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FIXED POINT RESULTS FOR GENERALIZED α-ψ-CONTRACTIONS IN METRIC-LIKE SPACES AND APPLICATIONS

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Abstract. In this article, we introduce the concept of generalized α-ψ-contraction in the context of metric-like spaces and establish some related fixed point theorems. As consequences, we obtain some known fixed point results in the literature. Some examples and an application on two-point boundary value problems for second order differential equation are also considered.

1. Introduction and preliminaries

The notion of metric-like (dislocated) metric spaces was introduced by Hitzler and Seda [9] in 2000 as a generalization of a metric space. They generalized the Banach Contraction Principle [5] in such spaces. Metric-like spaces were discovered by Amini-Harandi [4] who established some fixed point results. Very recently, Karapinar and Salimi [14] established some fixed point theorems for cyclic contractions in the setting of metric-like spaces. Many other (common) fixed point results in the context of metric-like (quasi) spaces have been proved, see for example [1, 2, 14, 15, 21, 25, 26].

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}_0^+$ and $\mathbb{N}^*$ will denote the set of real numbers, the set of nonnegative real numbers and the set of positive integer numbers, respectively.

Definition 1.1 ([4]). Let X be a nonempty set. A function $\sigma : X \times X \to \mathbb{R}_0^+$ is said to be a dislocated (metric-like) metric on X if for any $x, y, z \in X$, the following conditions hold:

(S1) $\sigma(x, y) = \sigma(x, x) = 0 \Longrightarrow x = y$;
(S2) $\sigma(x, y) = \sigma(y, x)$;
(S3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is then called a dislocated (metric-like) metric space.

Example 1.2. A trivial example of a metric-like space is the pair $(\mathbb{R}_0^+, \sigma)$, where $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined as $\sigma(x, y) = \max\{x, y\}$. Here, σ is also a partial metric [16].

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Example 1.3. Take $X = \mathbb{R}$ and define the σ metric-like as

$$\sigma(x,y) = \frac{|x-y|+|x|+|y|}{2} \quad \text{for all } x,y \in X.$$

Notice that σ is not a metric. Particularly, if $X = \mathbb{R}_0^+$, we have $\sigma(x,y) = \max\{x,y\}$ and so we return to Example 1.2. But, if $X = \mathbb{R}$, we have $\sigma(x,y) \neq \max\{x,y\}$.

As it is well known, a partial metric [16] is a metric-like. The converse is not true. The following example concerns this statement.

Example 1.4. Take $X = \{1,2,3\}$ and consider the metric-like $\sigma : X \times X \to \mathbb{R}_0^+$ given by

$$\sigma(1,1) = 0, \quad \sigma(2,2) = 1, \quad \sigma(3,3) = \frac{2}{3}, \quad \sigma(1,2) = \sigma(2,1) = \frac{9}{10},$$
$$\sigma(2,3) = \sigma(3,2) = \frac{4}{5}, \quad \sigma(1,3) = \sigma(3,1) = \frac{7}{10}.$$

Since $\sigma(2,2) \neq 0$, so σ is not a metric and since $\sigma(2,2) > \sigma(1,2)$, so σ is not a partial metric.

Each metric-like σ on X generates a T_0 topology τ_σ on X which has as a base the family open σ-balls $\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe that a sequence $\{x_n\}$ in a metric-like space (X,σ) converges to a point $x \in X$, with respect to τ_σ, if and only if $\sigma(x,x) = \lim_{n\to\infty} \sigma(x,x_n)$.

Definition 1.5 ([4])**.** Let (X,σ) be a metric-like space.

(a) A sequence $\{x_n\}$ in X is a Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite.

(b) (X,σ) is complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_σ to a point $x \in X$; that is,
$$\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m).$$

Definition 1.6 ([4])**.** Let (X,σ) be a metric-like space. A mapping $T : (X,\sigma) \to (X,\sigma)$ is continuous if for any sequence $\{x_n\}$ in X such that $\sigma(x_n,x) \to \sigma(x,x)$ as $n \to \infty$, we have $\sigma(Tx_n,Tx) \to \sigma(Tx,Tx)$ as $n \to \infty$.

Lemma 1.7 ([14])**.** *Let (X,σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ where $x \in X$ and $\sigma(x,x) = 0$. Then, for all $y \in X$, we have*
$$\lim_{n\to\infty} \sigma(x_n,y) = \sigma(x,y).$$

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$.

Note that if $\psi \in \Psi$, we have $\psi(t) < t$ for all $t > 0$.

In 2012, Samet et al [23] introduced the class of α-admissible mappings.

Definition 1.8. [23] *For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be given mappings. We say that T is α-admissible if for all $x,y \in X$, we have*
$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1. \tag{1.1}$$

The notion of $\alpha-\psi$-contractive mappings is also defined in the following way.

Definition 1.9 ([23])**.** Let (X,d) be a metric space and $T:X\to X$ be a given mapping. We say that T is a $\alpha-\psi$ contractive mapping if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y)),\ \text{for all } x,y\in X. \tag{1.2}$$

Many authors have proved fixed point results for generalized contractions using the function α, see for instance [3, 6, 7, 13]. Now, we state in the following definition a generalization of the notion of $\alpha-\psi$ contractive mappings in the context of a metric-like space.

Definition 1.10. Let (X,σ) be a metric-like space and $T:X\to X$ be a given mapping. We say that T is a generalized $\alpha-\psi$ contractive mapping of type A if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)\sigma(Tx,Ty)\le\psi(M(x,y)),\quad \text{for all } x,y\in X, \tag{1.3}$$

where

$$M(x,y)=\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}. \tag{1.4}$$

Our aim in this article is to provide some fixed point results for variant generalized $\alpha-\psi$ contractive mappings in the setting of metric-like spaces. We support our obtained theorems by some concrete examples and an application.

2. Main results

Our first fixed point result read as follows.

Theorem 2.1. *Let (X,σ) be a complete metric-like space and $T:X\to X$ be a generalized $\alpha-\psi$ contractive mapping of type A. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;*
(iii) *T is continuous.*

Then there exists a $u\in X$ such that $\sigma(u,u)=0$. Assume in addition that

(H1) *If $\sigma(x,x)=0$ for some $x\in X$, then $\alpha(x,x)\ge1$.*

Then such u is a fixed point of T, that is, $Tu=u$.

Proof. By assumption (ii), there exists a point $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$. We define a sequence $\{x_n\}$ in X by $x_{n+1}=Tx_n=T^{n+1}x_0$ for all $n\ge0$. Suppose that $x_{n_0}=x_{n_0+1}$ for some n_0. So the proof is completed since $u=x_{n_0}=x_{n_0+1}=Tx_{n_0}=Tu$. Consequently, throughout the proof, we assume that

$$x_n\ne x_{n+1}\quad \text{for all } n. \tag{2.1}$$

Observe that

$$\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\ge1\Rightarrow\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\ge1,$$

since T is α-admissible. By repeating the process above, we derive that

$$\alpha(x_n,x_{n+1})\ge1,\quad \text{for all } n=0,1,\dots. \tag{2.2}$$

Step 1: We shall prove that

$$\lim_{n\to\infty}\sigma(x_n,x_{n+1})=0. \tag{2.3}$$

Combining (1.3) and 2.2, we find that

$$\sigma(x_n,x_{n+1})=\sigma(Tx_{n-1},Tx_n)\le\alpha(x_{n-1},x_n)\sigma(Tx_{n-1},Tx_n)\le\psi(M(x_{n-1},x_n)), \tag{2.4}$$

for all $n\ge 1$, where

$$\begin{aligned}
&M(x_{n-1},x_n)\\
&=\max\{\sigma(x_{n-1},x_n),\sigma(x_{n-1},Tx_{n-1}),\sigma(x_n,Tx_n),\frac{\sigma(x_{n-1},Tx_n)+\sigma(x_n,Tx_{n-1})}{4}\}\\
&=\max\{\sigma(x_{n-1},x_n),\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1}),\frac{\sigma(x_{n-1},x_{n+1})+\sigma(x_n,x_n)}{4}\}\\
&\le\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1}),\frac{\sigma(x_{n-1},x_n)+3\sigma(x_n,x_{n+1})}{4}\}\\
&=\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}.
\end{aligned} \tag{2.5}$$

If for some n, $\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}=\sigma(x_n,x_{n+1})(\neq 0)$, then (2.4) and (2.5) turn into

$$\sigma(x_n,x_{n+1})\le\psi(M(x_{n-1},x_n))\le\psi(\sigma(x_n,x_{n+1}))<\sigma(x_n,x_{n+1}),$$

which is a contradiction. Hence, $\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}=\sigma(x_{n-1},x_n)$ for all $n\in\mathbb{N}^*$ and (2.4) becomes

$$\sigma(x_n,x_{n+1})\le\psi(\sigma(x_{n-1},x_n))\quad\text{for all } n\ge 1. \tag{2.6}$$

This yields

$$\sigma(x_n,x_{n+1})<\sigma(x_{n-1},x_n)\quad\text{for all } n\ge 1. \tag{2.7}$$

By (2.6), we find that

$$\sigma(x_n,x_{n+1})\le\psi^n(\sigma(x_0,x_1)),\quad\text{for all } n\in\mathbb{N}. \tag{2.8}$$

By the properties of ψ, we have

$$\lim_{n\to\infty}\sigma(x_n,x_{n+1})=0.$$

Step 2: We shall prove that $\{x_n\}$ is a Cauchy sequence. First, by using (S3) and (2.8)

$$\begin{aligned}
\sigma(x_n,x_{n+k})&\le\sigma(x_n,x_{n+1})+\sigma(x_{n+1},x_{n+2})+\ldots+\sigma(x_{n+k-1},x_{n+k})\\
&\le\sum_{p=n}^{n+k-1}\psi^p(\sigma(x_0,x_1))\\
&\le\sum_{p=n}^{+\infty}\psi^p(\sigma(x_0,x_1))\to 0\quad\text{as } n\to\infty.
\end{aligned} \tag{2.9}$$

Thus, by the symmetry of σ, we obtain

$$\lim_{n,m\to\infty}\sigma(x_n,x_m)=0. \tag{2.10}$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X,σ). Since (X,σ) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, u) = \sigma(u,u) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.11}$$

Since T is continuous, from (2.11) we obtain that

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \lim_{n\to\infty} \sigma(Tx_n, Tu) = \sigma(Tu, Tu), \tag{2.12}$$

On the other hand, by (2.11) and Lemma 1.7

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tu) = \sigma(u, Tu). \tag{2.13}$$

Comparing (2.12) and (2.13), we get $\sigma(u,Tu) = \sigma(Tu,Tu)$. By (1.3),

$$\alpha(u,u)\sigma(Tu,Tu) \le \psi(M(u,u)),$$

where

$$\begin{aligned} M(u,u) &= \max\{\sigma(u,u), \sigma(u,Tu), \sigma(u,Tu), \frac{\sigma(u,Tu)+\sigma(u,Tu)}{4}\} \\ &= \max\{0, \sigma(u,Tu)\} = \sigma(u,Tu). \end{aligned}$$

From hypothesis (H1) and the fact that $\sigma(u,u)=0$, we have $\alpha(u,u) \ge 1$. Therefore, by (1.3)

$$\sigma(u,Tu) \le \alpha(u,u)\sigma(u,Tu) \le \psi(\sigma(u,Tu)),$$

which holds unless $\sigma(u,Tu)=0$, that is $Tu=u$. So u is a fixed point of T. □

Theorem 2.1 remains true if we replace the continuity hypothesis by the following property:

> If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.

This statement is given as follows.

Theorem 2.2. *Let (X,d) be a complete metric-like space and $T: X \to X$ be a generalized $\alpha-\psi$ contractive mapping of type A. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.*

Then, there exists $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem 2.1, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \ge 0$ is Cauchy in (X,σ) and converges to some $u \in X$. Also, (2.11)) holds, so

$$\lim_{k\to\infty} \sigma(x_{n(k)+1}, Tu) = \sigma(u, Tu). \tag{2.14}$$

We shall show that $Tu = u$. Suppose, on the contrary, that $Tu \ne u$, i.e, $\sigma(Tu,u) > 0$. From (2.2) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \ge 1$ for all k.

By applying (1.3), we obtain

$$\sigma(x_{n(k)+1}, Tu) \le \alpha(x_{n(k)}, u)\sigma(Tx_{n(k)}, Tu) \le \psi(M(x_{n(k)}, u)) \tag{2.15}$$

where

$$
\begin{aligned}
&M(x_{n(k)},u)\\
&=\max\{\sigma(x_{n(k)},u),\sigma(x_{n(k)},Tx_{n(k)}),\sigma(u,Tu),\frac{\sigma(x_{n(k)},Tu)+\sigma(u,Tx_{n(k)})}{4}\}\\
&=\max\{\sigma(x_{n(k)},u),\sigma(x_{n(k)},x_{n(k)+1}),\sigma(u,Tu),\frac{\sigma(x_{n(k)},Tu)+\sigma(u,x_{n(k)+1})}{4}\}.
\end{aligned}
\tag{2.16}
$$

By (2.3) and (2.14), we have

$$\lim_{k\to\infty} M(x_{n(k)},u)=\sigma(u,Tu). \tag{2.17}$$

Letting $k\to\infty$ in (2.15))

$$\sigma(u,Tu)\le\psi(\sigma(u,Tu))<\sigma(u,Tu), \tag{2.18}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T, that is, $Tu=u$. □

Definition 2.3. Let (X,σ) be a metric-like space and $T:X\to X$ be a given mapping. We say that T is a generalized $\alpha-\psi$ contractive mapping of type B if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)\sigma(Tx,Ty)\le\psi(M_0(x,y)),\ \text{for all }\ x,y\in X, \tag{2.19}$$

where

$$M_0(x,y)=\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}. \tag{2.20}$$

Theorem 2.4. *Let (X,d) be a complete metric-like space and $T:X\to X$ be a generalized $\alpha-\psi$ contractive mapping of type B. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$;*
(iii) *T is continuous.*

Then there exists a $u\in X$ such that $\sigma(u,u)=0$. If in addition (H1) *holds, then such u is a fixed point of T, that is, $Tu=u$.*

Proof. Along the lines of the proof of Theorem 2.1, we get the desired result. Because of the analogy, we skip the details of the proof. □

Theorem 2.5. *Let (X,d) be a complete metric-like space and $T:X\to X$ be a generalized $\alpha-\psi$ contractive mapping of type B. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\ge1$ for all k.*

Then, there exists $u\in X$ such that $Tu=u$.

We omit the proof because of the similarity to Theorem 2.2.

Definition 2.6. Let (X,σ) be a metric-like space and $T:X\to X$ be a given mapping. We say that T is a $\alpha-\psi$ contractive mapping if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)\sigma(Tx,Ty)\le\psi(\sigma(x,y)),\quad \text{for all } x,y\in X, \tag{2.21}$$

Theorem 2.7. *Let (X,d) be a complete metric-like space and $T : X \to X$ be a $\alpha - \psi$ contractive mapping. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then there exists a $u \in X$ such that $\sigma(u,u) = 0$. If in addition (H1) *holds, then such u is a fixed point of T, that is, $Tu = u$.*

The above theorem is a simple consequence of Theorem 2.1.

Theorem 2.8. *Let (X,d) be a complete metric-like space and $T : X \to X$ be a $\alpha - \psi$ contractive mapping. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then, there exists $u \in X$ such that $Tu = u$.

The above theorem follows from Theorem 2.2.

3. Consequences of the main results

In the following, we present some illustrated consequences of our obtained results given by Theorem 2.1 and Theorem 2.2.

3.1. Standard fixed point results in metric-like spaces.

Corollary 3.1. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be such that*

$$\sigma(Tx, Ty) \leq \psi(M(x,y)) \quad \text{for all } x, y \in X$$

where $M(x,y)$ is defined by (1.4)*. Then, T has a fixed point.*

To prove the above corollary it suffices to take $\alpha(x,y) = 1$ in Theorem 2.2.

Corollary 3.2. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be such that*

$$\sigma(Tx, Ty) \leq \lambda\, M(x,y) \quad \text{for all } x, y \in X$$

where $\lambda \in [0,1)$. Then, T has a fixed point.

Proof. To prove the above corollary it suffices to take $\psi(t) = \lambda t$ in Corollary 3.1. □

Corollary 3.3. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be such that*

$$\sigma(Tx, Ty) \leq \psi(M_0(x,y)) \quad \text{for all } x, y \in X$$

where $M_0(x,y)$ is defined by (2.20)*. Then, T has a fixed point.*

To prove the above corollary it suffices to take $\alpha(x,y) = 1$ in Theorem 2.4.

Corollary 3.4. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be such that*

$$\sigma(Tx, Ty) \leq \lambda\, M_0(x,y) \quad \text{for all } x, y \in X$$

where $\lambda \in [0,1)$. Then, T has a fixed point.

To prove the above corollary it suffices to take $\psi(t) = \lambda t$ in Corollary 3.3.

Corollary 3.5. *Let (X, σ) be a complete metric-like space and $T : X \to X$ be such that*
$$\sigma(Tx, Ty) \le \psi(\sigma(x, y)) \quad \textit{for all } x, y \in X.$$
Then, T has a fixed point.

To prove the above corollary it suffices to take $\alpha(x, y) = 1$ in Theorem 2.8.

Corollary 3.6. *Let (X, σ) be a complete metric-like space and $T : X \to X$ be such that*
$$\sigma(Tx, Ty) \le \lambda\, \sigma(x, y) \quad \textit{for all } x, y \in X$$
where $\lambda \in [0, 1)$. Then, T has a fixed point.

To prove the above corollary it suffices to take $\psi(t) = \lambda t$ in Corollary 3.5.

3.2. **Standard fixed point results in partial metric spaces.** The partial metric spaces were introduced by Matthews [16] as a part of the study of denotational semantics of data for networks.

Definition 3.7 ([16])**.** A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:

- (P1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$,
- (P2) $p(x, x) \le p(x, y)$,
- (P3) $p(x, y) = p(y, x)$,
- (P4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}_0^+$ given by
$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{3.1}$$
is a metric on X.

Lemma 3.8. *Let (X, p) be a partial metric space. Then, (a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p), (b) X is complete if and only if the metric space (X, d_p) is complete.*

Corollary 3.9. *Let (X, σ) be a complete partial space and $T : (X, p) \to (X, p)$ be such that*
$$\alpha(x, y) p(Tx, Ty) \le \psi(N(x, y)) \quad \textit{for all } x, y \in X$$
where $N(x, y)$ is defined as
$$N(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$
Suppose that

- (i) *T is α-admissible;*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
- (iii) *T is continuous.*

Then there exists a $u \in X$ such that $p(u, u) = 0$. If in addition (H1) *holds, then such u is a fixed point of T, that is, $Tu = u$.*

Proof. It suffices to replace the metric-like σ in Theorem 2.1 by the partial metric p which itself a metric-like. Note that we considered in $N(x,y)$ the fourth term $\frac{p(x,Ty)+p(y,Tx)}{2}$ instead of $\frac{p(x,Ty)+p(y,Tx)}{4}$ due to the inequality $p(x,x) \leq p(x,y)$. Its proof is evident. $\square$

Similar to Corollary 3.9, from Theorem 2.2 we deduce the following result.

Corollary 3.10. *Let (X,σ) be a complete partial space and $T:(X,p)\to(X,p)$ be such that*
$$\alpha(x,y)p(Tx,Ty) \leq \psi(N(x,y)) \quad \text{for all } x,y \in X.$$
Suppose that

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \geq 1$ for all k.*

Then, there exists $u \in X$ such that $Tu = u$.

Remark 3.11. It is clear that one can easily state the analog of Theorem 2.4, Theorem 2.5, Theorem 2.7 and Theorem 2.8 in the setting of partial metric spaces.

3.3. **Fixed point results with a partial order.** The study of the existence of fixed points on metric spaces endowed with a partial order can be considered as one of the very interesting improvements in the field of fixed point theory. This trend was initiated by Turinici [24] in 1986, but it became one of the core research subject after the publications of Ran and Reurings in [20] and Nieto and Rodríguez-López [17].

Definition 3.12. Let $(X,\preceq)$ be a partially ordered set and $T:X\to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if
$$x,y \in X,\ x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 3.13. Let $(X,\preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 3.14. *Let $(X,\preceq)$ be a partially ordered set and σ be a metric-like on X. We say that $(X,\preceq,\sigma)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Corollary 3.15. *Let $(X,\preceq)$ be a partially ordered set and σ be a metric-like on X such that (X,σ) is complete. Let $T:X\to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*
$$\sigma(Tx,Ty) \leq \psi(M(x,y)),$$
for all $x,y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
(ii) *(T is continuous and the property (H) holds) or ($(X,\preceq,\sigma)$ is regular).*

Then, T has a fixed point.

Proof. Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, T is a generalized $\alpha - \psi$ contractive mapping of type A; that is,

$$\alpha(x,y)\sigma(Tx,Ty) \le \psi(M(x,y)),$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \ge 1$. Moreover, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x,y) \ge 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx,Ty) \ge 1.$$

Thus, T is α-admissible. Now, if T is continuous and the hypothesis (H1) holds, the existence of a fixed point follows from Theorem 2.1. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies from the definition of α that $\alpha(x_{n(k)}, x) \ge 1$ for all k. In this case, the existence of a fixed point follows from Theorem 2.2. □

Remark 3.16. Notice that we may obtain the analog of Theorem 2.4, Theorem 2.5, Theorem 2.7, Theorem 2.8 and the results of Subsection 3.1 and Subsection 3.2 in the setting of partially ordered metric-like spaces.

3.4. **Fixed point results for cyclic contractions.** Kirk, Srinivasan and Veeramani [12] proved very interesting generalizations of the Banach Contraction Mapping Principle by introducing a cyclic contraction. This remarkable paper [12] has been appreciated by many several researchers (see, for example, [10, 11, 18, 19, 22] and the related reference therein). In this subsection, we derive some fixed point theorems for cyclic contractive mappings in the setting of metric-like spaces.

Corollary 3.17. *Let $\{A_i\}_{i=1}^2$ be nonempty closed subsets of a complete metric-like space (X,σ) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) *$T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;*
(II) *there exists a function $\psi \in \Psi$ such that*

$$\sigma(Tx,Ty) \le \psi(M(x,y)), \text{ for all } (x,y) \in A_1 \times A_2.$$

Then T has a fixed point that belongs to $A_1 \cap A_2$.

Proof. Since A_1 and A_2 are closed subsets of the complete metric-like space (X,σ), then (Y,σ) is complete. Define the mapping $\alpha : Y \times Y \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 & \text{otherwise.} \end{cases}$$

From (II) and the definition of α, we can write

$$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)),$$

for all $x, y \in Y$. Thus T is a generalized $\alpha - \psi$ contractive mapping of type A. Let $(x,y) \in Y \times Y$ such that $\alpha(x,y) \ge 1$.

If $(x,y) \in A_1 \times A_2$, from (I), $(Tx,Ty) \in A_2 \times A_1$, which implies that $\alpha(Tx,Ty) \ge 1$.

If $(x, y) \in A_2 \times A_1$, from (I), $(Tx, Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx, Ty) \geq 1$.

Hence, in all cases, we conclude that $\alpha(Tx, Ty) \geq 1$ which yields that that T is α-admissible.

Notice also that, from (I), for any $a \in A_1$, we have $(a, Ta) \in A_1 \times A_2$, which implies that $\alpha(a, Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. This implies from the definition of α that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \quad \text{for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the metric-like σ, we get that

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. Thus we get immediately from the definition of α that $\alpha(x_n, x) \geq 1$ for all n.

Now, all the hypotheses of Theorem 2.2 are satisfied and T has a fixed point in Y. $\square$

Note that Corollary 3.17 is a generalization of [14, Corollary 1.10].

4. Examples

We present the following two concrete examples to support our results.

Example 4.1. Consider $X = \{0, 1, 2\}$. Take the metric-like $\sigma : X \times X \to \mathbb{R}_0^+$ defined by

$$\begin{gathered}\sigma(0,0) = \sigma(1,1) = 0, \quad \sigma(2,2) = \frac{9}{20}, \\ \sigma(0,2) = \sigma(2,0) = \frac{2}{5}, \quad \sigma(1,2) = \sigma(2,1) = \frac{3}{5}, \\ \sigma(0,1) = \sigma(1,0) = \frac{1}{2}.\end{gathered}$$

Note that $\sigma(2,2) \neq 0$, so σ is not a metric and $\sigma(2,2) > \sigma(0,2)$, so σ is not a partial metric. Clearly, (X, σ) is a complete metric-like space. Given $T : X \to X$ as $T0 = T1 = 0$ and $T2 = 1$. Take $\psi(t) = 5t/6$ for each $t \geq 0$. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

First, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. By the definition of α, this implies that $x = 0$ and since $T0 = 0$, so $\alpha(Tx, Ty) = 1$ for each $y \in X$, that is, T is α-admissible. We distinguish two cases:

Case 1: If ($x = 0$ and $y = 0$) or ($x = 0$ and $y = 1$), we have

$$\alpha(Tx, Ty)\sigma(Tx, Ty) = \sigma(Tx, Ty) = 0.$$

Case 2: If $x = 0$ and $y = 2$, we have

$$\begin{aligned}\alpha(Tx, Ty)\sigma(Tx, Ty) = \sigma(Tx, Ty) = \sigma(0, 1) = \frac{1}{2} = \frac{5}{6}\sigma(2, 1) \\ = \psi(\sigma(y, Ty))\end{aligned}$$

$$\leq \psi(M(x,y))$$

where $M(x,y)$ is defined by (1.4). It is also obvious that hypothesis (iii) of Theorem 2.2 is satisfied. Thus, we map apply Theorem 2.2 and so T has a fixed point, which is $u = 0$.

Example 4.2. Let $X = [0,\infty)$ be endowed with the metric-like σ given as $\sigma(x,y) = \max\{x,y\}$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}x^2 & \text{if } x \in [0,1], \\ 3x - 1 & \text{otherwise.} \end{cases}$$

Consider $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \leq t < 1, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Obviously, $\psi \in \Psi$. Consider $\alpha : X \times X \to [0,\infty)$ as

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

First, let $x, y \in X$ such that $\alpha(x,y) \geq 1$, so $x, y \in [0,1]$. In this case,

$$\alpha(Tx,Ty) = \alpha(\frac{1}{2}x^2, \frac{1}{2}y^2) = 1;$$

that is, T is α-admissible. Here we also have

$$\begin{aligned} \alpha(Tx,Ty)\sigma(Tx,Ty) &= \sigma(Tx,Ty) = \sigma(\frac{1}{2}x^2, \frac{1}{2}y^2) \\ &= \sigma(\psi(x),\psi(y)) = \max(\psi(x),\psi(y)) \\ &= \psi(\max(x,y)) = \psi(\sigma(x,y)) \\ &\leq \psi(M(x,y)). \end{aligned}$$

Note that hypothesis (iii) of Theorem 2.2 is also satisfied. Applying Theorem 2.2, T has a fixed point in X, which is $u = 0$.

5. Applications

Here, we consider the following two-point boundary-value problem for the second-order differential equation

$$\begin{gathered} -\frac{d^2x}{dt^2} = f(t,x(t)), \quad t \in [0,1] \\ x(0) = x(1) = 0, \end{gathered} \tag{5.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Recall that the Green's function associated to (5.1) is

$$G(t,s) = \begin{cases} t(1-s) & \text{if } 0 \leq t \leq s \leq 1 \\ s(1-t) & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{5.2}$$

Let $X = \mathcal{C}(I)(I = [0,1])$ be the space of all continuous functions defined on I. We consider on X, the metric-like σ given by

$$\sigma(x,y) = \|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty \quad \text{for all } x, y \in X,$$

where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$ for each $u \in X$.

Note that σ is also a partial metric on X and since

$$d_\sigma(x, y) := 2\sigma(x, y) - \sigma(x, x) - \sigma(y, y) = 2\|x - y\|_\infty,$$

so by Lemma 3.8, (X, σ) is complete since the metric space $(X, \|\cdot\|_\infty)$ is complete.

It is well known that $x \in C^2(I)$ is a solution of (5.1) is equivalent to that $x \in X = C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t, s)f(s, x(s))ds, \quad \text{for all } t \in I. \tag{5.3}$$

Theorem 5.1. *Suppose the following conditions hold:*

- *there exists a continuous function $p : I \to \mathbb{R}_0^+$ such that*
$$|f(s, a) - f(s, b)| \le 8\, p(s)\, |a - b|,$$
for each $s \in I$ and $a, b \in \mathbb{R}$;
- *there exists a continuous function $q : I \to \mathbb{R}_0^+$ such that*
$$|f(s, a)| \le 8\, q(s)\, |a|,$$
for each $s \in I$ and $a \in \mathbb{R}$;
- $\sup_{s\in I} p(s) = \lambda_1 < \frac{1}{3}$;
- $\sup_{s\in I} q(s) = \lambda_2 < \frac{1}{3}$.

Then problem 5.1 has a solution $u \in X = \mathcal{C}(I, \mathbb{R})$.

Proof. Consider the mapping $T : X \to X$ defined by

$$Tx(t) = \int_0^1 G(t, s)f(s, x(s))ds.$$

for all $x \in X$ and $t \in I$. Then, problem (5.1) is equivalent to finding $u \in X$ that is a fixed point of T.

Now, let $x, y \in X$. We have

$$\begin{aligned}
|Tx(t) - Ty(t)| &= |\int_0^1 G(t, s)f(s, x(s))ds - \int_0^1 G(t, s)f(s, y(s))ds| \\
&\le \int_0^1 G(t, s)|f(s, x(s)) - f(s, y(s))|\, ds \\
&\le 8 \int_0^1 G(t, s)p(s)\, |x(s) - y(s)|\, ds \\
&\le 8\lambda_1 \|x - y\|_\infty \sup_{t\in I} \int_0^1 G(t, s)\, ds \\
&= \lambda_1 \|x - y\|_\infty.
\end{aligned}$$

In the above equality, we used that for each $t \in I$, we have $\int_0^1 G(t, s)\, ds = -\frac{t^2}{2} + \frac{t}{2}$, and so $\sup_{t\in I} \int_0^1 G(t, s)\, ds = \frac{1}{8}$. Therefore,

$$\|Tx - Ty\|_\infty \le \lambda_1 \|x - y\|_\infty. \tag{5.4}$$

Again, we have

$$|Tx(t)| = |\int_0^1 G(t, s)f(s, x(s))ds|$$

$$
\begin{aligned}
&\le \int_0^1 G(t,s)\,|f(s,x(s))|\,ds\\
&\le 8\int_0^1 G(t,s)\,q(s)\,|x(s)|\,ds\\
&\le 8\lambda_2\|x\|_\infty \sup_{t\in I}\int_0^1 G(t,s)\,ds\\
&\le \lambda_2\|x\|_\infty.
\end{aligned}
$$

Thus

$$\|Tx\|_\infty \le \lambda_2\|x\|_\infty. \tag{5.5}$$

Proceeding similarly,

$$\|Ty\|_\infty \le \lambda_2\|y\|_\infty. \tag{5.6}$$

Take $\lambda = \lambda_1 + 2\lambda_2$. Under assumptions in Theorem 5.1, we have $\lambda < 1$. Summing (5.4) to (5.6), we find

$$
\begin{aligned}
\sigma(Tx,Ty) &= \|Tx-Ty\|_\infty + \|Tx\|_\infty + \|Ty\|_\infty\\
&\le \lambda_1\|x-y\|_\infty + \lambda_2\|x\|_\infty + \lambda_2\|y\|_\infty\\
&\le (\lambda_1+2\lambda_2)(\|x-y\|_\infty + \|x\|_\infty + \|y\|_\infty)\\
&= \lambda\sigma(x,y) \le \lambda M(x,y).
\end{aligned}
$$

So all hypotheses of Corollary 3.2 are satisfied, and so T has a fixed point $u \in X$, that is, the problem (5.1) has a solution $u \in C^2(I)$. $\square$

Conclusion. All fixed point results presented in this article are also valid for metric spaces. Consequently, our results extend and unify several results from the literature.

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