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APPROXIMATION OF THE SINGULARITY COEFFICIENTS OF AN ELLIPTIC EQUATION BY MORTAR SPECTRAL ELEMENT METHOD

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Abstract. In a polygonal domain, the solution of a linear elliptic problem is written as a sum of a regular part and a linear combination of singular functions multiplied by appropriate coefficients. For computing the leading singularity coefficient we use the dual method which based on the first singular dual function. Our aim in this paper is the approximation of this leading singularity coefficient by spectral element method which relies on the mortar decomposition domain technics. We prove an optimal error estimate between the continuous and the discrete singularity coefficient. We present numerical experiments which are in perfect coherence with the analysis.

1. Introduction

If the data are smooth, the solution of an elliptic partial differential equation is not regular when the domain is polygonal. For a Dirichlet problem of the Laplace operator, we define some singular functions depending only on the geometry of the domain. The solution is written as the sum of a regular part and singular functions multiplied by appropriate coefficients [6, 5]. For approximating the leading singularity coefficients we use two algorithms. The first one is Strang and Fix algorithm [9], which consists to add the leading singularity function to the discrete space see [4]. For the second algorithm we apply the dual method. The numerical computation of the leading singularity coefficient has been performed by finite elements, see Amara and Moussaoui [1, 2]. This coefficient is physically meaningful in solid mechanics (crack propagation). We use the mortar spectral element method: the domain is decomposed in a union of finite number of disjoint rectangles, the discrete functions are polynomials of high degree on each rectangle and are enforced to satisfy a matching condition on the interfaces. This technique is nonconforming because the discrete functions are not continuous. We refer to Bernardi, Maday and Patera [3] for the introduction of the mortar spectral element method.

An outline of this article is as follows. In the second section, we give the dual singular function and the formula for finding the leading coefficient of the singularity. This coefficient does not depend on the data function or the geometry of the domain but it just depends on the solution. In section 3, we present the discrete

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problem and the discrete leading singularity coefficient. The section 4 is devoted to the estimation of the error and we prove the optimality. Finally, a results of a numerical test are given in Section 5.

2. Dual singular function and the coefficient of the singularity

In the rest of the paper, we assume that our domain Ω is a polygon of $\mathbb{R}^2$ such that there exists a finite number of open rectangles $\Omega_k, 1 \le k \le K$, for which

$$\bar{\Omega} = \cup_{k=1}^{K} \bar{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_l = \emptyset \quad \text{for } k \neq l. \tag{2.1}$$

We suppose also that the intersection of each $\bar{\Omega}_k$ (for $1 \le k \le K$) with the boundary $\partial\Omega$ is either empty or a corner or one or several entire edges of Ω_k. We choose the coordinate axis parallel to the edge of the Ω_k.

Handling the singularity is a local process. Therefore, it is not restricted to suppose that Ω has a unique non-convex corner $\mathbf{a}$ with an angle ω equal either to $3\pi/2$ or to 2π (case of the crack). We choose the origin of the coordinate axis at the point $\mathbf{a}$, we introduce a system of polar coordinates (r, θ) where r stands for the distance from $\mathbf{a}$ and θ is such that the line $\theta = 0$ contains an edge of $\partial\Omega$. For more technical proof that will appear later we need to make the following conformity assumption: if the intersection of $\bar{\Omega}_k$ and $\bar{\Omega}_l, k \neq l$ contains $\mathbf{a}$, then it contains either $\mathbf{a}$ or both an edge of Ω_k and Ω_l. Let Σ be the open domain in Ω such that $\bar{\Sigma}$ is the union of the $\bar{\Omega}_k$ which contain $\mathbf{a}$.

The model equation under consideration is the following Dirichlet problem for the Laplace operator:

$$\begin{gathered} -\Delta u = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.2}$$

If the data f belongs to $H^{s-2}(\Omega)$, then the above problem admits a unique solution u belongs to $H^s(\Omega)$. This solution is decomposed as:

$$u = u_r + \lambda S_1, \tag{2.3}$$

where the function u_r belongs to $H^s(\Omega)$ for $s < 1 + \frac{2\pi}{\omega}$ such that

$$\|u_r\|_{H^s(\Omega)} + |\ \lambda\ | < C\|f\|_{H^{s-2}(\Omega)}.$$

Here λ is the leading singularity coefficient and S_1 is the first singular function given by

$$S_1(r, \theta) = \chi(r) r^{\frac{\pi}{\omega}} \sin(\pi\theta/\omega),$$

with χ is a smooth cut-off function with supported on $\bar{\Sigma}$ and is equal to 1 in a neighborhood of $\mathbf{a}$ [7].

Since the Laplace operator with the homogeneous Dirichlet boundary conditions is self-adjoint operator, we define the dual singular function associated with S_1 by:

$$S_1^*(r, \theta) = \chi(r) r^{-\pi/omega} \sin(\pi\theta/\omega).$$

We remark that this function does not belong to $H^1(\Omega)$, however ΔS_1^* belongs to the space $L^2(\Omega)$. This allows us to define the function φ^* in $H^1(\Omega)$ solution of the following problem

$$\int_\Omega \nabla\varphi^* \nabla\psi \, dx\, dy = \int_\Omega \Delta S_1^* \psi dx \quad \text{for all } \psi \in H_0^1(\Omega). \tag{2.4}$$

More details are given in [8] and [7].

Proposition 2.1. *Let u, respectively φ^* be the solution of the problem* (2.2)*, respectively* (2.4)*, then the coefficient of the singularity λ satisfies*

$$\lambda\pi = \int_\Omega f S_1^* \, dx + \int_\Omega u \Delta S_1^* \, dx = \int_\Omega f(S_1^* - \varphi^*) \, dx. \tag{2.5}$$

Proof. Let $C_1 = \Omega \cap B(\mathbf{a}, R)$ where $B(\mathbf{a}, R)$ is the ball of center $\mathbf{a}$ and the radius R. Since the cut off function χ is equal to 1 in a neighborhood of $\mathbf{a}$, we can choose R such that: $\Delta S_1 = \Delta S_1^* = 0$ in C_1. Then, using (2.2) and (2.3) we have

$$\int_\Omega f S_1^* \, dx = \int_{C_1} -\Delta u_r S_1^* \, dx + \int_{\Omega \setminus C_1} -\Delta u S_1^* \, dx.$$

Integrating by parts, this yields

$$\int_\Omega f S_1^* \, dx + \int_{C_1} u \Delta S_1^* \, dx = \int_0^\omega \Big(\partial_r (u - u_r) S_1^* - (u - u_r) \partial_r (S_1^*) \Big)(R, \theta) r \, d\theta.$$

Replacing $u - u_r$ with λS_1, we obtain

$$\int_\Omega f S_1^* \, dx + \int_\Omega u \Delta S_1^* \, dx = \lambda \int_0^\omega \Big(\partial_r (S_1) S_1^* - S_1 \partial_r (S_1^*) \Big)(R, \theta) = \pi \lambda.$$

To get the second equality, we replace ψ by u in the problem (2.4) and we integrate by parts. □

3. Approximation of the singularity coefficient

In this section we recall some basic notion concerning the spectral element method and the mortar matching condition. Since the discretization is essentially a Galerkin method relying on the variational formulation (2.4), we need to define the discrete space and give the quadrature formula which is used to compute the integrals of polynomials [10].

The discretization parameter is a K-tuple of integers $(N_1, \dots, N_K)$ larger than or equal to 2 denoted by δ. For any nonnegative integer n and for $1 \le k \le K$, we denote by $\mathbb{P}_n(\Omega_k)$ the space of polynomials on Ω_k such that their degree with respect to each variable x and y is less than or equal to n. The restriction of discrete functions to Ω_k will belong to $\mathbb{P}_{N_k}(\Omega_k)$.

Let us recall the Gauss-Lobatto quadrature formula: for any positive integer n, there exists a unique set of $(n+1)$ nodes ξ_j in $[-1, 1]$ and of $(n+1)$ positive weights ρ_j (for $0 \le j \le n$) such that the following equality holds for any polynomial ϕ with degree less than or equal to $2n - 1$:

$$\int_{-1}^1 \phi(z) dz = \sum_{j=0}^n \phi(\xi_j) \rho_j.$$

If T^k denotes an affine mapping from $(-1, 1)^2$ onto Ω_k, we define a bilinear form on continuous functions u and v on $\bar{\Omega}_k$ as follow:

$$(u, v)_{N_k} = \frac{\mid \Omega_k \mid}{4} \sum_{i=0}^{N_k} \sum_{j=0}^{N_k} (uoT^k)(\xi_i, \xi_j)(voT^k)(\xi_i, \xi_j) \rho_i \rho_j,$$

where $\mid \Omega_k \mid$ is the area of Ω_k.

We need some more notations to enforce the matching condition. Let ν be the set of all the corners of the Ω_k for $1 \le k \le K$. We denote by $\Gamma^{k,j}$ for $1 \le j \le 4$ the edges of Ω_k and $\gamma^{kl} = \bar{\Omega}_k \cap \bar{\Omega}_l$ for $k \ne l$. We make the further assumption that the boundary $\partial\Omega$ consists of entire edges of the Ω_k. Next we introduce the skeleton of the decomposition:

$$S = (\cup_{k=1}^{K} \partial\Omega_k) \setminus \partial\Omega,$$

and we assume that it is a disjoint union of mortars (γ_m), $1 \leq m \leq M$ (M is positive integer)

$$S = \cup_{m=1}^{M} \gamma_m \text{ and } \gamma_m \cap \gamma_{m'} = \emptyset \quad \text{for } m \neq m',$$

where each mortar γ_m is an entire edge of one rectangle Ω_k, denoted by $\Omega_{k(m)}$.

For any nonnegative integer n and for any segment γ, we denote by $\mathbb{P}_n(\gamma)$ the space of polynomials with degree less or equal to n on γ. The mortar space W_δ is then defined by

$$W_\delta = \{\phi \in L^2(S), \text{ such that for all } m,\ 1 \leq m \leq M,\ \phi/_{\gamma_m} \in \mathbb{P}_{N_{k(m)}}(\gamma_m)\}.$$

The space X_δ is then defined as in the standard mortar method [3]. It is the space of function v_δ in $L^2(\Omega)$ such that

- The restriction of v_δ to Ω_k, for $1 \leq k \leq K$, belongs to $\mathbb{P}_{N_k}(\Omega_k)$.
- The function v_δ vanishes on $\partial\Omega$.
- The mortar function φ defined on S by

$$\varphi/_{\gamma_m} = v_\delta/\Omega_{k(m)}/\gamma_m, \text{ for } 1 \leq m \leq M,$$

satisfies that for $1 \leq k \leq K$ and for any edge Γ of Ω_k contained in S,

$$\forall \psi \in \mathbb{P}_{N_k - 2}(\Gamma), \int_\Gamma (v_\delta/_{\Omega_k} - \varphi)(\tau)\psi(\tau)d\tau = 0.$$

Later we will need to define the space

$$X_\delta^- = \{v_\delta \in X_\delta \text{ such that } v_{\delta/\Omega_k} \in \mathbb{P}_{N_k - 1}(\Omega_k) \text{ for } 1 \leq k \leq K\}.$$

Remark 3.1. In the general case of nonconforming decomposition, the space X_δ is not embedded in $H^1(\Omega)$ since the functions in X_δ are not necessarily continuous through the interfaces. To ensure the continuity in a neighbourhood of $\mathbf{a}$, we assume the following conformity assumption:

> Conformity assumption: If $\mathbf{a}$ is the extremity of an edge $\Gamma^{k(m),j(m)}$, this edge necessarily coincides with the edge of an another rectangle Ω_l.

Then we suppose that $N_{k(m)} \leq N_l$.

Now, we define the scalar product on Ω as follows:

$$(\varphi, \psi)_\delta = \sum_{k=1}^{K} (\varphi, \psi)_{N_k} \quad \text{for all } \varphi, \psi \in \mathcal{C}^0(\bar{\Omega}_k).$$

Our goal now is to give a more accurate approximation than the Strang and Fix algorithm for calculating the singularity coefficient λ. In fact, let $X_\delta^* = X_\delta + \mathbb{R}S_1$ be the augmented discrete space. We consider the following two discrete problems:

- Find a function $u_\delta^* = u_\delta + \lambda S_1$ in X_δ^* such that

$$a_\delta^*(u_\delta^*, v_\delta^*) = \sum_{k=0}^{K} \int_{\Omega_k} f/_{\Omega_k} v_\delta^*/_{\Omega_k} \, dx\, dy \quad \text{for all } v_\delta^* = v_\delta + \mu S_1 \in X_\delta^*. \tag{3.1}$$

- Find φ_δ^* in X_δ^* such that
$$a_\delta^*(\varphi_\delta^*, \psi_\delta^*) = (\Delta S_1^*; \psi_\delta^*)_\delta \quad \text{for all } \psi_\delta^* \in X_\delta^*, \tag{3.2}$$
where
$$\begin{aligned} a_\delta^*(u_\delta^*, v_\delta^*) = \sum_{k=1}^K \Big(& (\nabla u_k, \nabla v_k)_{N_k} + \lambda \int_{\Omega_k} \nabla S_1 \nabla v_k \, dx\, dy \\ & + \mu \int_{\Omega_k} \nabla u_k \nabla S_1 \, dx\, dy + \lambda\mu \int_{\Omega_k} (\nabla S_1)^2 \, dx\, dy \Big). \end{aligned}$$

We refer to [4] for the numerical analysis and the implementation by the mortar spectral element method of this problems.

Proposition 3.2. *Let u, φ^*, u_δ^* and φ_δ^* be the respective solutions of the problems* (2.2),(2.4), (3.1) *and* (3.2)*, then*
$$\pi\lambda_\delta^* = \int_\Omega f S_1^* \, dx + \int_\Omega u_\delta^* \Delta S_1^* \, dx) = \int_\Omega f(S_1^* - \varphi_\delta^*) \, dx \tag{3.3}$$
and
$$\begin{aligned} \pi(\lambda - \lambda_\delta^*) = & \sum_{k=1}^K \int_{\Omega_k} \nabla(u - u_\delta^*)_{|\Omega_k} \nabla(\varphi^* - \varphi_\delta^*)/_{\Omega_k} \, dx \\ & + \sum_{1 \le k < l \le K} \int_{\gamma^{kl}} (\frac{\partial u}{\partial n_k})(\varphi_\delta^*/_{\Omega_k} - \varphi_\delta^*/_{\Omega_l}) - (\frac{\partial \varphi^*}{\partial n_k})(u_\delta^*/_{\Omega_k} - u_\delta^*/_{\Omega_l}) \, d\tau. \end{aligned} \tag{3.4}$$

Proof. To obtain (3.3), we proceed in the same way as in Proposition 2.1. Now, using (2.5) and (3.3), we obtain
$$\pi(\lambda - \lambda_\delta^*) = \int_\Omega f(\varphi_\delta^* - \varphi^*) \, dx = \sum_{k=1}^K \int_{\Omega_k} \Delta u_{/\Omega_k} (\varphi^* - \varphi_\delta^*)_{/\Omega_k} \, dx.$$
Integrating by part yields
$$\pi(\lambda - \lambda_\delta^*) = \sum_{k=1}^K \int_{\Omega_k} \nabla u \nabla(\varphi^* - \varphi_\delta^*) \, dx + \sum_{1 \le k < l \le K} \int_{\gamma^{kl}} (\frac{\partial u}{\partial n_k})(\varphi^*_{\delta/\Omega_k} - \varphi^*_{\delta/\Omega_l}) \, d\tau. \tag{3.5}$$
Let $\varphi_\delta^* = \varphi_\delta + \mu S_1$ and $u_\delta^* = u_\delta + \lambda S_1$ be in X_δ^*. Since
$$\begin{aligned} a_\delta^*(\varphi_\delta^*, u_\delta^*) = & \sum_{k=1}^K (\nabla \varphi_{\delta/\Omega_k}, \nabla u_{\delta/\Omega_k})_{N_k} + \lambda \int_{\Omega_k} \nabla S_1 \nabla u_\delta \, dx \\ & + \mu \int_{\Omega_k} \nabla \varphi_\delta \nabla S_1 \, dx + \lambda\mu \int_{\Omega_k} (\nabla S_1)^2 \, dx. \end{aligned}$$
If u_δ is in X_δ^- then
$$\sum_{k=1}^K (\nabla \varphi_\delta/_{\Omega_k}, \nabla u_\delta/_{\Omega_k})_{N_k} = \sum_{k=1}^K \int_{\Omega_k} \nabla \varphi_{\delta/\Omega_k} \nabla u_{\delta/\Omega_k} \, dx.$$
Using (3.2), we deduce that
$$a_\delta^*(\varphi_\delta^*, u_\delta^*) = \sum_{k=1}^K \int_{\Omega_k} \nabla \varphi_\delta^* \nabla u_\delta^* \, dx = \sum_{k=1}^K \int_{\Omega_k} \Delta S_1^* u_\delta^* \, dx. \tag{3.6}$$

From (3.6), $\Delta\varphi^* = \Delta S_1^*$ in the distributions sense and since $\varphi^* = 0$ on $\partial\Omega$, this yields

$$a_\delta^*(\varphi_\delta^*, u_\delta^*) = \sum_{k=1}^K \int_{\Omega_k} \Delta\varphi^*_{/\Omega_k} u^*_{\delta/\Omega_k}\, dx,$$

hence, integration by part yields

$$a_\delta^*(\varphi_\delta^*, u_\delta^*) = \sum_{k=1}^K \int_{\Omega_k} \nabla\varphi^*_{/\Omega_k} \nabla u^*_{\delta/\Omega_k}\, dx - \sum_{1\le k<l\le K} \int_{\gamma^{kl}} (\frac{\partial\varphi^*}{\partial n_k})(u_\delta^*/_{\Omega_k} - u_\delta^*/_{\Omega_l})\, d\tau. \tag{3.7}$$

Thanks to (3.6) and (3.7), we conclude that

$$\sum_{k=1}^K \int_{\Omega_k} \nabla(\varphi^* - \varphi_\delta^*)_{/\Omega_k} \nabla u^*_{\delta/\Omega_k}\, dx = \sum_{1\le k<l\le K} \int_{\gamma^{kl}} (\frac{\partial\varphi^*}{\partial n_k})(u_\delta^*/_{\Omega_k} - u_\delta^*/_{\Omega_l})\, d\tau.$$

Adding this equality to (3.6), we obtain de desired result. □

4. Error estimation

In the following theorem, we give our result concerning the error on the approximation of the singularity coefficient.

Theorem 4.1. *Let $s \ge 0$ and $f \in H^{s-2}(\Omega)$. The error between λ and λ_δ^* satisfies the following estimation: For $\varepsilon > 0$*

$$|\lambda - \lambda_\delta^*| \le CN^{-1}\Big(\sum_{k=1}^K N_k^{-\sigma_k}\Big)\|f\|_{H^{s-2}(\Omega)},$$

where $N = \inf_{1\le k\le K} N_k$ and

$$\sigma_k = \begin{cases} s-1 & \text{if } \bar\Omega^k \text{ contains no corners of } \bar\Omega,\\ \inf\{s-1, 8-\varepsilon\} & \text{if } \bar\Omega^k \text{ contains corners different from } \mathbf{a},\\ \inf\{s-1, \frac{4\pi}{\omega}-\varepsilon\} & \text{if } \bar\Omega^k \text{ contains } \mathbf{a}. \end{cases}$$

Proof. To estimate $|\lambda - \lambda_\delta^*|$, we have to estimate each term of the inequality (3.4). For the first term, using Cauchy-Schwarz and Poincaré-Friedrichs inequalities we deduce that

$$\big|\sum_{k=1}^K \int_{\Omega_k} \nabla(u - u_\delta^*)\nabla(\varphi^* - \varphi_\delta^*)\, dx\big| \le C\|u - u_\delta^*\|_*\|\varphi^* - \varphi_\delta^*\|_*. \tag{4.1}$$

Since u (respectively u_δ^*) is the solution of the continuous problem (2.2) (respectively the discrete problem (3.1)). As the same for φ and φ_δ^*, are respectively the solutions of the problems (2.4) and (3.2) with second member equal to ΔS_1^* in $L^2(\Omega)$, we conclude by [4, result (5.16)].

From the continuity of S_1, we deduce that for any function φ_δ^* in X_δ^* the jump $\varphi^*_{\delta/\Omega_k} - \varphi^*_{\delta/\Omega_l}$ is equal to $\varphi_{\delta/\Omega_k} - \varphi_{\delta/\Omega_l}$ which vanishes on Σ due to the conformity assumption.

We note also that $u = u_r$ on $\Omega\setminus\bar\Sigma$. Hence

$$\int_{\gamma^{kl}} (\frac{\partial u}{\partial n_k})(\varphi^*_{\delta/\Omega_k} - \varphi^*_{\delta/\Omega_l})\, d\tau = \int_{\gamma^{kl}} (\frac{\partial u_r}{\partial n_k})(\varphi^*_{\delta/\Omega_k} - \phi)\, d\tau - \int_{\gamma^{kl}} (\frac{\partial u_r}{\partial n_k})(\varphi^*_{\delta/\Omega_l} - \phi)\, d\tau,$$

where ϕ is the mortar function associated to φ_δ. So, the estimation of this quantity can be evaluated as in [4, result (5.24)]. If Γ^k is not a mortar then

$$
\begin{aligned}
&|\sum_{1\le k<l\le K}\int_{\gamma^{kl}}(\frac{\partial u}{\partial n_k})(\varphi^*_{\delta/\Omega_k}-\varphi^*_{\delta/\Omega_l})\,d\tau|\\
&\le C\sum_{k=1}^{K}\sum_{j=1}^{4}\inf_{\psi_{k,j}\in\mathbb{P}_{N_k-2}(\Gamma^{kj})}\|\frac{\partial u_r}{\partial n_k}-\psi_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}\|\varphi-\varphi^*\|_*,
\end{aligned}
\tag{4.2}
$$

by taking $\psi_{k,j}=\Pi_{N_k-2}(\frac{\partial u_r}{\partial n_k})$, where Π_{N_k-2} is the orthogonal projection operator from $L^2(\Gamma^k)$ in $\mathbb{P}_{N_{k-2}}(\Gamma^k)$. Now, let us consider

$$
u_r=\widetilde{u}_r+\widetilde{\lambda}S_2,
$$

where S_2 is the second singular function [4], its proved in the same way that

$$
\begin{aligned}
&|\sum_{1\le k<l\le K}\int_{\gamma^{kl}}(\frac{\partial u}{\partial n_k})(\varphi^*_{\delta/\Omega_k}-\varphi^*_{\delta/\Omega_l})\,d\tau|\\
&\le C\sum_{k=1}^{K}\sum_{j=1}^{4}\inf_{\psi_{k,j}\in\mathbb{P}_{N_k-2}(\Gamma^{kj})}\|\frac{\partial \widetilde{u}_r}{\partial n_k}-\psi_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}\|\varphi-\varphi^*\|_*.
\end{aligned}
$$

Using the fact that $\widetilde{u}_r$ belongs to $H^s(\Omega)$ for s between $3/2$ and $1+\frac{3\pi}{\omega}$, we take $\psi_{kj}=\Pi_{N_k-2}(\frac{\partial\widetilde{u}_r}{\partial n_k})$ in $\mathbb{P}_{N_k-2}(\Gamma^{kj})$ to deduce the desired estimation.

Now, to get the estimation of the third term, we note that ΔS_1^* belongs to $L^2(\Omega)$. Since the function φ^* is the sum of $\widetilde{\varphi}_r^*$ in $H^2(\Omega)$ and a linear combination of S_1 and S_2 then, for the same reasons as above we have

$$
\begin{aligned}
&|\sum_{1\le k<l\le K}\int_{\gamma^{kl}}(\frac{\partial \varphi^*}{\partial n})(u^*_{\delta/\Omega_k}-u^*_{\delta/\Omega_l})\,d\tau|\\
&\le C(\inf_{v_\delta\in X_\delta}\|u_r-v_\delta\|_*)\sum_{k=1}^{K}\sum_{j=1}^{4}\inf_{\mathcal{X}_{k,j}\in\mathbb{P}_{N_k-2}(\Gamma^{kj})}\|\frac{\partial\widetilde{\varphi}_r}{\partial n_k}-\mathcal{X}_{kj}\|_{(H^{1/2}(\Gamma^{kj}))'}.
\end{aligned}
\tag{4.3}
$$

Since $\frac{\partial\widetilde{\varphi}_r}{\partial n_k}$ belongs to $H^{1/2}(\Gamma^{kj})$, we can choose $\mathcal{X}_{k,j}=\Pi_{N_k-2}(\frac{\partial\widetilde{\varphi}_r}{\partial n_k})$ in $\mathbb{P}_{N_k-2}(\Gamma^{kj})$ and consider the fact that

$$
\inf_{v_\delta\in X_\delta}\|u_r-v_\delta\|_*\le\inf_{v_\delta\in X_\delta}\|\widetilde{u}_r-v_\delta\|_*+|\,\widetilde{\lambda}\,|\inf_{v_\delta\in X_\delta}\|S_2-v_\delta\|_*.
$$

We complete the proof by combining (4.1), (4.2) and (4.3). □

Remark 4.2. Note that the convergence order is as $N^{\epsilon-3}$ in the case of the crack. However, in the case $\omega=\frac{3\pi}{2}$, the convergence order is as $N^{\epsilon-\frac{11}{3}}$. This proves the high accuracy of the method.

5. Implementation and numerical results

To write the algebraic system of the problem (3.1), it is necessary to choose a basis of the discrete space X_δ^*. This basis is naturally defined by the local basis (on each sub-domain). Let h_j^N be the Lagrange polynomials

$$
h_j^N\in\mathbb{P}([-1,1]),\quad h_j^N(\xi)=\delta_{ij}\quad\text{for }0\le i,j\le N.
$$

Here ξ_i are the $(N+1)$ Gauss-Lobatto nodes. Then for all v_δ in X_δ

$$v_\delta(x,y)_{/\Omega_k} = \sum_{i=0}^{N_k}\sum_{j=0}^{N_k} v_{N_k}^{i,j} h_i^{N_k}(x) h_j^{N_k}(y),$$

where $v_{N_k}^{i,j} = v_\delta(\xi_i^k, \xi_j^k)_{/\Omega_k}$, the ξ_i^k, respectively $h_i^{N_k}$ deduced from ξ_i, respectively from h_i^N by translation and dilatation.

Now, for v_δ^* in X_δ^* there exists a function (v_δ, λ) belongs to $X_\delta \times \mathbb{R}$ such that $v_\delta^* = v_\delta + \lambda S_1$ and

$$v_\delta^*(x,y)_{/\Omega_k} = \sum_{i=0}^{N_k}\sum_{j=0}^{N_k} v_{N_k}^{i,j} h_i^{N_k}(x) h_j^{N_k}(y) \quad + \quad \lambda S_{1/\Omega_k}.$$

Note that the internal degrees of freedom $v_{N_k}^{i,j}$ for $1 \le i,j \le (N_k - 1)$ are free. However, the boundary degrees of freedom found by the transformation of the mortar functions, they are linked by the following integral matching condition

$$\int_{\Gamma^{k,l}} (v_{\delta/\Omega^k} - \phi)(\tau)\psi(\tau)d\tau = 0 \quad \text{for all } \psi \in \mathbb{P}(\Gamma^{k,l}). \tag{5.1}$$

Let Q be the matrix reflecting the transmission condition on the interfaces of the sub-domains. Q^T its transpose purges the vector of unknowns from the false degrees of freedom. The calculation of this matrix is local for each pair edge-mortar.

Now, let ϕ be a mortar function such that

$$\phi_{/\gamma_m} = \sum_{i=0}^{N_{k(m)}} \phi_i^m h_i^{N_{k(m)}}(s), \quad 1 \le m \le M,$$

here s is a local coordinate defined on the interval $[-1,1]$.

At this stage, we need to determine a basis of the space of test functions $\mathbb{P}_{N_k-2}(\Gamma^{k,l})$ identified with $\mathbb{P}_{N_k-2}(]-1,1[)$. Then

$$\psi_{/\Gamma^{k,l}} = \sum_{q=1}^{N_k-1} \beta_q \eta_q^{N_k-2}(\tilde{s})$$

where

$$\eta_q^{N_k-2}(z) = \frac{(-1)^{N_k-q} L_{N_k}(z)}{(\xi_q - z)}, \quad z \in]-1,1[,\ q \in \{1,\dots,N_k-1\},$$

with $\tilde{s}$ is a local variable in $(-1,1)$ and L_N is a Legendre polynomial of order N (more details are given in [10]). Hence, the integral matching condition (5.1) is given as $v_\delta = Q\phi$.

The matrix Q is not built and the computation of the vector v_δ on the spectral nodes of the edge $\Gamma^{k,l}$, $1 \le k \le K$, $1 \le l \le 4$ depends only on the value of ϕ/γ^m. Then, we introduce the matching global matrix

$$\underbrace{\begin{pmatrix} v_{ij}^k/\text{internal} \\ v_{ij}^k/\text{boundary} \\ \lambda \end{pmatrix}}_{v_\delta^*} = \underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & \widetilde{Q} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} v_{ij}^k/\text{internal} \\ \phi_j^m \\ \lambda \end{pmatrix}}_{\tilde{v}_\delta^*},$$

for $1 \le k \le K$, and $1 \le m \le M$.

After giving the basis of the space X_δ^* and the matrix Q, we can write the discrete problem (3.1) in matrix form

$$\widetilde{A}u_\delta^* = F \tag{5.2}$$

where $\widetilde{A}$ takes the form

$$\begin{pmatrix}
(\nabla(h_ih_j);\nabla(h_ph_q))_{N_1} & 0 & . & . & 0 & \int_{\Omega_1}\nabla S_1\nabla(h_ph_q)\\
0 & & & & & \vdots\\
\vdots & & & & & \int_{\Omega_{N_\Delta}}\nabla S_1\nabla(h_ph_q)\\
\vdots & & & & & 0\\
\vdots & & & & & \vdots\\
0 & . & . & 0 & (\nabla(h_ih_j);\nabla(h_ph_q))_{N_k} & 0\\
\int_{\Omega_1}\nabla S_1\nabla(h_ih_j) & . & \int_{\Omega_{N_\Delta}}\nabla S_1\nabla(h_ih_j) & 0 & . & \int_\Omega(\nabla S_1)^2
\end{pmatrix}$$

The k-th block $(\nabla(h_ih_j);\nabla(h_ph_q))_{N_k}$ for $1 \le i,j \le N_k - 1$ and $1 \le p,q \le N_k - 1$, represents the Neumann Laplace operator on the sub-domain Ω_k, and F is the second member given by

$$F = \begin{pmatrix} (h_ph_q, f)_{N_1} \\ \vdots \\ (h_ph_q, f)_{N_k} \\ \int_\Omega fS_1dx \end{pmatrix}.$$

Finally, u_δ^* is the vector of admissible unknown. It is constitute by the values of the solution at a collocation nodes in sub-domains and their respective boundary.

Note that we do not solve the system (5.2) because it has false degrees of freedom. However, the global system that we solve is the following

$$Q^T\widetilde{A}Q\widetilde{u}_\delta^* = Q^TF, \tag{5.3}$$

where $\widetilde{u}_\delta^*$ is the vector composed from the unknown on the internal nodes and the value of the mortar function ϕ on the skeleton S. The matrix $A = Q^T\widetilde{A}Q$ is symmetric and positive defined. Then, we will use the gradient conjugate method to solve the problem (5.3).

For this, let u_0 arbitrary, $r_0 = Q^TF - Au_0$, $q_0 = r_0$ and

$$\begin{gathered}
\alpha_n = \frac{(r_n, r_n)}{(q_n, Aq_n)}\,, \quad u_{n+1} = u_n + \alpha_n q_n\,,\\
r_{n+1} = r_n - \alpha_n Aq_n\,, \quad \beta_n = \frac{(r_{n+1}, r_{n+1})}{(r_n, r_n)}\,,\\
q_{n+1} = r_{n+1} + \beta_n q_n\,.
\end{gathered}$$

Although the resolution by the gradient algorithm is global, the calculation is made locally. Indeed, the product matrix-vector which is the most expensive induces the following elementary products

$$\widetilde{A}^k_{ijmn}u^k_m + C^k_{ij}\lambda = \delta_{jn}\rho_q\rho_nD_{qj}u^k_{mn} + \delta_{im}\rho_m\rho_qD_{qi}D_{qn}u^k_{mn} + C^k_{ij}\lambda, \tag{5.4}$$

where

$$\begin{gathered}
\widetilde{A}^k_{ijmn} = (\nabla(h_ih_j), \nabla(h_mh_n))_{N_k},\\
C^k_{ij} = \int_{\Omega_k}\nabla S_1\nabla(h_ih_j)dx, \quad D_{ij} = \frac{dh_j}{dz}(\xi_i)
\end{gathered}$$

is the monodimensional derivative operator and δ_{ij} is the Kronecker symbol.

The local matrices $\widetilde{A}^k_{ijmn}$ are full, thus the coast is as $O(N^4)$ operations and $O(N^4)$ memory space. The tensorisation which consists to write the product matrix-vector as:

$$\widetilde{A}^k_{ijmn}u^k_m = [\delta_{jn}D_{qj}[\rho_q\rho_n[D_{qm}u^k_{mn}]]] + [\delta_{in}D_{qi}[\rho_q\rho_m[D_{qn}u^k_{mn}]]]$$

reduces the cost of operations to $O(N^3)$ and the memory space to $O(N^2)$ by sub-domain.

We present now some numerical results on the approximation of the solution of the problem (3.1) and the singularity coefficient by applying the dual method. It consists of varying the parameter of discretization N and the problem data.

The test cases are done in a neighborhood of the singular corner **a**, i.e. four sub-domains in the case of the crack and three ones in the case of $\omega = 3\pi/2$ (see Figure 1).

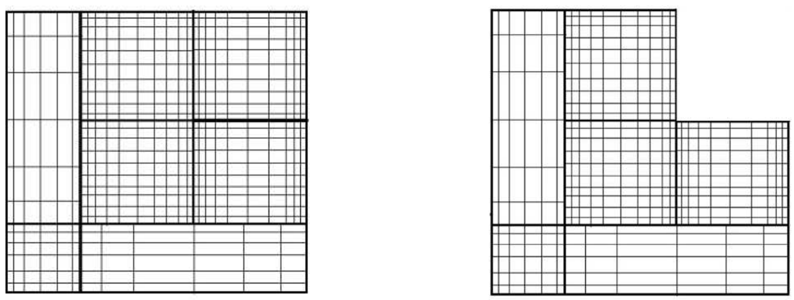

FIGURE 1. The domain when $\omega = 2\pi$ and $\omega = 3\pi/2$.

We present below some numerical results related to the calculation of the discrete solution of the problem (3.1) and the leading singularity coefficient by the dual method. In the following examples, λ^*_δ denotes the discrete singularity coefficient.

Example 1: $u(x,y) = \sin(\pi x^2)\sin(\pi y^2)$ and $\omega = 3\pi/2$.

N	5	10	15	20	30
λ^*_δ	$8.0\,10^{-4}$	$8.127\,10^{-7}$	$-2.731\,10^{-13}$	$-5.021\,10^{-14}$	$7.481\,10^{-14}$

Example 2: $u(r,\theta) = r^{1/2}\cos(\theta/2)$ and $\omega = 2\pi$.

N	5	15	20	30
λ^*_δ	0.9995.	0.9999.	1.	1.

Example 3: $u(r,\theta) = r^{2/3}\cos(2\theta/3)$ and $\omega = 3\pi/2$.

N	5	10	15	20	30
λ^*_δ	0.9987.	0.9996.	0.9999.	1.	1.

Figure 2 presents the error curves on the solution of the problem (3.1) (curves in blue) and the curves of error on the singularity coefficient (curves in red) in both $\omega = 2\pi$ and $\omega = 3\pi/2$. The continuous solutions are equal to the first singular function which corresponds to a singularity coefficient equal to 1 (Example 2 and Example 3). Error curves are made in logarithmic scale which permits the

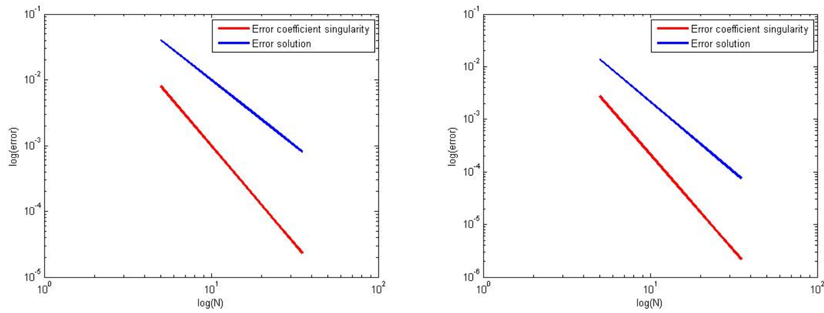

Figure 2. The error on the solution and the singularity coefficient.

computing of the convergence order corresponding to the slope of the curve. We remark that the convergence order on the singularity coefficient is better than the one of the solution. This order is equal to 2.8976 for the crack and to 1.9779 for the L-domain. However in the case of the solution, this order is equal to 1.9997 for the crack and to 0.9994 for the L-domain.

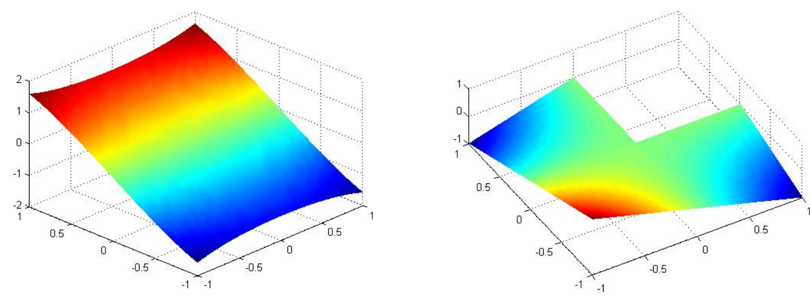

Figure 3. The discrete solution $\omega = 2\pi$ and $\omega = 3\pi/2$.

Let $\Gamma_0 = \{(r, \theta)$ such that $\theta = 0$ and $\theta = \omega\}$. Figure 3 presents the iso-values of the discrete solution in the case when $\omega = \frac{3\pi}{2}$ for the following Dirichlet problem

$$\begin{gathered} -\Delta u = 0 \quad \text{in } \Omega \\ u = x \quad \text{on } \partial\Omega/\bar{\Gamma}_0 \\ u = 0 \quad \text{on } \Gamma_0 \end{gathered}$$

and in the case when $\omega = 2\pi$ the discrete solution corresponding to the problem:

$$\begin{gathered} -\Delta u = 1 \quad \text{in } \Omega \\ u = x^2 \quad \text{on } \partial\Omega \end{gathered}$$

We consider in the Example 4 the calculation of the leading singularity coefficient in the case of the crack for the following Neumann problem:

$$-\Delta u = f \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega/\bar{\Gamma}_0$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_0.$$

Under the following condition assuring the existence of the solution

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\tau = 0.$$

Example 4: $f = 0$, $g = x$ on $\partial\Omega/\Gamma_0$, $g = 0$ on Γ_0 and $\omega = 2\pi$.

N	10	15	20	30	40
λ_δ^*	0.2680.	0.2701.	0.2752.	0.2787.	0.2787.

For Example 4, if we compare these results with those found in the case of a discretization with finite element method (see [2]), we obtain a better precision. This is due to the accuracy of the spectral method.

Conclusion. In this work we dealt with the approximation of the leading singularity coefficient by mortar spectral element method. The results obtained using the dual method are better than those obtained by Strang and Fix algorithm (see [4]). This confirm the theory since the dual method gives us an optimal error estimate and bring to light the efficiency of spectral discretization of this type of problem.

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