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NONEXISTENCE OF POSITIVE SOLUTIONS FOR A NONPOSITONE SYSTEM IN A BALL

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Abstract. In this article, we prove the nonexistence of positive solutions for a nonpositone system in a ball when the nonlinearities may have more than one zero.

1. Introduction

Reaction-diffusion systems model many phenomena in biology, chemical reaction, population dynamics etc. A typical example of these models is the boundary value problem

$$\begin{gathered} -\Delta u(x) = \lambda f(u(x)), \quad x \in \Omega \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.1}$$

The fact that the reaction term f may be negative at the origin makes it very challenging problem in showing the positivity of the solution. In the case of systems, it is even more difficult since we have to the positivity of every component. In this work we restrict our analysis to the system

$$\begin{gathered} -\Delta u(x) = \lambda f(v(x)), \quad x \in \Omega \\ -\Delta v(x) = \mu g(u(x)), \quad x \in \Omega \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

where $\min(\lambda, \mu) \geq \varepsilon_0 > 0$, $\Omega = B(0, R)$ is a ball in $\mathbb{R}^N$ with radius R, $N \geq 2$, f and g are smooth functions that grow at least linearly at infinity. When f and g are a monotone nondecreasing nonlinearities and have only one zero, problem (1.2) has been studied by Hai, Oruganti and Shivaji [6] in a ball, and by Hakimi [9] in an annulus.

Let (u, v) be a positive solution of (1.2). Then u, v are radial, decreasing and satisfy

$$\begin{gathered} -(r^{N-1}u')' = \lambda r^{N-1} f(v), \quad 0 < r < R \\ -(r^{N-1}v')' = \mu r^{N-1} g(u), \quad 0 < r < R \\ u'(0) = v'(0) = 0 \\ u(R) = v(R) = 0. \end{gathered} \tag{1.3}$$

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In this note, we shall prove that the nonexistence result of positive solutions of (1.2) remains valid when f and g have more than one zero (without loss of generality, we assume that f and g have exactly three zeros) and are not strictly increasing entirely $[0,+\infty)$; see [6, Theorem 1.1]. To be precise, we shall make the following assumptions

- (H1) $f, g \in C^1([0,+\infty),\mathbb{R})$ such that f and g have three zeros $\alpha_1 < \alpha_2 < \alpha_3$ and $\beta_1 < \beta_2 < \beta_3$ respectively with $f'(\alpha_i) \neq 0$, $g'(\beta_i) \neq 0$ for all $i \in \{1,2,3\}$. Moreover, $f' \geq 0$ on $[0,\alpha_1] \cup [\alpha_3,+\infty)$, $g' \geq 0$ on $[0,\beta_1] \cup [\beta_3,+\infty)$ and $F(\alpha_3) < 0$, $G(\beta_3) < 0$ where $F(x) = \int_0^x f(t)dt$ and $G(x) = \int_0^x g(t)dt$.
- (H2) $f(0) < 0$ and $g(0) < 0$.
- (H3) There exist two positive real numbers a_i and b_i, $i = 1,2$ such that

$$f(z) \geq a_1 z - b_1, \quad g(z) \geq a_2 z - b_2, \quad \forall z \geq 0.$$

2. Main result

Our main result is the following theorem.

Theorem 2.1. *Assume that* (H1)–(H3) *are satisfied. Then there exists a positive real number σ such that* (1.2) *has no positive solution for $\lambda\mu > \sigma$.*

Remark. Existence result for positive solutions with superlinearities satisfying (H1), (H2), $\lambda = \mu$ and λ small can be found in [4, 5]. For the single equation case, see [1, 3, 7, 10] for existence results and [1, 2, 8] for nonexistence results.

To prove Theorem 2.1, we need the next three lemmas. We note that the proofs of the first and the second lemma are analogous to those of [6, lemma 2.1, theorem B]. On the other hand, the proof of the last is different from that of [6, Lemma 2.2]. This is so because our f and g have no constant sign in $(\alpha_1,+\infty)$ and $(\beta_1,+\infty)$ respectively. Here we use ideas adapted from Hai, Oruganti and Shivaji [6].

Let $t_1 \in (0,R)$. We have the following result.

Lemma 2.2. *There exists a positive constant C such that for $\lambda\mu$ large,*

$$u(t_1) + v(t_1) \leq C.$$

Proof. Let λ_1 be the first eigenvalue of the $-\Delta$ with Dirichlet boundary conditions. Multiplying the first equation in (1.3) by a positive eigenfunction, say ϕ corresponding to λ_1, and using (H3) we obtain

$$-\int_0^R (r^{N-1}u')'\phi dr \geq \int_0^R \lambda(a_1 v - b_1)\phi r^{N-1} dr;$$

that is,

$$\int_0^R \lambda_1 u r^{N-1}\phi dr \geq \int_0^R \lambda(a_1 v - b_1)\phi r^{N-1} dr. \tag{2.1}$$

Similarly, using the second equation in (1.3) and (H3), we obtain

$$\int_0^R \lambda_1 v r^{N-1}\phi dr \geq \int_0^R \mu(a_2 u - b_2)\phi r^{N-1} dr. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$\int_0^R [\lambda_1 - \lambda\mu\frac{a_1 a_2}{\lambda_1}] v\phi r^{N-1} dr \geq \int_0^R \mu[-\lambda\frac{a_2 b_1}{\lambda_1} - b_2]\phi r^{N-1} dr.$$

Now, if $\frac{\lambda\mu a_1 a_2}{2} \geq \lambda_1^2$, then

$$\int_0^R \mu[-\lambda a_2 b_1 - b_2\lambda_1]\phi r^{N-1} dr \leq \int_0^R -\frac{\lambda\mu}{2} a_1 a_2 v\phi r^{N-1} dr;$$

that is,

$$\int_0^R \frac{a_1 a_2}{2} v\phi r^{N-1} dr \leq \int_0^R [a_2 b_1 + \frac{b_2\lambda_1}{\varepsilon_0}]\phi r^{N-1} dr, \tag{2.3}$$

(because $\min(\lambda,\mu) \geq \varepsilon_0$). Similarly

$$\int_0^R \frac{a_1 a_2}{2} u\phi r^{N-1} dr \leq \int_0^R [a_1 b_2 + \frac{b_1\lambda_1}{\varepsilon_0}]\phi r^{N-1} dr. \tag{2.4}$$

Adding (2.3) and (2.4), we obtain the inequality

$$\int_0^R (u+v)\phi r^{N-1} dr \leq \frac{2}{a_1 a_2}\int_0^R [a_1 b_2 + \frac{b_1\lambda_1}{\varepsilon_0} + a_2 b_1 + \frac{b_2\lambda_1}{\varepsilon_0}]\phi r^{N-1} dr.$$

Then

$$\begin{aligned}(u+v)(t_1)\int_0^{t_1} \phi r^{N-1} dr &\leq \int_0^{t_1} (u+v)\phi r^{N-1} dr \\ &\leq \int_0^R (u+v)\phi r^{N-1} dr \\ &\leq \frac{2}{a_1 a_2}\int_0^R [a_1 b_2 + \frac{b_1\lambda_1}{\varepsilon_0} + a_2 b_1 + \frac{b_2\lambda_1}{\varepsilon_0}]\phi r^{N-1} dr,\end{aligned}$$

because u and v are decreasing. The proof is complete. □

Now, assume that there exists $z \geq 0$ ($z \not\equiv 0$) on $\overline{I}$ where $I = (a,b)$, and a constant γ such that

$$-(r^{N-1}z')' \geq \gamma r^{N-1} z, \quad r \in I. \tag{2.5}$$

Let $\lambda_1 = \lambda_1(I) > 0$ denote the principal eigenvalue of

$$\begin{gathered}-(r^{N-1}\psi')' = \lambda r^{N-1}\psi, \quad r \in (a,b) \\ \psi(a) = 0 = \psi(b),\end{gathered} \tag{2.6}$$

where $0 < a < b \leq 1$.

Lemma 2.3. *Let* (2.5) *hold. Then* $\gamma \leq \lambda_1(I)$.

Proof. Multiplying (2.5) by ψ ($\psi > 0$), an eigenfunction corresponding to the principal eigenvalue $\lambda_1(I)$, and integrating by parts (twice) we obtain

$$\int_a^b [\gamma - \lambda_1(I)] r^{N-1} z\psi dr \leq b^{N-1}\psi'(b)z(b) - a^{N-1}\psi'(a)z(a). \tag{2.7}$$

But $\psi'(b) < 0$ and $\psi'(a) > 0$. Hence the right-hand side of (2.7) is less than or equal to zero. Then $\gamma \leq \lambda_1(I)$, and the proof is complete. □

Now, we define

$$t_0 = t_1 + \frac{R - t_1}{3}, \quad t_2 = t_1 + \frac{2(R-t_1)}{3}.$$

Lemma 2.4. *For $\lambda\mu$ sufficiently large, $u(t_2) \leq \beta_3$ or $v(t_2) \leq \alpha_3$.*

Proof. We argue by contradiction. Suppose that $u(t_2) > \beta_3$ and $v(t_2) > \alpha_3$.

Case 1: $u(t_0) > \rho_2$ or $v(t_0) > \rho_1$, where $\rho_1 = \frac{\alpha_3+\theta_1}{2}$ and $\rho_2 = \frac{\beta_3+\theta_2}{2}$ (θ_1 and θ_2 are the greatest zeros of F and G respectively. If $u(t_0) > \rho_2$ then

$$-(r^{N-1}v')' = \mu r^{N-1}g(u) \geq \varepsilon_0 r^{N-1}g(\rho_2) \quad \text{in } J = (t_1, t_0)$$

and $v(r) \geq \alpha_3$ on $\bar{J}$. Let ω be the unique solution of

$$\begin{gathered} -(r^{N-1}\omega')' = \varepsilon_0 r^{N-1}g(\rho_2) \quad \text{in } J \\ \omega = \alpha_3 \quad \text{on } \partial J. \end{gathered}$$

Then by comparison arguments, $v(r) \geq \omega(r) = \varepsilon_0 g(\rho_2)\omega_0(r) + \alpha_3$ in $\bar{J}$, where ω_0 is the unique (positive) solution of

$$\begin{gathered} -(r^{N-1}\omega_0')' = r^{N-1} \quad \text{in } J \\ \omega_0 = 0 \quad \text{on } \partial J. \end{gathered}$$

In particular, there exists $\overline{\alpha}_3 > \alpha_3$ ($f(\overline{\alpha}_3) \neq 0$) such that

$$v(t_1 + \frac{2(t_0-t_1)}{3}) \geq \omega(t_1 + \frac{2(t_0-t_1)}{3}) \geq \overline{\alpha}_3$$

in $J^* = (t_1 + \frac{t_0-t_1}{3}, t_1 + \frac{2(t_0-t_1)}{3})$. Then

$$\begin{aligned} -(r^{N-1}(u-\beta_3)')' &= \lambda r^{N-1}f(v) \\ &\geq \lambda r^{N-1}f(\overline{\alpha}_3) \\ &\geq (\frac{\lambda f(\overline{\alpha}_3)}{C})r^{N-1}(u-\beta_3) \quad \text{in } J^*, \end{aligned}$$

(where C is as in Lemma 2.2). Since $u - \beta_3 > 0$ in $\bar{J}^*$, it follows that

$$\frac{\lambda f(\overline{\alpha}_3)}{C} \leq \lambda_1(J^*), \tag{2.8}$$

where $\lambda_1(J^*)$ is the principal eigenvalue of (2.6) (with $(a,b) = J^*$).

Next we consider

$$\begin{aligned} (r^{N-1}(v-\alpha_3)')' &= \mu r^{N-1}g(u) \\ &\geq \mu r^{N-1}g(\rho_2) \\ &\geq (\frac{\mu g(\rho_2)}{C})r^{N-1}(v-\alpha_3) \quad \text{in } J. \end{aligned}$$

Since $v - \alpha_3 > 0$ in $\bar{J}$, it follows that

$$\frac{\mu g(\rho_2)}{C} \leq \lambda_1(J), \tag{2.9}$$

where $\lambda_1(J)$ is the principal eigenvalue of (2.6) (with $(a,b) = J$). Combining (2.8) and (2.9), we obtain

$$\frac{\lambda\mu f(\overline{\alpha}_3)g(\rho_2)}{C^2} \leq \lambda_1(J^*)\lambda_1(J),$$

but $f(\overline{\alpha}_3)$, $g(\rho_2)$ and C are fixed positive constants. This is a contradiction for $\lambda\mu$ large. A similar contradiction can be reached for the case $v(t_0) > \rho_1$.

Case 2: $u(t_0) \leq \rho_2$ and $v(t_0) \leq \rho_1$. Then $\beta_3 < u \leq \rho_2$ and $\alpha_3 < v \leq \rho_1$ in $J_1 = [t_0, t_2]$. Then by the mean value theorem, there exist $c_1, c_2 \in (t_0, t_2)$ such that

$$|u'(c_2)| \leq \frac{3\rho_2}{R - t_1}, \quad |v'(c_1)| \leq \frac{3\rho_1}{R - t_1}.$$

Since $-(r^{N-1}u')' \geq 0$ on $[t_0, t_2)$, it follows that
$$-r^{N-1}u'(r) \leq -c_2^{N-1}u'(c_2) \quad \text{on } J_2 = [t_0, c_2);$$
thus
$$|u'(r)| \leq \frac{c_2^{N-1}}{r^{N-1}}|u'(c_2)| \leq (\frac{t_2}{t_0})^{N-1}\frac{3\rho_2}{R-t_1} \quad \text{in } J_2.$$
Similarly, we obtain
$$|v'(r)| \leq (\frac{t_2}{t_0})^{N-1}\frac{3\rho_1}{R-t_1} \quad \text{in } J_3 = [t_0, c_1).$$
Hence there exists $r_0 \in [t_0, R)$ such that
$$|u'(r_0)| \leq \widetilde{c}, \quad |v'(r_0)| \leq \widetilde{c},$$
where
$$\widetilde{c} = \frac{3}{R-t_1}(\frac{t_2}{t_0})^{N-1}\max(\rho_2, \rho_1).$$
Now, define the energy function
$$E(r) = u'(r)v'(r) + \lambda F(v(r)) + \mu G(u(r)).$$
Then
$$E'(r) = -\frac{2(N-1)}{r}u'(r)v'(r) \leq 0,$$
and hence $E \geq 0$ in $[0, R]$, since $E(R) = u'(R)v'(R) \geq 0$. However,
$$E(r_0) \leq \widetilde{c}^2 + \lambda F(\rho_1) + \mu G(\rho_2), \tag{2.10}$$
and $F(\rho_1) < 0$ and $G(\rho_2) < 0$. Hence $E(r_0) < 0$ for $\lambda\mu$ large which is a contradiction. The proof is complete. $\square$

Proof of Theorem 2.1. Assume $\lambda\mu$ is large enough so that both lemmas 2.2, 2.4 hold. We take the case when $u(t_2) \leq \beta_3$ (we assume that $u(t_2) \leq \beta_1$, unless we can choose $t_2^* > t_2$ such that $u(t_2^*) \leq \beta_1$). Then
$$\begin{gathered} -(r^{N-1}v')' = \mu r^{N-1}g(u) \leq 0 \quad \text{in } J_3 = (t_2, R) \\ v(t_2) \leq C, \quad v(R) = 0, \end{gathered}$$
hence, by comparison arguments, $v(r) \leq \widetilde{\omega}(r)$, where $\widetilde{\omega}$ is the solution of
$$\begin{gathered} -(r^{N-1}\widetilde{\omega}')' = 0 \quad \text{in } J_3 \\ \widetilde{\omega}(t_2) = C, \quad \widetilde{\omega}(R) = 0. \end{gathered}$$
However, $\widetilde{\omega}(r) = C\int_r^R s^{1-N}ds/\int_{t_2}^R s^{1-N}ds$ decreases from C to 0 on $[t_2, R]$, hence there exists $r_1 \in (t_2, R)$ (independent of $\lambda\mu$) such that $\widetilde{\omega}(r_1) = \frac{\alpha_1}{2}$.

Hence $v(r_1) \leq \alpha_1/2$, and
$$\begin{aligned} -(r^{N-1}(\beta_3 - u)')' &= -\lambda r^{N-1}f(v) \\ &\geq -\lambda r^{N-1}f(\frac{\alpha_1}{2}) \\ &\geq \lambda\big(-f(\frac{\alpha_1}{2})\big)r^{N-1}\frac{\beta_3 - u}{\beta_3} \quad \text{in } J_4 = (r_1, R). \end{aligned}$$
Since $\beta_3 - u > 0$ in $\bar{J}_4$, we have
$$\frac{\lambda\widetilde{K}_1}{\beta_3} \leq \lambda_1(J_4), \tag{2.11}$$

where $\widetilde{K}_1 = -f(\alpha_1/2)$ and $\lambda_1(J_4)$ is the principal eigenvalue of (2.6) (with $(a,b) = J_4$). Similarly, there exists $r_2 \in (r_1, R)$ (independent of $\lambda\mu$) such that

$$v(r_2) < \frac{\alpha_1}{2}.$$

Hence

$$\begin{gathered} -(r^{N-1}u')' = \mu r^{N-1} f(v) \le 0 \quad \text{in } J_5 = (r_2, R) \\ u(r_2) \le C, \quad u(R) = 0, \end{gathered}$$

then, by comparison arguments we obtain

$$u(r) \le \omega_1(r) = \frac{C}{\int_{r_2}^R s^{1-N} ds} \int_r^R s^{1-N} ds;$$

which satisfies

$$\begin{gathered} -(r^{N-1}\omega_1')' = 0, \quad \text{in } J_5, \\ \omega_1(r_2) = C, \quad \omega_1(R) = 0. \end{gathered}$$

Arguing as before there exists $r_3 \in (r_2, R)$ (independent of $\lambda\mu$) such that

$$u(r_3) \le \omega_1(r_3) \le \frac{\beta_1}{2} < C.$$

Hence

$$\begin{aligned} -(r^{N-1}(\alpha_3 - v)')' &= -\mu r^{N-1} g(u) \\ &\ge -\mu r^{N-1} g(\frac{\beta_1}{2}) \\ &\ge \mu\big(- g(\frac{\beta_1}{2})\big) r^{N-1} \frac{\alpha_3 - v}{\alpha_3} \quad \text{on } J_6 = (r_3, R). \end{aligned}$$

Since $\alpha_3 - v > 0$ in $\bar{J}_6$, it follows that

$$\frac{\mu \widetilde{K}_2}{\alpha_3} \le \lambda_1(J_6), \tag{2.12}$$

where $\widetilde{K}_2 = -g(\frac{\beta_1}{2})$ and $\lambda_1(J_6)$ is the principal eigenvalue of (2.6) (with $(a,b) = J_6$). Combining (2.11) and (2.12), we obtain

$$\frac{\lambda\mu \widetilde{K}_1 \widetilde{K}_2}{\alpha_3\beta_3} \le \lambda_1(J_4)\lambda_1(J_6),$$

which is a contradiction to $\lambda\mu$ being large.

A similar contradiction can be reached for the case $v(t_2) \le \alpha_3$. Hence Theorem 2.1 is proven. $\square$

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