n^i (1)^{n-i} \prod_{k=0}^{i-1} M_{kc}(z) - e^{P(z)} + (-1)^n, \]
\[ B(z) = \sum_{i=1}^{n} C_n^i (-1)^{n-i} N^{i-1}(z) - a(z). \]

It is clear that \( A(z) \) and \( B(z) \) are small functions with respect to \( f(z) \). If \( A(z) \neq 0 \), then (3.11) yields the contradiction
\[ T(r, f) = T(r, g) = T(r, \frac{B}{A}) = S(r, f). \]

Suppose now that \( A(z) \equiv 0 \), rewrite the equation \( A(z) \equiv 0 \) as
\[ \sum_{i=1}^{n} C_n^i (-1)^{n-i} \prod_{k=0}^{i-1} e^{-P(k+1)c} (e^{Pc} + e^{Qc}) = e^P - (-1)^n. \]

We can rewrite the left side of above equality as
\[
\begin{align*}
\sum_{i=1}^{n} C_n^i (-1)^{n-i} & e^{-\sum_{k=1}^{i} P_k c} \prod_{k=0}^{i-1} (e^{P_k c} + e^{Q_k c}) \\
& = \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{-\sum_{k=1}^{i} P_k c} \prod_{k=0}^{i-1} (1 + e^{Q_k c - P_k c}) \\
& = \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P_k c} \prod_{k=0}^{i-1} (1 + e^{Q_k c - P_k c}).
\end{align*}
\]

So
\[ \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P_k c} \prod_{k=0}^{i-1} (1 + e^{h_{kc}}) = e^P - (-1)^n, \tag{3.12} \]
where \( h_{kc} = Q_{kc} - P_{kc} \). On the other hand, let \( \Omega_i = \{0, 1, \ldots, i-1\} \) be a finite set of \( i \) elements, and
\[ P(\Omega_i) = \{\emptyset, \{0\}, \{1\}, \ldots, \{i-1\}, \{0, 1\}, \{0, 2\}, \ldots, \Omega_i\}, \]
where \( \emptyset \) is the empty set. It is easy to see that
\[ \prod_{k=0}^{i-1} (1 + e^{h_{kc}}) = 1 + \sum_{A \in P(\Omega_i) \setminus \{\emptyset\}} \exp \left( \sum_{j \in A} h_{jc} \right) \tag{3.13} \]
\[ = 1 + e^h + e^{h+c} + \ldots + e^{(i-1)c} \]
\[ + e^{h+hc} + e^{h+hc} + \ldots + [e^{h+h\ldots+hc+(i-1)c}]. \]

We divide the proof into two parts:

Part (1). \( h(z) \) is non-constant polynomial. Suppose that \( h(z) = a_m z^m + \cdots + a_0 \) (\( a_m \neq 0 \)), since \( P(\Omega_i) \subset P(\Omega_{i+1}) \), then by (3.12) and (3.13) we have
\[
\sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P_k c} + \alpha_1 a_m z^m + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{n a_m z^m} = e^P - (-1)^n
\]
which is equivalent to
\[
\alpha_0 + \alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{n a_m z^m} = e^P, \tag{3.14}
\]
where $\alpha_i$ ($i = 0, \ldots, n$) are entire functions of order less than $m$. Moreover,

\[
\alpha_0 = \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{P-P_ie} + (-1)^n
\]

\[
e^P \left( \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{-P_ie} + (-1)^n e^{-P} \right)
\]

\[
e^P \Delta_n^e e^{-P}.
\]

(i) If $\text{deg } P > m$, then we obtain from (3.14) that $\text{deg } P \leq m$ which is a contradiction.

(ii) If $\text{deg } P < m$, then by using Lemma 2.1 and (3.14) we obtain

\[
\text{deg } P = \rho(e^P) = \rho \left( \alpha_0 + \alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{na_m z^m} \right) = m
\]

which is also a contradiction.

(iii) If $\text{deg } P = m$, then we suppose that $P(z) = dz^m + P^*(z)$ where $\text{deg } P^* < m$. We have to study two subcases:

(*) If $d \neq ia_m$ ($i = 1, \ldots, n$), then

\[
\alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + \alpha_n e^{na_m z^m} - e^{P^*} dz^m = -\alpha_0.
\]

By using Lemma 2.2 we obtain $e^{P^*} \equiv 0$, which is impossible.

(**) Suppose now that there exists at most $j \in \{1, 2, \ldots, n\}$ such that $d = ja_m$.

Without loss of generality, we assume that $j = n$. Then we rewrite (3.14) as

\[
\alpha_1 e^{a_m z^m} + \alpha_2 e^{2a_m z^m} + \cdots + (\alpha_n - e^{P^*}) e^{na_m z^m} = -\alpha_0.
\]

By using Lemma 2.2 we have $\alpha_0 \equiv 0$, so $\Delta_n^e e^{-P} = 0$. Thus

\[
\sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-P_ie} \equiv 0.
\]

(3.15)

Suppose that $\text{deg } P = \text{deg } h = m > 1$ and

\[
P(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0, \quad (b_m \neq 0).
\]

Note that for $j = 0, 1, \ldots, n$, we have

\[
P(z + jc) = b_m z^m + (b_{m-1} + mb_m jc) z^{m-1} + \beta_j(z),
\]

where $\beta_j(z)$ are polynomials with degree less than $m - 1$. Rewrite (3.15) as

\[
e^{-\beta_m(z)} e^{-b_m z^m - (b_{m-1} + mb_m kc) z^{m-1}} \]

\[
- n e^{-\beta_{m-1}(z)} e^{-b_m z^m - (b_{m-1} + mb_m (n-1)c) z^{m-1}} + \cdots
\]

\[
+ (-1)^n e^{-\beta_0(z)} e^{-b_m z^m - b_{m-1} z^{m-1}} \equiv 0.
\]

(3.16)

For any $0 \leq l < k \leq n$, we have

\[
\rho(e^{-b_m z^m - (b_{m-1} + mb_m kc) z^{m-1}}) = \rho(e^{-mb_m (l-k)z^{m-1}}) = m - 1,
\]

and for $j = 0, 1, \ldots, n$, we see that

\[
\rho(e^{\beta_j}) \leq m - 2.
\]
By this, together with (3.16) and Lemma 2.2, we obtain $e^{-\beta_n(z)} \equiv 0$, which is impossible. Suppose now that $P(z) = \mu z + \eta (\mu \neq 0)$ and $Q(z) = \alpha z + \beta$ because if $\deg Q > 1$, then we go back to case (ii). It easy to see that

$$
\Delta^n e^{-P} = \sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-\mu(z+i\eta)}
$$

$$
= e^{-P} \sum_{i=0}^{n} C_n^i (-1)^{n-i} e^{-\mu c}
$$

$$
= e^{-P} (e^{-\mu c} - 1)^n.
$$

This together with $\Delta^n e^{-P} \equiv 0$ gives $(e^{-\mu c} - 1)^n \equiv 0$, which yields $e^{-\mu c} \equiv 1$. Therefore, for any $j \in \mathbb{Z}$,

$$
e^{P(z+jc)} = e^{\mu z + \mu j c + \eta} = (e^{\mu c})^j e^{P(z)} = e^{P(z)}.
$$

To prove that $e^Q(z)$ is also periodic entire function with period $c$, we suppose the contrary, which means that $e^{\alpha c} \neq 1$. Since $e^{P(z)}$ is of period $c$, then by (3.14), we obtain

$$
\alpha_1 e^{(\alpha-\mu)z} + \alpha_2 e^{2(\alpha-\mu)z} + \cdots + \alpha_n e^{n(\alpha-\mu)z} = e^{\mu z + \eta},
$$

(3.17)

where $\alpha_i (i = 1, \ldots, n)$ are constants. In particular,

$$
\alpha_n = e^{n(\beta-\eta) + \alpha c} 
$$

and

$$
\alpha_1 = \left[ \sum_{i=1}^{n} C_n^i (-1)^{n-i} + \sum_{i=2}^{n} C_n^i (-1)^{n-i} e^{\alpha c} 
+ \sum_{i=3}^{n} C_n^i (-1)^{n-i} e^{2\alpha c} + \cdots + e^{(n-1)\alpha c} \right] e^{(\beta-\eta)}
$$

$$
= \left[ C_n^1 (-1)^{n-1} + C_n^2 (-1)^{n-2}(1 + e^{\alpha c}) + C_n^3 (-1)^{n-3}(1 + e^{\alpha c} + e^{2\alpha c}) 
+ \cdots + C_n^n (-1)^{n-n}(1 + e^{\alpha c} + \cdots + e^{(n-1)\alpha c}) \right] e^{(\beta-\eta)}
$$

$$
= \left[ C_n^1 (-1)^{n-1} \frac{e^{\alpha c} - 1}{e^{\alpha c} - 1} + C_n^2 (-1)^{n-2} \frac{2 e^{\alpha c} - 1}{e^{\alpha c} - 1} + C_n^3 (-1)^{n-3} \frac{3 e^{\alpha c} - 1}{e^{\alpha c} - 1} 
+ \cdots + C_n^n (-1)^{n-n} \frac{e^{(n-1)\alpha c} - 1}{e^{\alpha c} - 1} \right] e^{(\beta-\eta)}
$$

$$
= \left[ C_n^1 (-1)^{n-1} (e^{\alpha c} - 1) + C_n^2 (-1)^{n-2} (2 e^{\alpha c} - 1) + C_n^3 (-1)^{n-3} (3 e^{\alpha c} - 1) 
+ \cdots + C_n^n (-1)^{n-n} (e^{n(\alpha c) - 1}) \right] e^{(\beta-\eta)}
$$

$$
= \left[ \sum_{i=1}^{n} C_n^i (-1)^{n-i} e^{i\alpha c} - (-1)^n - \sum_{i=1}^{n} C_n^i (-1)^{n-i} \right] e^{(\beta-\eta)}
$$

$$
= (e^{\alpha c} - 1)^{n-1} e^{(\beta-\eta)}.
$$

Rewrite (3.17) as

$$
\alpha_1 e^{(\alpha-2\mu)z} + \alpha_2 e^{2(\alpha-3\mu)z} + \cdots + \alpha_n e^{(n\alpha-(n+1)\mu)z} = e^\eta,
$$

(3.18)
it is clear that for each \( 1 \leq l < m \leq n \), we have
\[
\rho(e^{(m\alpha-(m+1)\mu-l\alpha+(l+1)\mu)z}) = \rho(e^{(m-l)(\alpha-\mu)z}) = 1.
\]

We have the following two cases:

(i) If \( j\alpha - (j+1)\mu \neq 0 \) for all \( j \in \{1, 2, \ldots, n\} \), which means that
\[
\rho(e^{(j\alpha-(j+1)\mu)z}) = 1, \quad 1 \leq j \leq n
\]
then, by applying Lemma 2.2 we obtain \( e^\eta = 0 \), which is a contradiction.

(ii) If there exists (at most one) an integer \( j \in \{1, 2, \ldots, n\} \) such that \( j\alpha - (j+1)\mu = 0 \). Without loss of generality, assume that \( e^{(n\alpha-(n+1)\mu)z} = 1 \), the equation (3.18) will be
\[
\alpha_1e^{(\alpha-2\mu)z} + \alpha_2e^{(2\alpha-3\mu)z} + \cdots + \alpha_{n-1}e^{((n-1)\alpha-n\mu)z} = e^\eta - e^{n(\beta-\eta)+\alpha c^{\frac{n(n-1)}{2}}}
\]
and by applying Lemma 2.2 we obtain \( \alpha_1 = (e^{\alpha c} - 1)^{n-1}e^{(\beta-\eta)} \equiv 0 \), which is impossible. So, by (ii) and (ii), we deduce that \( e^{\alpha c} \equiv 1 \). Therefore, for any \( j \in \mathbb{Z} \) we have
\[
e^{P(z+c)} = e^{\alpha z+\beta(e^{\alpha c})^j} = e^{Q(z)},
\]
which implies that \( e^{P} \) is periodic of period \( c \). Since \( e^{P(z)} \) is of period \( c \), then by (3.1), we obtain
\[
\Delta_c^{n+1}f(z) = e^{P(z)}\Delta_c f(z),
\]
then \( \Delta_c^{n+1}f(z) \) and \( \Delta_c f(z) \) share 0 CM. Substituting (3.19) into the second equation (3.2), we obtain
\[
e^{P(z)}\Delta_c f(z) = e^{Q(z)}(f(z) - a(z)) + a(z).
\]
Since \( e^{P(z)} \) and \( e^{Q(z)} \) are of period \( c \), then by (3.20), we obtain
\[
\Delta_c^{n+1}f(z) = e^{Q-P}\Delta_c^n f(z).
\]
So, \( \Delta_c^{n+1}f(z) \) and \( \Delta_c^n f(z) \) share \( 0, a(z) \) CM, combining (3.1), (3.2) and (3.21), we deduce that
\[
\frac{\Delta_c^{n+1}f(z) - a(z)}{\Delta_c^n f(z) - a(z)} = \frac{\Delta_c^{n+1}f(z)}{\Delta_c^n f(z)},
\]
and we obtain
\[
\Delta_c^{n+1}f(z) = \Delta_c^n f(z)
\]
which is a contradiction. Suppose now that \( P = c_1 \) and \( Q = c_2 \) are constants \( (e^{c_1} \neq e^{c_2}) \). By (3.8) we have
\[
g_c(z) = (e^{c_2-c_1} + 1)g(z) + a(z)e^{-c_1}
\]
by the same,
\[
g_{c_2}(z) = (e^{c_2-c_1} + 1)^2g(z) + a(z)e^{-c_1}((e^{c_2-c_1} + 1) + 1).
\]
By induction, we obtain
\[
g_{nc}(z) = (e^{c_2-c_1} + 1)^ng(z) + a(z)e^{-c_1} \sum_{i=0}^{n-1} (e^{c_2-c_1} + 1)^i
\]
\[
= (e^{c_2-c_1} + 1)^ng(z) + a(z)e^{-c_2}((e^{c_2-c_1} + 1)^n - 1).
\]
Rewrite the equation (3.6) as
\[
\Delta_c^n g(z) = \sum_{i=0}^{n} C_i^n(-1)^{n-i}[(e^{c_2-c_1} + 1)^ig(z) + a(z)e^{-c_2}((e^{c_2-c_1} + 1)^i - 1)]
\]
\[ e^{c_2}g(z) + a(z). \]

Since \( A(z) \equiv 0 \), we have
\[
\sum_{i=0}^{n} C_n^i (-1)^{n-i} (e^{c_2-c_1} + 1)^i = e^{c_1},
\]
\[
\sum_{i=0}^{n} C_n^i (-1)^{n-i} ((e^{c_2-c_1} + 1)^i - 1) = e^{c_2}
\]
which are equivalent to
\[
e^{n(c_2-c_1)} = e^{c_1},
\]
\[
e^{n(c_2-c_1)} = e^{c_2}
\]
which is a contradiction.

**Part (2).** \( h(z) \) is a constant. We show first that \( P(z) \) is a constant. If \( \deg P > 0 \), from the equation (3.12), we see
\[
\deg P \leq \deg P - 1,
\]
which is a contradiction. Then \( P(z) \) must be a constant and since \( h(z) = Q(z) - P(z) \) is a constant, we deduce that both of \( P(z) \) and \( Q(z) \) is constant. This case is impossible too (the last case in Part (1)), and we deduced that \( h(z) \) cannot be a constant. Thus, the proof complete.

**Proof of the Theorem 1.10.** Setting \( g(z) = f(z) + b(z) - a(z) \), we can remark that
\[
g(z) - b(z) = f(z) - a(z),
\]
\[
\Delta^n g(z) - b(z) = \Delta^n f(z) - b(z),
\]
\[
\Delta^{n+1} g(z) - b(z) = \Delta^{n+1} f(z) - b(z), n \geq 2.
\]
Since \( f(z) - a(z), \Delta^n f(z) - b(z) \) and \( \Delta^{n+1} f(z) - b(z) \) share 0 CM, it follows that \( g(z), \Delta^n g(z) \) and \( \Delta^{n+1} g(z) \) share \( b(z) \) CM. By using Theorem 1.7, we deduce that \( \Delta^{n+1} g(z) = \Delta^n g(z) \), which leads to \( \Delta^{n+1} f(z) = \Delta^n f(z) \) and the proof complete.

**Proof of the Theorem 1.11.** Note that \( f(z) \) is a nonconstant entire function of finite order. Since \( f(z), \Delta^n f(z) \) and \( \Delta^{n+1} f(z) \) share 0 CM, it follows that
\[
\frac{\Delta^n f(z)}{f(z)} = e^{P(z)},
\]
\[
\frac{\Delta^{n+1} f(z)}{f(z)} = e^{Q(z)},
\]
where \( P \) and \( Q \) are polynomials. If \( Q - P \) is a constant, then we can get easily from (3.22) and (3.23)
\[
\Delta^{n+1} f(z) = e^{Q(z) - P(z)} \Delta^n f(z) \equiv C \Delta^n f(z).
\]
This completes the proof. If \( Q - P \) is a not constant, with a similar arguing as in the proof of Theorem 1.7, we can deduce that the case \( \deg P = \deg(Q - P) > 1 \) is impossible. For the case \( \deg P = \deg(Q - P) = 1 \), we can obtain that \( e^{P(z)} \) is periodic entire function with period \( c \). This together with (3.22) yields
\[
\Delta^{n+1} f(z) = e^{P(z)} \Delta^n f(z)
\]
which means that \( f(z), \Delta_c f(z) \) and \( \Delta_{n+1} f(z) \) share 0 CM. Thus, by Theorem 1.6 we obtain
\[
\Delta_{n+1} f(z) \equiv C \Delta_c f(z)
\]
which is a contradiction to (3.22) and \( \deg P = 1 \). Theorem 1.11 is thus proved. \( \square \)

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