EXISTENCE AND UNIQUENESS OF LOCAL WEAK SOLUTIONS FOR THE EMDEN-FOWLER WAVE EQUATION IN ONE DIMENSION

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Abstract. In this article we consider the existence and uniqueness of local weak solutions to the Emden-Fowler type wave equation

$$t^2 u_{tt} - u_{xx} = |u|^{p-1}u \quad \text{in } [1,T] \times (a,b)$$

with initial-boundary value conditions in a finite time interval.

1. Introduction

In this article we focus on the existence and uniqueness of weak solutions in $H_2 := C^1([1,T), H^1_0(a,b)) \cap C^2([1,T), L^2(a,b))$ for the Emden-Fowler type wave equation

$$t^2 u_{tt} - u_{xx} = |u|^{p-1}u \quad \text{in } [1,T] \times (a,b) \quad (1.1)$$

subject to zero boundary values and initial values

$$u(1) = u_0 \in H^2(a,b) \cap H^1_0(a,b), \quad u_t(1) = u_1 \in H^1_0(a,b).$$

Here $p > 1$, and $a$ and $b$ are real numbers.

The study of the Emden-Fowler ordinary differential equation is derived from earlier theories concerning gas dynamics in astrophysics developed at the turn of the 20th century. The fundamental problem in the study of stellar structures at that time was to study the equilibrium configuration of the mass of spherical gas clouds. The equation

$$\frac{d}{dt} \left( t^2 \frac{du}{dt} \right) + t^2 u^p = 0, \quad (1.2)$$

is generally referred to as the Lane-Emden equation. Astrophysicists were interested in the behavior of the solution of (1.2) which satisfies the initial condition $u(0) = 1$, $u'(0) = 0$. The mathematical foundation for the investigation of such an equation and also of the more general equation

$$\frac{d}{dt} \left( t^p \frac{du}{dt} \right) + t^q u^r = 0, \quad t \geq 0, \quad (1.3)$$

was made by Fowler [15, 16, 17, 18] in a series of four papers from 1914 to 1931. The Emden-Fowler equation also arises in the study of gas dynamics and fluid

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mechanics \cite{12}, there the solutions of physical interest are bounded non-oscillatory ones which possess a positive zero. The zero of such a solution corresponds to an equilibrium state in a fluid with spherical distribution of density and under mutual attraction of its particles. The Emden-Fowler equations also appear in the study of relativistic mechanics, nuclear physics and also in the study of chemically reacting systems \cite{1 7 10 11 13 14 19 29 31}. The Emden-Fowler equation (1.3) can be transformed into a first order nonlinear autonomous system, in fact a quadratic system, and information concerning its solutions can be obtained from the associated quadratic systems through phase plane analysis. This approach was first used by Emden in his analysis of the Lane-Emden equation (1.2).

For a first comprehensive study on the generalized Emden-Fowler equation \[ u'' + q(t)u^n = 0, \] was made by Atkinson \cite{2, 3, 4, 5, 6}. Recently, in \cite{27}, we considered positive solutions of the Emden-Fowler equation \( t^2 u'' = u^p \) and obtained some results on the non-existence of global solutions, the estimates for the life-spans and the asymptotic behavior of solutions.

About the semilinear wave equation, Jörgens \cite{21} published the first global existence theorem for the equation

\[ \square u + f(u) = 0 \] in case \( \Omega = \mathbb{R}^n \), \( n = 3 \) and \( f(u) = g(u^2)u \). His result can be applied to the equation \( \square u + u^3 = 0 \); and Browder \cite{9} generalized Jörgens’ result for \( n > 2 \).

For local Lipschitz \( f \), Li \cite{25, 26} proved the nonexistence of global solutions of the initial-boundary value problem for the semilinear wave equation (1.4) in a bounded domain \( \Omega \subset \mathbb{R}^n \) under the assumption

\[ E(0) = \|D\|_2^2(0) + 2 \int_\Omega f(u)(0,x)dx \leq 0, \]

\[ \eta f(\eta) - 2(1 + 2\alpha) \int_0^\eta f(r)dr \leq \lambda_1 \alpha \eta^2 \quad \forall \eta \in \mathbb{R} \]

with \( \alpha > 0 \), \( \lambda_1 := \sup \{\|u\|_2/\|\nabla u\|_2 : u \in H^1_0(\Omega)\} \) and \( a'(0) > 0 \). There we have obtained a rough estimate for the life-span

\[ T \leq \beta_2 := 2[1 - (1 - k_2 a(0)^{-\alpha})^{1/2}]/(k_1 k_2) \]

with

\[ k_1 := \alpha a(0)^{-\alpha - 1} \sqrt{a^*(0)^2 - 4E(0)a(0)}, \quad k_2 := (-4\alpha^2 E(0)/k_1^2)^{\alpha/(1+2\alpha)}. \]

For \( n = 3 \) and \( f(u) = -u^3 \), there exist global solutions of (1.4) for small initial data \cite{24}; but if \( E(0) < 0 \) and \( a'(0) > 0 \) then the solutions are only local, i.e. \( T < \infty \) \cite{20}. John \cite{22} showed the nonexistence of solutions of the initial-boundary value problem for the wave equation \( \square u = A|u|^p \), \( A > 0 \), \( 1 < p < 1 + \sqrt{2} \), \( \Omega = \mathbb{R}^3 \). This problem was considered by Glassey \cite{20} in the two dimensional case \( n = 2 \); for \( n > 3 \) Sideris \cite{32} showed the nonexistence of global solutions under the conditions \( \|u_0\|_1 > 0 \) and \( \|u_1\|_1 > 0 \). According to this result Strauss \cite{33} p. 27 guessed that the solutions for the above mentioned wave equation are global for \( \Omega = \mathbb{R}^n \), \( p \geq p_0(n) = \lambda \), which is the positive root of the quadratic equation \( (n - 1)\lambda^2 - (n + 1)\lambda - 2 = 0 \).
Lemma 2.1. For we need a fundamental Lemma from \[28, p. 95\], \[23, p. 96\].

We shall set-up the fundamental lemmas in section 2 and prove the main the control of the boundedness of successive approximations solutions of equation (2.1). We want to extend our results \[27\] on ordinary differential equations and the wave equation \[25\] to the equation \([1,1]\), therefore we will deal with the existence and uniqueness theme of Emden-Fowler type wave equation \([1,1]\) with zero boundary values and initial values \(u(1) = u_0 \in H^2(a, b) \cap H^1_0(a, b)\) and \(u_t(1) = u_1 \in H^1_0(a, b)\), where \(p > 1\), and \(a, b\) are real numbers.

We are not aware of any other paper discussing this theme. We make a substitution \(t = e^s\), \(u(t, x) = v(s, x)\) to avoid degeneration of the equation \([1,1]\) which can be transferred into a nonlinear wave equation with negative linear damping \((2.1)\) below. The main difficulties in constructing our existence result for equation \((2.1)\) are the use of the Banach Fixed Point Theorem in a suitable solution space and the control of the boundedness of successive approximations solutions of equation \((2.1)\). We shall set-up the fundamental lemmas in section 2 and prove the main result in section 3.

2. Fundamental lemmas

To obtain the existence of solutions to \((1.1)\) with zero boundary values, \(u(t, a) = u(t, b) = 0\), and initial values \(u(1) = u_0 \in H^2(a, b) \cap H^1_0(a, b)\), \(u_t(1) = u_1 \in H^1_0(a, b)\), we need a fundamental Lemma from \([28, p. 95], [23, p. 96]\).

**Lemma 2.1.** For \(f \in W^{1,1}([t_0, T), L^2(a, b))\) the linear wave equation

\[
\begin{align*}
\Box u &:= u_{tt} - u_{xx} = f(t, x) \quad \text{in } [1, T) \times (a, b) \\
u(t_0, \cdot) &:= u_0 \in H^2(a, b) \cap H^1_0(a, b), \\
u_t(t_0, \cdot) &:= u_1 \in H^1_0(a, b),
\end{align*}
\]

possesses exactly one solution \(u \in H^2 := C^1([t_0, T), H^1_0(a, b)) \cap C^2([t_0, T), L^2(a, b))\) with \(u(t) \in H^2(a, b)\) for all \(t \in [t_0, T]\). Furthermore,

\[
\frac{d}{dt} \int_a^b (u_t^2 + |\nabla u|^2) dx - 2 \int_a^b u_t f(t, x) dx = 0 \quad \text{a.e. in } [t_0, T).
\]

To prove the existence of a local weak solution of \((1.1)\) in \(H^2\), we make the substitution \(s = \ln t\), \(u(t, x) = v(s, x)\), then \((1.1)\) can be transformed into

\[
\begin{align*}
v_{ss} - v_{xx} &= v_s + |v|^{p-1} v := -h(v), \\
v(0, x) &= u(1, x) = u_0(x) := v_0(x), \\
v_s(0, x) &= u_1(x) := v_1(x).
\end{align*}
\]

For \(T > 0, S = \ln T\) and \(v \in H^2 = C^1([0, S), H^1_0(a, b)) \cap C^2([0, S), L^2(a, b))\), we want to prove that \(h(v) \in W^{1,1}([0, S), L^2(a, b))\), thus we build the following Lemma.

**Lemma 2.2.** For \(T > 0, S = \ln T, v \in H^2\), we have \(h(v) \in W^{1,1}([0, S), L^2(a, b))\);

that is,

\[
\int_0^S \left( \|h(v)\|_2 + \|\frac{\partial}{\partial s} h(v)\|_2 \right) ds < \infty \quad \text{if } \|v\|_{H^2} < \infty.
\]

**Proof.** By the definition of \(h(v)\) we have

\[
\|h(v)\|_{W^{1,1}}^2 = \left( \int_0^S \left( \|h(v)\|_2 + \|\frac{\partial}{\partial s} h(v)\|_2 \right) ds \right)^2
\]

\[
\begin{align*}
&\leq \int_0^S 1 \, ds \int_0^S \left( \| h(v) \|_2^2 + \| \frac{\partial}{\partial s} h(v) \|_2^2 \right) ds \\
&\leq 2S \int_0^S \left( \| h(v) \|_2^2 + \| \frac{\partial}{\partial s} h(v) \|_2^2 \right) ds,
\end{align*}
\]

and
\[
\left( \int_0^S \left( \| h(v) \|_2^2 + \| \frac{\partial}{\partial s} h(v) \|_2^2 \right) ds \right)^2
\leq 2S \int_0^S \left( \int_a^b (v_s + |v|^{p-1} v) dx + \int_a^b (v_{ss} + p|v|^{p-1} v_s) dx \right) ds
\leq 4S \int_0^S \int_a^b \left( v_s^2 + |v|^{2p} + v_{ss}^2 + p|v|^{2p-2} v_s^2 \right) dx ds
= 4S \int_0^S \int_a^b (v_s^2 + v_{ss}^2) dx ds + I + II,
\]

where
\[
I = \int_0^S \int_a^b |v|^{2p} dx ds, \quad II = p \int_0^S \int_a^b |v|^{2p-2} v_s^2 dx ds.
\]

The boundedness of \( \| h(v) \|_{W^{1,1}}^2 \) is equivalent to show the boundedness of these two integrals \( I \) and \( II \) for some small \( S \), near zero, and this can be deduced using the Sobolev inequality, since that for any fixed \( s \in [0, S] \) and \( x \in [a, b] \) we have the following estimates

\[
|v|^{2p}(s, x) \leq \left( \int_a^x |v_x|(s, \eta) d\eta \right)^{2p} \leq (x - a)^p \left( \int_a^x v_s^2(s, \eta) d\eta \right)^p,
\]

\[
\int_a^b |v|^{2p}(s, x) dx \leq \int_a^b \left( \int_a^x |v_x|(s, \eta) d\eta \right)^{2p} dx
\]
\[
\leq \int_a^b (x - a)^p \left( \int_a^x v_s^2(s, \eta) d\eta \right)^p dx
\]
\[
\leq \frac{1}{p + 1} (b - a)^{p+1} \left( \max_{s \in [0, S]} \int_a^x v_s^2(s, \eta) d\eta \right)^p,
\]

\[
I \leq \frac{1}{p + 1} (b - a)^{p+1} S \left( \max_{s \in [0, S]} \int_a^b v_s^2(s, \eta) d\eta \right)^p,
\]

\[
|v|^{2p-2} v_s^2(s, x) \leq 2|v|^{2p-2}(s, x) \left( v_1^2(x) + \left( \int_0^s v_{ss}(\xi, x) d\xi \right)^2 \right)
\]
\[
\leq 2|v|^{2p-2}(s, x) \left( v_1^2(x) + s \int_0^s v_{ss}^2(\xi, x) d\xi \right),
\]

\[
II = p \int_0^S \int_a^b |v|^{2p-2} v_s^2(s, x) ds dx
\]
\[
\leq 2p \int_a^b \int_0^S |v|^{2p-2}(s, x) \left( v_1^2(x) + s \int_0^s v_{ss}^2(\xi, x) d\xi \right) ds dx
= III + IV,
\]
where

\[ III = 2p \int_a^b \int_0^S v_s^2(x)|v|^{2p-2}(s,x) \, ds \, dx, \]

\[ IV = 2p \int_a^b \int_0^S |v|^{2p-2}(s,x) \left( \int_0^s v_{ss}^2(\xi,x) \, d\xi \right) \, ds \, dx. \]

By \[2.4\] and \[2.5\] we obtain

\[ III \leq 2p \int_0^S \left( \int_a^b |v|^{2p}(s,x) \, dx \right)^{\frac{p-1}{p}} \left( \int_a^b v_1^2(x) \, dx \right)^{1/p} ds \]

\[ \leq 2p \int_0^S \left( \max_{s \in [0,S]} \int_a^b v_s^2(s,\eta) \, d\eta \right)^{\frac{p-1}{p}} \left( \int_a^b v_1^2(\eta) \, d\eta \right)^{1/p} ds \]

\[ = \frac{4p^2}{p+1} (b-a)^{p+1} S \left( \int_a^b v_1'(\eta)^2 d\eta \right)^{p-1} \]

and

\[ IV \leq 2pS \int_a^b \int_0^S \left( (x-a)^{p-1} \left( \int_a^x v_s^2(s,\eta) \, d\eta \right)^{p-1} \right) \left( \int_0^s v_{ss}^2(\xi,x) \, d\xi \right) \, ds \, dx \]

\[ \leq 2pS(b-a)^{p-1} \left( \max_{s \in [0,S]} \int_a^b v_s^2(s,x) \, dx \right)^{p-1} \]

\[ \times \int_a^b \int_0^S \left( \int_0^s v_{ss}^2(\xi,x) \, d\xi \right) ds \, dx \]

\[ \leq 2pS^2(b-a)^{p-1} \left( \max_{s \in [0,S]} \int_a^b v_s^2(s,x) \, dx \right)^{p-1} \int_a^b \int_0^S v_{ss}^2(\xi,x) \, d\xi \, dx. \]

The following Lemma is easy to check, we omit its proof.

**Lemma 2.3.** Suppose that \( X \) is a Banach space and \( f_n : [0,T) \to X \) are differentiable functions and the sequence \( f_n(t) \) converges uniformly to \( f(t) \). If the sequence \( df_n(t)/dt \) converges to \( g(t) \), then \( f : [0,T) \to X \) is differentiable and \( df(t)/dt = g(t) \) in \( X \).

3. Existence of Solutions for the Emden-Fowler Type Wave Equation

From the three lemmas above, we can obtain the following local existence result.

**Theorem 3.1.** Suppose that \( p > 1 \), \( u_0 \in H^2(a,b) \cap H_0^1(a,b) \) and \( u_1 \in H_0^1(a,b) \), then the initial-boundary value problem for the semilinear wave equation (1.1) with \( u(1,x) = u_0(x) \), \( u_t(1,x) = u_1(x) \) and \( u(t,a) = 0 = u(t,b) \) on \([1,T]\), possesses exactly one solution in \( \mathcal{H}_2 \) for some \( T > 1 \).

**Proof.** Proof the existence of a local solution in

\[ \mathcal{H}_1 := C([1,T), H^2_0(a,b)) \cap C^1([1,T), L^2(a,b)). \]

By using the substitution \( s = \ln t \), \( u(t,x) = v(s,x) \), equation (1.1) can be transformed to

\[ v_{ss} - v_{xx} = v_s + |v|^{p-1} v := -h(v), \]

\[ v(0,x) = u(1,x) = u_0(x) := v_0(x), \]

\[ v(1,x) = u_1(x). \]
\[ v_s(0, x) = u_1(x) := v_1(x). \]

(1) For \( T > 0 \) and \( v \in H_2 = C^1([0, S), H^1_0(a, b)) \cap C^2([0, S), L^2(a, b)) \), by Lemma 2.2 we have that \( h(v) \in W^{1,1}([0, S), L^2(a, b)) \).

According to Lemma 2.1, let \( w := Tv \) be the solution of initial-boundary value problem for the equation

\[ \square w + h(v) = 0, \]

\[ w(0, \cdot) = v_0(\cdot) \in H^2(a, b) \cap H^1_0(a, b), \]

\[ w_s(0, \cdot) = u_1(\cdot) = v_1(\cdot) \in H^1_0(a, b), \]

we have \( w \in H_2, w(s) \in H^2(a, b) \) for all \( s \in [0, S) \) and

\[ \frac{d}{ds} ||Dw||^2_{L_2} + 2 \int_a^b w_s h(v(s, x))dx = 0. \]

Suppose that \( v_2 := su_0 \), then by Lemma 2.2 we get \( -h(v_2) = -h(su_0) = u_0 + su_0 |v_0|^{p-1} \in W^{1,1}([0, S), L^2(a, b)) \) and therefore, there exists a function \( v_3 \in H_2 \) which satisfies the initial-boundary value problem for the equation

\[ \square w + h(v_2) = 0, \]

\[ w(0, \cdot) := u_0(\cdot) \in H^2(a, b) \cap H^1_0(a, b), \]

\[ w_s(0, \cdot) := v_1(\cdot) \in H^1_0(a, b). \]

Let \( v_{m+1} := Tv_m, m \geq 2 \) be the solution of the initial-boundary value problem for the linear equation

\[ \square v_{m+1} + h(v_m) = 0 \quad \text{in} \quad [0, S) \times (a, b), \]

\[ v_{m+1}(0, \cdot) = u_0(\cdot) \in H^2(a, b) \cap H^1_0(a, b), \]

\[ (v_{m+1})_s(0, \cdot) = v_1(\cdot) \in H^1_0(a, b). \]

Therefore, by Lemma 2.1, we have \( v_{m+1}(s) \in H^2(a, b) \) for all \( s \in [0, S), v_{m+1} \in H_2, m \in \mathbb{N} \) and

\[ \frac{d}{ds} \int_a^b |Dv_{m+1}(s, x)|^2dx + 2 \int_a^b (v_{m+1})_s h(v_m(s, x))dx = 0 \quad \text{a.e. in} \quad [0, S), \quad (3.1) \]

where \( |Dv|^2 := u_x^2 + |v_x|^2 \). Set \( A_{m+1}(s) := ||Dv_{m+1}(s)||_2 \). Then by (2.5) we find that

\[ (A_{m+1}(s))^2' \leq 2 A_{m+1}(s)^2 ||h(v_m(s))||_2 \quad \text{a.e. in} \quad [0, S)), \quad (3.2) \]

\[ A_{m+1}(s) \leq A_{m+1}(0) + \int_0^s ||h(v_m(r))||_2 dr \quad (3.3) \]

and

\[ A_{m+1}(s) \]

\[ \leq ||u_1||_2 + ||u_0||_2 + \int_0^s ||((v_m)_s + |v_m|^{p-1}v_m)||_2 dr \]

\[ \leq ||u_1||_2 + ||u_0||_2 + \int_0^s ((v_m)_s + |v_m|^{p})_2 + \int_0^s \left( \frac{1}{p+1} (b-a)^{\frac{p+1}{p}} \right) ||(v_m)_s||_2^p dr \quad (3.4) \]
for every $m - 1 \in \mathbb{N}$, almost everywhere in $[0, S)$, besides $A_{m+1}(s) = 0$.

(2) Since that $h(s v_0) = h(v_2) \in W^{1,1}([0, S), L^2(a, b))$, we get $v_3 \in H_2$ and by the inequality $(3.3)$, we obtain $-h(v_2) = u_0 + s u_0 |su_0|^{p-1},$

$$A_3(s) \leq A_3(0) + \int_0^s \|h(ru_0)\|_2 dr$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \int_0^s \|u_0 + ru_0|ru_0|^{p-1}\|_2 dr$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \int_0^s \left( \int_a^b (u_0^2 + r^2 u_0^{2p}) dx \right)^{1/2} dr$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \|\sqrt{2} \int_0^s \left( \int_a^b u_0^2 dx \right)^{1/2} + r^p \int_0^b u_2^p \right)$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \sqrt{2} S \left( \|u_0\|_2 + \frac{1}{p+1} S^p \right)$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \sqrt{2} S \left( \|u_0\|_2 + \frac{1}{p+1} S^p \right).$$

Set

$$\|v\|_{\infty, S} := \sup \{ \|Dv(s)\|_2 : 0 \leq s \leq S \},$$

$$M := 1 + 2(\|u_1\|_2 + \|u_0\|_2),$$

$$S := \frac{1}{2} \min \left\{ \frac{1}{1 + \sqrt{p}(b-a)^{2p} + 1 + \|u_0\|_2}, \frac{1 + \|u_1\|_2 + \|u_0\|_2}{2(1 + \|u_0\|_2) + M(1 + (b-a)^p)} \right\}.$$

Then using $(3.5)$, we obtain

$$A_3(s) = \|Dv_3\|_2(s) \leq A_3(0) + \int_0^s \|h(ru_0)\|_2 dr$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \sqrt{2}(S - 0)(\|u_0\|_2 + \frac{1}{p+1}(S - 0)^p \|u_0\|_2) \leq M.$$

Consequently $\|v_3\|_{\infty, S} \leq M$. Suppose that $\|v_m\|_{\infty, S} \leq M$, then by the definition of $S$ and $(3.4)$,

$$A_{m+1}(s) \leq A_3(0) + \int_0^s \|h(ru_0)\|_2 dr$$

$$\leq \|u_1\|_2 + \|u_0\|_2 + \int_0^s \left( \|\|v_m\|_2 + \sqrt{2}(S - 0)(\|v_m\|_2 + \frac{1}{p+1}(b-a)^{p+1} \|v_m\|_2) \right) dr \leq M.$$

Thus we get $\|v_{m+1}\|_{\infty, S} \leq M$ for all $m \in \mathbb{N}$.

(3) We claim that $v_m$ is a Cauchy sequence in

$$H_3 := C([0, S), H^1_0(a, b)) \cap C^1([0, S), L^2(a, b)).$$

By Lemma 2.1 and $(3.3)$,

$$\|D(v_{m+1} - v_m)(s)\|_2(s)$$
\[ \leq \int_0^s \| h(v_m) - h(v_{m-1}) \|_2 dr \]

\[ \leq \int_0^s \left( \| (v_m - v_{m-1})_s \|_2 + \frac{\sqrt{D}}{2} (b - a)^p K_v \| v_m - v_{m-1} \|_2 \right) dr, \]

where \( K_v = \| (v_m)_x \|_2^{p-1} + \| (v_{m-1})_x \|_2^{p-1} \), therefore

\[ \| D(v_{m+1} - v_m)(s) \|_2(s) \]

\[ \leq s \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \| D(v_m - v_{m-1})(s) \|_2 \]

\[ \leq S \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \| v_m - v_{m-1} \|_\infty, S \quad \forall s \in [0, S), \]

and

\[ \| v_{m+1} - v_m \|_\infty, S \leq S \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \| v_m - v_{m-1} \|_\infty, S. \]

It follows that

\[ \| v_{m+\tilde{c}} - v_m \|_\infty, S \leq \frac{S(1 + \sqrt{D}(b - a)^{1 + \frac{p}{2} M^{p-1}}) m - 2}{1 - S(1 + \sqrt{D}(b - a)^{1 + \frac{p}{2} M^{p-1}})} \to 0 \quad (3.7) \]

as \( m \to \infty \). Since

\[ S \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \leq \frac{1}{2}, \]

(ii) We prove the uniqueness of the solutions in \( \mathcal{H}_1 \). Suppose that \( w \) is the limit of \( v_m \), and \( v \in \mathcal{H}_1 \) is an another solution for (2), then

\[ \frac{d}{ds} \int_a^b | Dv_{m+1}(s, x) - Dv(s, x) |^2 dx \]

\[ \leq 2 \int_a^b \| (v_{m+1})_s - v_s \| (h(v_m) - h(v)) dx \]

\[ \leq 2 \| D(v_{m+1} - v) \|_2(s) \| h(v_m) - h(v) \|_2(s), \]

\[ \| D(v_{m+1} - v)(s) \|_2(s) \leq \int_0^s \| h(v_m) - h(v) \|_2(r) dr \]

\[ \leq s \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \| D(v_m - v)(s) \|_2 \]

\[ \leq S \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \| v_m - v \|_\infty, S \quad \forall s \in [0, S). \]

Thus

\[ \| v_{m+1} - v \|_\infty, S \leq S \left( 1 + \frac{\sqrt{D}}{2} (b - a)^{1 + \frac{p}{2} M^{p-1}} \right) \| v_m - v \|_\infty, S. \]

It follows that

\[ \| w - v \|_\infty, S \leq \| w - v_{m+1} \|_\infty, S + \| v_{m+1} - v \|_\infty, S \to 0 \]

as \( m \to \infty \), so \( w \equiv v \) in \( \mathcal{H}_1 \).

(iii) Now we show the local existence \( u \) in \( \mathcal{H}_2 \). Let \( S \) and \( M \) be the same as above.
(1) For a $S > 0$ and $v_m \in H_2$, we have $h(v_m) \in W^{1,1}([0, S], L^2(a,b))$. According to Lemma 2.1 we have $v_{m+1}(s) \in H^2(a,b)$ for all $s \in [0, S)$, $m \in \mathbb{N}$ and
\[
\frac{d}{ds} \int_{a}^{b} |Dv_{m+1}(s,x)|^2 dx = -2 \int_{a}^{b} (v_{m+1})_s h(v_m) dx \quad \text{a.e. in } [0, S)
\]
also
\[
\frac{d}{ds} \|Dv_{m+1}\|_2^2(s) + \frac{d^2}{ds^2} \int_{a}^{b} v_{m+1}^2 dx = -2 \int_{a}^{b} (v_{m+1})_s h(v_m) dx + 2 \int_{a}^{b} (v_{m+1}(v_{m+1})_s + (v_{m+1})_x^2) dx
\]
\[
= -2 \int_{a}^{b} (v_{m+1})_s h(v_m) dx + 2 \int_{a}^{b} ((v_{m+1})_s - (v_{m+1})_x^2) dx,
\]
and
\[
\frac{d}{ds} \|Dv_{m+1}\|_2^2(s) + \frac{d^2}{ds^2} \int_{a}^{b} v_{m+1}(s,x)^2 dx \leq 2(\|v_{m+1}\|_2 + \|(v_{m+1})_s\|_2)\|h(v_m)\|_2(s) + 2\|Dv_{m+1}\|_2^2(s),
\]
for every $m \geq 2$, almost everywhere in $[0, S)$.

(2) We claim that $v_m$ is a Cauchy sequence in $H_2$. By similar arguments as in establishing inequalities (3.6)–(3.8), we obtain
\[
\frac{d}{ds} \|D(v_{m+k} - v_m)\|_2^2(s) + \frac{d^2}{ds^2} \int_{a}^{b} (v_{m+k} - v_m)(s,x)^2 dx \leq 2(\|v_{m+k} - v_m\|_2 + \|(v_{m+k} - v_m)_s\|_2)\|h(v_{m+k-1}) - h(v_{m-1})\|_2(s)
\]
\[
+ 2\|D(v_{m+k} - v_m)\|_2^2(s) \rightarrow 0 \quad \text{as } m \rightarrow \infty.
\]
By (i), (ii) and Lemma 2.3 we obtain the assertions of Theorem 3.1. 

**References**


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