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UNILATERAL PROBLEMS FOR THE KLEIN-GORDON OPERATOR WITH NONLINEARITY OF KIRCHHOFF-CARRIER TYPE

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Abstract. This work concerns the unilateral problem for the Klein-Gordon operator

$$\mathbb{L} = \frac{\partial^2 u}{\partial t^2} - M(|\nabla u|^2)\Delta u + M_1(|u|^2)u - f.$$

Using an appropriate penalization, we obtain a variational inequality for a perturbed equation, and then show the existence and uniqueness of solutions.

1. Introduction

The one-dimensional nonlinear equation of motion of an elastic string of the length L (1.1) was proposed by Kirchhoff [12], in connection with some problems in nonlinear elasticity, and later rediscovered by Carrier [6],

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{2mL}\int_0^L \big(\frac{\partial u}{\partial x}\big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where τ_0 is the initial tension, m the mass of the string and k the Young's modulus of the material of the string. This model describes small vibrations of a stretched string when only the transverse component of the tension is considered, and for mathematical aspects of (1.1) see Bernstein [5] and Dickey [8].

Model (1.1) is a generalization of the linearized problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m}\frac{\partial^2 u}{\partial x^2} = 0,$$

obtained by d'Alembert and Euler. A particular case of (1.1) can be written, in general, as

$$\frac{\partial^2 u}{\partial t^2} - M\Big(\int_\Omega |\nabla u(x,t)|^2 dx\Big)\Delta u = 0, \tag{1.2}$$

or

$$\frac{\partial^2 u}{\partial t^2} + M\left(\|u(t)\|^2\right) Au = 0, \tag{1.3}$$

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in operator notation, where we consider the Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$, where V' is the dual of V with the immersions continuous and dense. By $\|\cdot\|$ we denote the norm in V and $A : V \to V'$ a bounded linear operator.

Problem (1.3) is called nonlocal because of the presence of the term

$$M(\|u(t)\|^2) = M\Big(\int_\Omega |\nabla u(x,t)|^2 dx\Big),$$

which implies that the equation is no longer a pointwise identity. The nonlocal term provokes some mathematical difficulties which makes the study of such a problem particulary interesting. On this subject, see a interesting work of Arosio-Panizzi [2], where was proved that (1.3) is well-posedness in the Hadamard sense (existence, uniqueness and continuous dependence of the local solution upon the initial data) in Sobolev spaces.

Nonlocal initial boundary value problems are important from the point of view of their practical application to modeling and investigation of various phenomena. For the last several decades, various types of equations have been employed as some mathematical model describing physical, chemical, biological and ecological systems. See for example the nonlocal reaction-diffusion system given in Raposo et all [19].

When we assume that $M : [0, \infty) \to \mathbb{R}$ real function, $M(\lambda) \geq m_0 > 0$, $M \in C^1(0, \infty)$, Pohozhaev [18] proved that the mixed problem for (1.2) has global solution in t when the initial data $u(x, 0), u_t(x, 0)$ are restricted the class of functions called Pohozhaev's Class. This result can also be found in Lions [15] to the operator given in (1.3), that was also analyzed by Arosio-Spagnolo [3] and Hazoya-Yamada [10] when $M(\lambda) \geq 0$ and many other authors, for example, Arosio-Espagnolo [3], Dickey [8], Hazoya-Yamada [10] and Medeiros-Límaco-Menezes [16].

Let Ω be a bounded and open set of $\mathbb{R}^n$, with smooth boundary Γ, and let T be a positive real number. Let $Q = \Omega \times]0, T[$ be the cylinder with lateral boundary $\Sigma = \Gamma \times]0, T[$. A unilateral mixed problem associated with a nonlinear perturbation

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2} - M(|\nabla u|^2)\Delta u + \theta \geq f, \quad \text{in } Q, \\
\theta_t - \Delta\theta + u' \geq g, \quad \text{in } Q, \\
u = \theta = 0 \quad \text{in } \Sigma, \\
u(0) = u_0; \quad u_t(0) = u_1; \quad \theta(0) = \theta_0,
\end{gathered}$$

where f, g, M are given real-valued functions with M positive, was studied by Clark-Lima in [7], where was proved existence and uniqueness of solution.

In this subject, we consider Ω a bounded open set of $\mathbb{R}^n$. A nonlinear perturbation of the problem (1.3), is given by

$$\rho\frac{\partial^2 u}{\partial t^2} + M(\|u(t)\|^2)Au \geq f,$$

where $\rho : \overline{\Omega} \times (0, T) \to \mathbb{R}$ and $f : \Omega \times (0, T) \to \mathbb{R}$ are real functions. The unilateral problem associated with this nonlinear perturbation was studied in Frota-Lar'kin [9] without geometrical restrictions and ρ a positive function.

In the case where ρ is a constant function equal to one, Medeiros-Milla [17] proved the local existence and uniqueness theorem in non-degenerated case. Lar'kin-Medeiros [13] under condition $M(\lambda) \geq m_0 > 0$ for all $\lambda \geq 0$ under Ω being a square $(0,1)\times(0,1)\subset\mathbb{R}^2$, showed the existence and uniqueness of a global solution theorem.

Unilateral problem is very interesting too, because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problem in elasticity and finite element method see Kikuchi-Oden [11] and reference there in. For Contact Problem Viscoelastic Materials see Rivera-Oquendo [20]. For dynamic contact problems with friction, for example problems involving unilateral contact with dry friction of Coulomb, see Ballard-Basseville [4].

For Ω be a bounded and open set of $\mathbb{R}^n$, with smooth boundary Γ, consider the Cauchy problem associated with the Klein-Gordon operator

$$\frac{\partial^2 u}{\partial t^2} - M(|\nabla u|^2)\Delta u + M_1(|u|^2)u = f,$$

with initial data

$$\begin{gathered} u(t) = u_0 \in H_0^1(\Omega)\cap H^2(\Omega) \\ u'(t) = u_1 \in H_0^1(\Omega), \\ u = 0 \quad \text{on } \Gamma, \end{gathered} \tag{1.4}$$

and $f\in L^2(0,T;H_0^1(\Omega))$, where

$$\begin{gathered} M, M_1 \in C^1([0,\infty);\mathbb{R}), \\ M(s)\geq m_0>0, \quad \forall s\in[0,\infty), \\ M_1(s)\geq m_1>0, \quad \forall s\in[0,\infty). \end{gathered} \tag{1.5}$$

For the problem above, the existence and uniqueness of solution was proved in [14] where the abstract model was considered.

Motivated by the problem (1.4)-(1.5) this work deals with a unilateral problem associated with the perturbed operator type Klein-Gordon

$$\frac{\partial^2 u}{\partial t^2} - M(|\nabla u|^2)\Delta u + M_1(|u|^2)u \geq f.$$

More precisely, here we study a unilateral problem, i.e. a variational inequality, see Lions [15], for the operator $\mathbb{L}$ under standard hypothesis on f, f', u_0 and u_1. Making use of the penalty method and Galerkin's approximations, we prove the existence and uniqueness of solution.

This work is organized as follows. In the section 2 we introduce the notation and functional spaces, we use the classical theory of Sobolev spaces as in Adams [1]. In the section 3 we present the main Theorem. In the section 4 prove the theorem of existence of solution and finally in the section 5 we prove the uniqueness of solution.

2. Notation and functional spaces

Let $T>0$ be a real number, Ω a bounded open set of $\mathbb{R}^n$ with boundary Γ regular and the cylinder $Q=\Omega\times]0,T[$ with lateral boundary $\Sigma=\Gamma\times]0,T[$.

We propose the variational inequality

$$u'' - M(|\nabla u|^2)\Delta u + M_1(|u|^2)u \geq f \quad \text{in } Q. \tag{2.1}$$

This inequality is satisfied by the unknown; that is, we formulate our problem as follows. Let $K=\{v\in L^2(\Omega); v\geq 0 \text{ a. e. in } \Omega\}$ be a closed and convex subset of

$H_0^1(\Omega) \cap L^2(\Omega)$, the variational problem consists in find the solution $u = u(x,t)$ satisfying

$$\int_Q (u'' - M(|\nabla u|^2)\Delta u + M_1(|u|^2)u - f)\,(v - u')\,dx\,dt \geq 0, \quad \forall v \in K,$$

with $u'(x,t) \in K$, a. e. on $[0,T]$ and taking the initial and boundary values

$$\begin{gathered} u = 0 \text{ on } \Sigma, \\ u' = 0 \text{ on } \Sigma, \\ u(x,0) = u_0(x), u'(x,0) = u_1(x) \quad \text{in } \Omega. \end{gathered} \tag{2.2}$$

To study the existence an uniqueness of solutions for the Problem (2.1)-(2.2), we consider the following hypothesis

- (H1) $M, M_1 \in C^1([0,\infty);\mathbb{R})$,
- (H2) $M(s) \geq m_0 > 0$ for all $s \in [0,\infty)$,
- (H3) $M_1(s) \geq m_1 > 0$ for all $s \in [0,\infty)$.

The notation for the functional spaces are contained in Lions [15]. We denote the inner product and norm in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively, by

$$((u,v)) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\,dx, \quad \|u\|^2 = \sum_{i=1}^n \int_\Omega \big(\frac{\partial u}{\partial x_i}(x)\big)^2 dx,$$
$$(u,v)_= \int_\Omega u(x)v(x)\,dx, \quad |u|^2 = \int_\Omega |u(x)|^2\,dx.$$

We introduce the bilinear form

$$a(u,v) = \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\,dx = ((u,v)) \quad \forall v \in H_0^1(\Omega). \tag{2.3}$$

By $\langle \cdot,\cdot \rangle$ we denote the duality V and V', where V' is the topological dual of the space V.

3. Existence and uniqueness of weak solution

For the rest of this article, C denotes various positive constants. Next, we present the main results of this paper.

Theorem 3.1. *If*

$$f \in L^2(0,T;H_0^1(\Omega)), \quad f' \in L^2(0,T;L^2(\Omega)), \tag{3.1}$$
$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad u_1 \in H_0^1(\Omega) \cap K, \tag{3.2}$$

and hypothesis (H1)–(H3) *hold, then there exists $T_0 < T$ and a unique function u such that*

$$u \in L^\infty(0,T_0;H_0^1(\Omega) \cap H^2(\Omega)), \tag{3.3}$$
$$u' \in L^\infty(0,T_0;H_0^1(\Omega)), \quad u'(t) \in K\ \forall t \in [0,T], \tag{3.4}$$
$$u'' \in L^\infty(0,T_0;L^2(\Omega)), \tag{3.5}$$

satisfying

$$
\begin{aligned}
&\int_0^T (u'', v-u')dt + \int_0^T a\left(M(|\nabla u|^2)u, v-u'\right)dt \\
&+ \int_0^T (M_1(|u|^2)u, v-u')dt \qquad (3.6)\\
&\geq \int_0^T (f, v-u')dt \quad \forall v \in K, \ \textit{a.e. in } t,
\end{aligned}
$$

$$
u(0) = u_0, \quad u'(0) = u_1, \qquad (3.7)
$$

where $a(M(|\nabla u|^2)u, v-u') = ((M(|\nabla u|^2)u, v-u')) = -(M(|\nabla u|^2)\Delta u, v-u')$.

The proof of Theorem 3.1 is made by the penalty method. The method consists in to consider a perturbation of the operator $\mathbb{L}$ with adding singular term, called penalization, depending on a parameter $\epsilon > 0$. We solve the mixed problem in Q for the penalized operator and the estimates obtained for the local solution of the penalized equation that allow to pass to the limit, when $\epsilon > 0$, in order to obtain a function u which is the solution of our problem.

First of all, let us consider the penalty operators $\beta : L^2(\Omega) \to L^2(\Omega)$ associated to the closed convex sets K, cf. Lions [15, p. 370]. The operator β is monotonous, hemicontinuous, takes bounded sets of $L^2(\Omega)$ into bounded sets of $L^2(\Omega)$, its kernel is K and $\beta : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ is equally monotone and hemicontinous.

The penalized problem associated with the variational inequality (2.1) and (2.2) consists in given $0 < \epsilon < 1$, find u_ϵ solution in Q of the mixed problem:

$$
\begin{gathered}
u_\epsilon'' - M(|\nabla u_\epsilon|^2)\Delta u_\epsilon + M_1(|u_\epsilon|^2)u_\epsilon + \frac{1}{\epsilon}\beta(u_\epsilon') = f \quad \text{in } Q,\\
u_\epsilon = 0 \quad \text{on } \Sigma,\\
u_\epsilon' = 0 \quad \text{on } \Sigma,\\
u_\epsilon(x,0) = u_{\epsilon 0}(x) \quad u_\epsilon'(x,0) = u_{\epsilon 1}(x) \quad \text{in } \Omega.
\end{gathered} \qquad (3.8)
$$

Definition 3.2. We suppose that $f \in L^2(0,T;H_0^1(\Omega))$, $f' \in L^2(0,T;L^2(\Omega))$, $u_{\epsilon_0} \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_{\epsilon_1} \in H_0^1(\Omega)$ and hypothesis $(H_1) - (H_3)$ hold. A weak solution to the boundary value problem(3.8) is a functions u_ϵ, such that for each $0 < \epsilon < 1$, $u_\epsilon \in L^\infty(0,T_0;H_0^1(\Omega) \cap H^2(\Omega))$, $u_\epsilon' \in L^\infty(0,T_0;H_0^1(\Omega))$, $u_\epsilon'' \in L^\infty(0,T_0;L^2(\Omega)))$, for $T_0 > 0$, satisfying

$$
\begin{gathered}
(u_\epsilon'', \varphi) + a(M(|\nabla u_\epsilon|^2 u_\epsilon, \varphi) + (M_1|u_\epsilon|^2 u_\epsilon, \varphi) + \frac{1}{\epsilon}(\beta(u_\epsilon'), \varphi)\\
= (f, \varphi), \quad \forall \varphi \in L^2(0,T_0;L^2(\Omega)),\\
u_\epsilon(0) = u_{\epsilon_0}, \ u_\epsilon'(0) = u_{\epsilon_1}.
\end{gathered} \qquad (3.9)
$$

The solution of (3.8) is given by the following theorem.

Theorem 3.3. *Assume that*

$$
f \in L^2(0,T;H_0^1(\Omega)), \quad f' \in L^2(0,T;L^2(\Omega)), \qquad (3.10)
$$

$$
u_{\epsilon_0} \in H_0^1(\Omega) \cap H^2(\Omega), \qquad (3.11)
$$

$$
u_{\epsilon_1} \in H_0^1(\Omega), \qquad (3.12)
$$

and hypothesis (H1)–(H3) *hold. Then for each* $0 < \epsilon < 1$ *there exists a unique weak solution of* (3.8)*.*

In the next section, we prove the Theorem 3.1. Our goal is first prove the penalized Theorem 3.3, applying Faedo-Galerkin method.

4. Proof of main results

Proof of Theorem 3.3. To prove Theorem 3.1, we first prove the penalized Theorem 3.3. We apply the Faedo-Galerkin method, noting that the immersions

$$H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

are continuous and dense and that $H_0^1(\Omega)$ is compact $L^2(\Omega)$. Let $\{w_\nu, \lambda_\nu\}$ eigenvectors and eigenfunctions of $-\Delta$. We consider $(w_\nu) \subset H_0^1(\Omega) \cap H^2(\Omega)$ a Hilbertian basis for Faedo-Galerkin method and $V_m = [w_1, w_2, \dots, w_m]$ the subspace generated by the vectors $w_1, w_2, \dots, w_m$. Let us consider

$$u_{\epsilon m} = \sum_{j=1}^{m} g_{\epsilon jm}(t) w_j$$

solution of the approximate problem

$$\begin{gathered}
(u''_{\epsilon_m}, w_j) + M(|\nabla u_{\epsilon_m}|^2((u_{\epsilon_m}, w_j)) + M_1(|u_{\epsilon_m}|^2)(u_{\epsilon_m}, w_j) \\
+ \frac{1}{\epsilon}\left(\beta(u'_{\epsilon_m}), w_j\right) = (f, w_j), \quad j = 1, 2, \dots m, \\
u_{\epsilon_m}(x,0) \to u_\epsilon(x,0) \quad \text{strongly in } H_0^1(\Omega) \cap H^2(\Omega), \\
u'_{\epsilon_m}(x,0) \to u'_\epsilon(x,0) \quad \text{strongly in } H_0^1(\Omega).
\end{gathered} \tag{4.1}$$

The system of ordinary differential equation (4.1) has a solution $u_{\epsilon_m}(t)$ defined in $[0, t_m[$, $0 < t_m \le T$. The next estimate implies that $u_{\epsilon_m}(t)$ is defined in the whole $[0, T]$.

To obtain a shorter notation, in the calculation of the following three estimates, we omit the parameter ϵ in the approximate problem .

First estimate. We consider $w_j = 2u'_m$ in (4.1) to obtain

$$\begin{aligned}
&\frac{d}{dt}\left|u'_m(t)\right|^2 + M(\|u_m(t)\|^2)\frac{d}{dt}\|u_m(t)\|^2 \\
&+ M_1(|u_m(t)|^2)\frac{d}{dt}|u_m(t)|^2 + \frac{2}{\epsilon}(\beta(u'_m), u'_m) = 2(f(t), u'_m(t)).
\end{aligned} \tag{4.2}$$

Now, if we consider

$$\widehat{M}(\lambda) = \int_0^\lambda M(s)\,ds \quad \text{and} \tag{4.3}$$

$$\widehat{M_1}(\lambda) = \int_0^\lambda M_1(s)\,ds, \tag{4.4}$$

from (4.2), (4.3) and (4.4) it follows that

$$\frac{1}{2}\frac{d}{dt}\left[|u'_m(t)|^2 + \widehat{M}(\|u_m(t)\|^2) + \widehat{M_1}(|u_m(t)|^2)\right] \le (f(t), u'_m(t)), \tag{4.5}$$

because $(\beta u'_m(t), u'_m(t)) \ge 0$.

Integrating (4.5) from 0 to t, we obtain

$$\begin{aligned} &\frac{1}{2}[|u_m'(t)|^2 + \widehat{M}(\|u_m(t)\|^2) + \widehat{M}_1(|u_m(t)|^2)] \\ &\le \int_0^t (f(t), u_m'(t)) + \frac{1}{2}[|u_m'(0)|^2 + \widehat{M}(\|u_m(0)\|^2) + \widehat{M}_1(|u_m(0)|^2)]. \end{aligned} \tag{4.6}$$

From (4.6), (4.1), (3.10) and Cauchy-Schwarz's inequality it follows that

$$|u_m'(t)|^2 + \|u_m(t)\|^2 + |u_m(t)|^2 \le C + C\int_0^t |u_m'(t)|^2 ds. \tag{4.7}$$

Then Gronwall's inequality implies

$$(u_m) \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)), \tag{4.8}$$

$$(u_m') \text{ is bounded } L^\infty(0,T;L^2(\Omega)). \tag{4.9}$$

Second estimate. Considering $w_j = -\Delta u_m'$ in (4.1), we obtain

$$\begin{aligned} &\frac{d}{dt}\|u_m'(t)\|^2 + M(\|u_m(t)\|^2)\frac{d}{dt}|\Delta u_m(t)|^2 \\ &+ M_1(|u_m(t)|^2)\frac{d}{dt}\|u_m(t)\|^2 + \frac{2}{\epsilon}(\beta u_m', -\Delta u_m') \\ &= 2((f(t), u_m'(t))). \end{aligned} \tag{4.10}$$

Observe that

$$\begin{aligned} &M(\|u_m(t)\|^2)\frac{d}{dt}|\Delta u_m(t)|^2 + M_1(|u_m(t)|^2)\frac{d}{dt}\|u_m(t)\|^2 \\ &= \frac{d}{dt}\big[M(\|u_m(t)\|^2)|\Delta u_m(t)|^2 + M_1(|u_m(t)|^2)\|u_m(t)\|^2\big] \\ &\quad - \frac{d}{dt}M(\|u_m(t)\|^2)|\Delta u_m(t)|^2 - \frac{d}{dt}M_1(|u_m(t)|^2)\|u_m(t)\|^2. \end{aligned} \tag{4.11}$$

On the other hand

$$\begin{aligned} &\frac{d}{dt}M(\|u_m(t)\|^2)|\Delta u_m(t)|^2 + \frac{d}{dt}M_1(|u_m(t)|^2)\|u_m(t)\|^2 \\ &= 2M'(\|u_m(t)\|^2)((u_m(t), u_m'(t)))|\Delta u_m(t)|^2 \\ &\quad + 2M_1'(|u_m(t)|^2)(u_m(t), u_m'(t))\|u_m(t)\|^2. \end{aligned} \tag{4.12}$$

Using $(\beta u_m', -\Delta u_m') \ge 0$ and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, follows from (4.10), (4.11) and (4.12) that

$$\begin{aligned} &\frac{d}{dt}\big[\|u_m'(t)\|^2 + M(\|u_m(t)\|^2)|\Delta u_m(t)|^2 + M_1(|u_m(t)|^2)\|u_m(t)\|^2\big] \\ &\le 2|M'(\|u_m(t)\|^2)|\|u_m(t)\|\|u_m'(t)\||\Delta u_m(t)|^2 \\ &\quad + 2|M_1'(|u_m(t)|^2)|\,|u_m(t)|C\|u_m'(t)\|\|u_m(t)\|^2 \\ &\quad + \|f(t)\|^2 + C\|u_m'(t)\|^2. \end{aligned} \tag{4.13}$$

Note that (4.8) implies $\|u_m(t)\| \le C$, therefore $\|u_m(t)\| \in [0,C]$, for each m and $t \in [0,t_m[$. Since $M \in C^1([0,\infty);\mathbb{R})$, this implies that

$$|M'(\|u_m(t)\|^2)| \le C, \quad \forall m, \forall t \in [0,t_m[, \tag{4.14}$$

and analogously for M_1. Therefore, using (4.14), (4.8), (4.9) and (3.10) we can write

$$\begin{aligned} &2|M'(\|u_m(t)\|^2)|\,\|u_m(t)\|\|u_m'(t)\||\Delta u_m(t)|^2 \\ &+2|M_1'(|u_m(t)|^2)|\,|u_m(t)|C\|u_m'(t)\|\|u_m(t)\|^2+\|f(t)\|^2+C\|u_m'(t)\|^2 \\ &\le C+C|\Delta u_m(t)|^2+2C\|u_m'(t)\|^2|\Delta u_m(t)|^2+C\|u_m'(t)\|^2 \\ &\le C+C\Big[\|u_m'(t)\|^2+|\Delta u_m(t)|^2+\big(\|u_m'(t)\|^2+|\Delta u_m(t)|^2\big)^2\Big]. \end{aligned} \tag{4.15}$$

Making

$$\varphi(t)=\|u_m'(t)\|^2+|\Delta u_m(t)|^2 \tag{4.16}$$

and using (4.12), (H2), (H3), (4.1), (4.15) we can write, after integration from 0 to t,

$$\varphi(t)\le C+C\int_0^t(\varphi(s)+\varphi(s)^2)ds. \tag{4.17}$$

Observe that $\varphi(t)$ is continuous in $[0,T_0]$, therefore there exists $T_0<T$ such that $\varphi(t)\le C$ for all m and all $t\in[0,T_0]$.

From (4.17) it follows that

$$\|u_m'(t)\|\le C,\quad \forall m,\ \forall t\in[0,T_0], \tag{4.18}$$

$$|\Delta u_m(t)|\le C,\quad \forall m,\ \forall t\in[0,T_0]. \tag{4.19}$$

That is,

$$(u_m')\text{ is bounded in }L^\infty(0,T_0;H_0^1(\Omega)), \tag{4.20}$$

$$(\Delta u_m)\text{ is bounded in }L^\infty(0,T_0;L^2(\Omega)). \tag{4.21}$$

Statements (4.8) and (4.21) imply that

$$u_m\text{ is bounded in }L^\infty(0,T_0;H_0^1(\Omega)\cap H^2(\Omega)). \tag{4.22}$$

Third estimate. Taking derivatives in the distribution sense in (4.1), we obtain

$$\begin{aligned} &(u_m'''(t),w_j)+\frac{d}{dt}M(\|u_m(t)\|^2)a(u_m(t),w_j)+M(\|u_m(t)\|^2)a(u_m'(t),w_j) \\ &+\frac{d}{dt}M_1(|u_m(t)|^2)(u_m(t),w_j)+M_1(|u_m(t)|^2)(u_m'(t),w_j)+\frac{1}{\epsilon}((\beta u_m'(t))',w_j) \\ &=(f'(t),w_j). \end{aligned}$$

Considering $w_j=2u_m''(t)$ in the above equation, we have

$$\begin{aligned} &\frac{d}{dt}|u_m''(t)|^2+2\frac{d}{dt}M(\|u_m(t)\|^2)a(u_m(t),u_m''(t))+M(\|u_m(t)\|^2)\frac{d}{dt}\|u_m'(t)\|^2 \\ &+2\frac{d}{dt}M_1(|u_m(t)|^2)(u_m(t),u_m''(t))+M_1(|u_m(t)|^2)\frac{d}{dt}|u_m'(t)|^2 \\ &+\frac{2}{\epsilon}((\beta u_m'(t))',u_m''(t)) \\ &=2(f'(t),u_m''(t)). \end{aligned} \tag{4.23}$$

Since

$$((\beta u_m'(t))',u_m''(t))=\lim_{h\to0}\Big(\frac{\beta(u_m'(t+h))-\beta(u_m'(t))}{h},\frac{u_m'(t+h)-u_m'(t)}{h}\Big)\ge0,$$

and because β is monotone, we have

$$\begin{aligned} &\frac{d}{dt}|u_m''(t)|^2 + M(\|u_m(t)\|^2)\frac{d}{dt}\|u_m'(t)\|^2 + M_1(|u_m(t)|^2)\frac{d}{dt}|u_m'(t)|^2 \\ &\leq 2|(f'(t), u_m''(t))| + 2\Big|\frac{d}{dt}M(\|u_m(t)\|^2)a(u_m(t), u_m''(t))\Big| \\ &\quad + 2\Big|\frac{d}{dt}M_1(|u_m(t)|^2)(u_m(t), u_m''(t))\Big|. \end{aligned} \tag{4.24}$$

Using (4.14), (4.8), (4.9) and (4.21), we conclude that

$$\begin{aligned} &\Big|\frac{d}{dt}M(\|u_m(t)\|^2)a(u_m(t), u_m''(t))\Big| \\ &= 2\left|M'(\|u_m(t)\|^2)(u_m(t), u_m'(t))(-\Delta u_m(t), u_m''(t))\right| \\ &\leq \left|M'(\|u_m(t)\|^2)(u_m(t), u_m'(t))\right| \left\{|\Delta u_m(t)|^2 + |u_m''(t)|^2\right\} \\ &\leq C + C\left|u_m''(t)\right|^2. \end{aligned} \tag{4.25}$$

Analogously,

$$\begin{aligned} &\Big|\frac{d}{dt}M_1(|u_m(t)|^2)(u_m(t), u_m''(t))\Big| \\ &= 2\left|M_1'\left(|u_m(t)|^2)(u_m(t), u_m'(t)\right)(u_m(t), u_m''(t))\right| \\ &\leq \left|M_1'\left(|u_m(t)|^2)(u_m(t), u_m'(t)\right)\right| \left\{|u_m(t)|^2 + |u_m''(t)|^2\right\} \\ &\leq C + C\left|u_m''(t)\right|^2. \end{aligned} \tag{4.26}$$

By Young's inequality,

$$2|(f'(t), u_m''(t))| \leq |f'(t)|^2 + |u_m''(t)|^2. \tag{4.27}$$

We observe that (4.3), (4.4), (4.24), (4.25), (4.26) and (4.27) lead to

$$\frac{d}{dt}\big\{|u_m''(t)|^2 + \widehat{M}(\|u_m'(t)\|^2) + \widehat{M}_1(|u_m'(t)|^2)\big\} \leq C + |f'(t)|^2 + C|u_m''(t)|^2. \tag{4.28}$$

Integrating (4.28) from 0 to $t < T_0$, using (H2), (H3), (3.1), (4.1) we have

$$|u_m''(t)|^2 + \|u_m'(t)\|^2 + |u_m'(t)|^2 \leq C + C\int_0^t |u_m''(s)|^2 ds + |u_m''(0)|^2. \tag{4.29}$$

We observe that, making $t = 0$ and $w_j = u_m''(0)$ in approximate problem (4.1) we obtain

$$|u_m''(0)|^2 \leq \big\{M(\|u_{0m}\|^2)|\Delta u_{0m}| + M_1(|u_{0m}|^2)|u_{0m}| + \frac{1}{\epsilon}|\beta(u_{1m})| + |f(0)|\big\}|u_m''(0)|;$$

that is,

$$|u_m''(0)| \leq C, \tag{4.30}$$

because $u_{1m} \to u_1$ in $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $\beta : L^2(\Omega) \to L^2(\Omega)$ is continuous.

From (4.29) and (4.30) it follows that

$$|u_m''(t)|^2 \leq C + C\int_0^T |u_m''(s)|^2 ds. \tag{4.31}$$

Using Gronwall's inequality we conclude that

$$(u_m'') \text{ is bounded in } L^\infty(0, T_0; L^2(\Omega)). \tag{4.32}$$

Now we return to the notation u_{ϵ_m}. The estimates above and Aubin-Lions compactness Theorem implies that the existence of a subsequence of (u_{ϵ_m}), still denoted by (u_{ϵ_m}), such that

$$u_{\epsilon_m} \to u_\epsilon \text{ weak star in } L^\infty(0,T_0;H_0^1(\Omega)\cap H^2(\Omega)), \tag{4.33}$$
$$u'_{\epsilon_m} \to u'_\epsilon \text{ weak star in } L^\infty(0,T_0;H_0^1(\Omega)), \tag{4.34}$$
$$u''_{\epsilon_m} \to u''_\epsilon \text{ weak star in } L^\infty(0,T_0;L^2(\Omega)), \tag{4.35}$$
$$u_{\epsilon_m} \to u_\epsilon \text{ strong in } L^2(0,T_0;L^2(\Omega)) \text{ and a.e in Q}, \tag{4.36}$$
$$u'_{\epsilon_m} \to u'_\epsilon \text{ strong in } L^2(0,T_0;L^2(\Omega)) \text{ and a.e in Q}. \tag{4.37}$$

Statements (4.36) and (4.37), the continuity of de norm and of β imply

$$\|u_{\epsilon_m}\| \to \|u_\epsilon\| \text{ a.e in } Q, \tag{4.38}$$
$$\beta(u'_{\epsilon_m}) \to \beta(u'_\epsilon) \text{ a.e in } Q. \tag{4.39}$$

Using (H1) in (4.38) it follows that

$$M(\|u_{\epsilon_m}\|^2) \to M(\|u_\epsilon\|)^2 \quad \text{a.e in } Q, \tag{4.40}$$

analogously,

$$M_1(|u_{\epsilon_m}|^2) \to M_1(|u_\epsilon|)^2 \text{ a.e in } Q. \tag{4.41}$$

The convergences above are sufficient to pass to the limit with $m \to \infty$ and then we prove the Theorem 3.3.

Proof of Theorem 3.1. Finally, we prove the main Theorem of this work. Let $v \in L^2(0,T_0;H_0^1(\Omega))$ be $v(t) \in K$ a. e. for $t \in (0,T_0)$. From $(3.8)_1$ it follows that

$$\begin{aligned}
&\int_0^{T_0} (u''_\epsilon, v-u'_\epsilon)dt + \int_0^{T_0} M(|\nabla u_\epsilon|)^2 a(u_\epsilon, v-u'_\epsilon)dt \\
&+ \int_0^{T_0} M_1(|u_\epsilon|^2)(u_\epsilon, v-u'_\epsilon)dt - \int_0^{T_0} (f, v-u'_\epsilon)dt \\
&= \frac{1}{\epsilon}\int_0^{T_0} (\beta(u'_\epsilon), u'_\epsilon - v)\,dt \\
&= \frac{1}{\epsilon}\int_0^{T_0} (\beta(u'_\epsilon) - \beta v, u'_\epsilon - v)\,dt \ge 0,
\end{aligned} \tag{4.42}$$

because $v \in K$ $(\beta(v) = 0)$ and β is monotone.

From (4.33)-(4.37) and the Banach-Steinhauss Theorem, it follows that there exists a subnet $(u_\epsilon)_{0<\epsilon<1}$, such that it converges to u as $\epsilon \to 0$, in the sense of (4.33)-(4.37); that is,

$$u_\epsilon \to u \text{ weak star in } L^\infty(0,T_0;H_0^1(\Omega)\cap H^2(\Omega)), \tag{4.43}$$
$$u'_\epsilon \to u' \text{ weak star in } L^\infty(0,T_0;H_0^1(\Omega)), \tag{4.44}$$
$$u''_\epsilon \to u'' \text{ weak star in } L^\infty(0,T_0;L^2(\Omega)), \tag{4.45}$$
$$u_\epsilon \to u \text{ strong in } L^2(0,T_0;L^2(\Omega)) \text{ and a.e in Q}, \tag{4.46}$$
$$u'_\epsilon \to u' \text{ strong in } L^2(0,T_0;L^2(\Omega)) \text{ and a.e in Q}. \tag{4.47}$$

The convergences above are sufficient to pass to the limit in (4.42) with $\epsilon \to 0$ to conclude that (3.6) is valid. To complete the proof of Theorem 3.1, it remains to show that $u'(t) \in K$. a. e..

In this position we observe that using convergences (4.33)-(4.37), leting $m \to \infty$ in (4.1), we can find u_ϵ such that

$$u''_\epsilon - M(|\nabla u_\epsilon|^2)\Delta u_\epsilon + M_1(|u_\epsilon|^2)u_\epsilon + \frac{1}{\epsilon}\beta(u'_\epsilon) = f \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{4.48}$$

On the other hand, from (4.48) we have

$$\beta(u'_\epsilon) = \epsilon[f - u''_\epsilon + M(|\nabla u_\epsilon|^2)\Delta u_\epsilon - M_1(|u_\epsilon|^2)u_\epsilon]. \tag{4.49}$$

Then

$$\beta(u'_\epsilon) \to 0 \quad \text{in } \mathcal{D}'(0,T;H^{-1}(\Omega)). \tag{4.50}$$

From (4.48) It follows that

$$\beta(u'_\epsilon) \text{ is bounded in } L^2(0,T;L^2(\Omega)). \tag{4.51}$$

Therefore,

$$\beta(u'_\epsilon) \to 0 \quad \text{weak in } L^2(0,T;L^2(\Omega)). \tag{4.52}$$

On the other hand we deduce from (4.49) that

$$0 \le \int_0^T (\beta(u'_\epsilon), u'_\epsilon)\, dt \le \epsilon\, C. \tag{4.53}$$

Thus

$$\int_0^T (\beta(u'_\epsilon), u'_\epsilon) dt \to 0. \tag{4.54}$$

We have

$$\int_0^T (\beta(u'_\epsilon) - \beta(\varphi), u'_\epsilon - \varphi)\, dt \ge 0, \quad \forall \varphi \text{ in } L^2(0,T;L^2(\Omega)),$$

because β is a monotonous operator. Thus,

$$\int_0^T (\beta(u'_\epsilon), u'_\epsilon)\, dt - \int_0^T (\beta(u'_\epsilon), \varphi)\, dt - \int_0^T (\beta(\varphi), u'_\epsilon - \varphi)\, dt \ge 0. \tag{4.55}$$

From (4.52) (4.54) and (4.55) we have

$$\int_0^T (\beta(\varphi), u'(t) - \varphi)\, dt \le 0. \tag{4.56}$$

Taking $\varphi = u' - \lambda v$, with $v \in L^2(0,T;L^2(\Omega))$ and $\lambda > 0$, using the hemicontinuity of β we deduce that

$$\beta(u'(t)) = 0, \tag{4.57}$$

and this implies that $u'(t) \in K$ a. e. and the proof of the existence of solution is complete.

In the next section we prove the uniqueness of solution to achieve our goal.

5. Uniqueness of solution

Suppose that u_1, u_2 are two solutions of (3.6) and set $w = u_2 - u_1$ and $t \in [0, T_0]$. Taking $v = u_1'$ (resp. u_2') in (3.6) relative to v_2 (resp. v_1) and adding up the results we obtain

$$\begin{aligned} &-\int_0^t (w'', w')dt + \int_0^t M(\|u_2\|^2 \Delta u_2, w')dt - \int_0^t M(\|u_1\|^2 \Delta u_1, w')dt \\ &-\int_0^t (M_1(|u_2|^2)u_2, w')dt + \int_0^t (M_1(|u_1|^2)u_1, w')dt \geq 0, \end{aligned} \tag{5.1}$$

or equivalently

$$\begin{aligned} &-\int_0^t (w'', w')ds + \int_0^t (M(\|u_2\|^2 \Delta u_2, w')ds - \int_0^t (M(\|u_2\|^2 \Delta u_1, w')ds \\ &+\int_0^t (M(\|u_2\|^2 \Delta u_1, w')ds - \int_0^t (M(\|u_1\|^2 \Delta u_1, w')ds \\ &-\int_0^t (M_1(|u_2|^2)u_2, w')ds + \int_0^t (M_1(|u_1|^2)u_1, w')ds \\ &-\int_0^t (M_1(|u_2|^2)u_1, w')ds + \int_0^t (M_1(|u_2|^2)u_1, w')ds \\ &= -\int_0^t (w'', w')ds + \int_0^t (M(\|u_2\|^2 \Delta w, w')ds - \int_0^t (M_1(|u_2|^2)w, w')ds \\ &\quad + \int_0^t ([M(\|u_2\|^2 - M(\|u_1\|^2)]\Delta u_1, w')ds \\ &\quad - \int_0^t ([M_1(|u_2|^2) - M_1(|u_1|^2)]u_1, w')ds \geq 0, \end{aligned} \tag{5.2}$$

By hypothesis (H1), we can use the Mean Value Theorem to write

$$\begin{aligned} &\int_0^t (w'', w')ds - \int_0^t (M(\|u_2\|^2 \Delta w, w')ds + \int_0^t (M_1(|u_2|^2)w, w')ds \\ &- \int_0^t (M'(\epsilon)[\|u_2\|^2 - \|u_1\|^2]\Delta u_1, w')ds \\ &+ \int_0^t (M_1'(\epsilon_1)[|u_2|^2 - |u_1|^2]u_1, w')ds \leq 0, \end{aligned} \tag{5.3}$$

where

$$\|u_1\|^2 \leq \epsilon \leq \|u_2\|^2, \quad |u_1|^2 \leq \epsilon \leq |u_2|^2. \tag{5.4}$$

From (5.3) It follows that

$$\begin{aligned} &\int_0^t \frac{d}{dt}|w'|^2 ds + \int_0^t M(\|u_2\|^2)\frac{d}{dt}\|w\|^2 ds + \int_0^t M_1(|u_2|^2)\frac{d}{dt}|w|^2 ds \\ &\leq 2\int_0^t |M'(\epsilon)|\,[(\|u_2\| - \|u_1\|)\,(\|u_2\| + \|u_1\|)]\,|\Delta u_1||w'|ds \\ &\quad + 2\int_0^t |M_1'(\epsilon_1)|\,[(|u_2| - |u_1|)\,(|u_2| + |u_1|)]\,|u_1||w'|ds, \end{aligned} \tag{5.5}$$

using an argument similar to that used in (4.11) and (4.12), from (5.5) it follows that

$$\begin{aligned} &\int_0^t \frac{d}{dt}\left\{|w'|^2 + M(\|u_2\|^2)\|w\|^2 + M_1(|u_2|^2)|w|^2\right\} ds \\ &\leq 2\int_0^t |M'(\epsilon)|\left[(\|u_2\| - \|u_1\|)(\|u_2\| + \|u_1\|)\right]|\Delta u_1||w'|ds \\ &\quad + 2\int_0^t |M_1'(\epsilon_1)|\left[(|u_2| - |u_1|)(|u_2| + |u_1|)\right]|u_1||w'|ds \\ &\quad + 2\int_0^t (|M'(\|u_2\|^2)|\|u_2\|\|u_2'\|\|w\|^2 + 2|M_1'(|u_2|^2)||u_2||u_2'||w|^2)ds. \end{aligned} \tag{5.6}$$

Using (H2), (H3), (4.18) (4.19) and (5.6), we conclude that

$$|w'|^2 + \|w\|^2 + |w|^2 \leq C\int_0^t \|w\||w'|ds + C\int_0^t |w||w'|ds + C\int_0^t |w|^2 ds. \tag{5.7}$$

This implies

$$|w'|^2 + \|w\|^2 + |w|^2 \leq C\int_0^t (|w'|^2 + C\|w\|^2 + |w|^2)ds. \tag{5.8}$$

Using Gronwall's inequality in (5.8), we conclude that $w(t) = 0$; therefore $u_1 = u_2$.

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